

Astro 448

Cosmic Microwave Background

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Astro 448

Acoustic Kinematics

Recombination

- Equilibrium number density distribution of a non-relativistic species

$$n_i = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} e^{-m_i/T}$$

- Apply to the $e^- + p \leftrightarrow H$ system: Saha Equation

$$\begin{aligned} \frac{n_e n_p}{n_H n_b} &= \frac{x_e^2}{1 - x_e} \\ &= \frac{1}{n_b} \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-B/T} \end{aligned}$$

where $B = m_e + m_p - m_H = 13.6\text{eV}$

- Naive guess of $T = B$ for recombination would put $z_* \approx 45000$.

Recombination

- But the photon-baryon ratio is very low

$$\eta_{b\gamma} \equiv n_b/n_\gamma \approx 3 \times 10^{-8} \Omega_b h^2$$

- Eliminate in favor of $\eta_{b\gamma}$ and B/T through

$$n_\gamma = 0.244T^3, \quad \frac{m_e}{B} = 3.76 \times 10^4$$

- Big coefficient

$$\frac{x_e^2}{1 - x_e} = 3.16 \times 10^{15} \left(\frac{B}{T} \right)^{3/2} e^{-B/T}$$

$$T = 1/3\text{eV} \rightarrow x_e = 0.7, \quad T = 0.3\text{eV} \rightarrow x_e = 0.2$$

- Further delayed by inability to maintain equilibrium since net is through 2γ process and redshifting out of line

Thomson Scattering

- Thomson scattering of photons off of free electrons is the most important CMB process with a cross section (averaged over polarization states) of

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25} \text{cm}^2$$

- Density of free electrons in a fully ionized $x_e = 1$ universe

$$n_e = (1 - Y_p/2)x_e n_b \approx 10^{-5} \Omega_b h^2 (1+z)^3 \text{cm}^{-3},$$

where $Y_p \approx 0.24$ is the Helium mass fraction, creates a high (comoving) Thomson **opacity**

$$\dot{\tau} \equiv n_e \sigma_T a$$

where dots are conformal time $\eta \equiv \int dt/a$ derivatives and τ is the optical depth.

Temperature Fluctuations

- Observe blackbody radiation with a temperature that differs at 10^{-5} coming from the surface of last scattering, with distribution function (specific intensity $I_\nu = 4\pi\nu^3 f(\nu)$ each polarization)

$$f(\nu) = [\exp(2\pi\nu/T(\hat{\mathbf{n}})) - 1]^{-1}$$

- Decompose the temperature perturbation in spherical harmonics

$$T(\hat{\mathbf{n}}) = \sum_{\ell m} T_{\ell m} Y_{\ell m}(\hat{\mathbf{n}})$$

- For Gaussian random fluctuations, the statistical properties of the temperature field are determined by the power spectrum

$$\langle T_{\ell m}^* T_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell$$

where the δ -function comes from statistical isotropy

Spatial vs Angular Power

- Take the radiation distribution at last scattering to also be described by an isotropic temperature field $T(\mathbf{x})$ and recombination to be instantaneous

$$T(\hat{\mathbf{n}}) = \int dD T(\mathbf{x}) \delta(D - D_*)$$

where D is the comoving distance and D_* denotes recombination.

- Describe the temperature field by its Fourier moments

$$T(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} T(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

with a power spectrum

$$\langle T(\mathbf{k})^* T(\mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') P_T(k)$$

Spatial vs Angular Power

- Note that the variance of the field

$$\begin{aligned}\langle T(\mathbf{x})T(\mathbf{x}) \rangle &= \int \frac{d^3k}{(2\pi)^3} P(k) \\ &= \int d \ln k \frac{k^3 P(k)}{2\pi^2} \equiv \int d \ln k \Delta_T^2(k)\end{aligned}$$

so it is more convenient to think in the log power spectrum $\Delta_T^2(k)$

- Temperature field

$$T(\hat{\mathbf{n}}) = \int \frac{d^3k}{(2\pi)^3} T(\mathbf{k}) e^{i\mathbf{k} \cdot D_* \hat{\mathbf{n}}}$$

- Expand out plane wave in spherical coordinates

$$e^{i\mathbf{k} D_* \cdot \hat{\mathbf{n}}} = 4\pi \sum_{\ell m} i^\ell j_\ell(k D_*) Y_{\ell m}^*(\mathbf{k}) Y_{\ell m}(\hat{\mathbf{n}})$$

Spatial vs Angular Power

- Multipole moments

$$T_{\ell m} = \int \frac{d^3 k}{(2\pi)^3} T(\mathbf{k}) 4\pi i^\ell j_\ell(kD_*) Y_{\ell m}(\mathbf{k})$$

- Power spectrum

$$\begin{aligned} \langle T_{\ell m}^* T_{\ell' m'} \rangle &= \int \frac{d^3 k}{(2\pi)^3} (4\pi)^2 (i)^{\ell-\ell'} j_\ell(kD_*) j_{\ell'}(kD_*) Y_{\ell m}^*(\mathbf{k}) Y_{\ell' m'}(\mathbf{k}) P_T(k) \\ &= \delta_{\ell\ell'} \delta_{mm'} 4\pi \int d \ln k j_\ell^2(kD_*) \Delta_T^2(k) \end{aligned}$$

with $\int_0^\infty j_\ell^2(x) d \ln x = 1/(2\ell(\ell + 1))$, slowly varying Δ_T^2

$$C_\ell = \frac{4\pi \Delta_T^2(\ell/D_*)}{2\ell(\ell + 1)} = \frac{2\pi}{\ell(\ell + 1)} \Delta_T^2(\ell/D_*)$$

so $\ell(\ell + 1)C_\ell/2\pi = \Delta_T^2$ is commonly used log power

Tight Coupling Approximation

- Near recombination $z \approx 10^3$ and $\Omega_b h^2 \approx 0.02$, the (comoving) mean free path of a photon

$$\lambda_C \equiv \frac{1}{\dot{\tau}} \sim 2.5 \text{Mpc}$$

small by cosmological standards!

- On scales $\lambda \gg \lambda_C$ photons are **tightly coupled** to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions
- Specifically, their bulk velocities are defined by a **single fluid velocity** $v_\gamma = v_b$ and the photons carry **no anisotropy** in the rest frame of the baryons
- \rightarrow No **heat conduction** or **viscosity** (anisotropic stress) in fluid

Zeroth Order Approximation

- Momentum density of a fluid is $(\rho + p)v$, where p is the pressure
- Neglect the momentum density of the **baryons**

$$R \equiv \frac{(\rho_b + p_b)v_b}{(\rho_\gamma + p_\gamma)v_\gamma} = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}$$
$$\approx 0.6 \left(\frac{\Omega_b h^2}{0.02} \right) \left(\frac{a}{10^{-3}} \right)$$

since $\rho_\gamma \propto T^4$ is fixed by the CMB temperature $T = 2.73(1 + z)\text{K}$
– OK substantially **before recombination**

- Neglect **radiation** in the **expansion**

$$\frac{\rho_m}{\rho_r} = 3.6 \left(\frac{\Omega_m h^2}{0.15} \right) \left(\frac{a}{10^{-3}} \right)$$

Number Continuity

- Photons are **not created** or destroyed. Without expansion

$$\dot{n}_\gamma + \nabla \cdot (n_\gamma \mathbf{v}_\gamma) = 0$$

but the **expansion** or Hubble flow causes $n_\gamma \propto a^{-3}$ or

$$\dot{n}_\gamma + 3n_\gamma \frac{\dot{a}}{a} + \nabla \cdot (n_\gamma \mathbf{v}_\gamma) = 0$$

- **Linearize** $\delta n_\gamma = n_\gamma - \bar{n}_\gamma$

$$(\delta n_\gamma)^\cdot = -3\delta n_\gamma \frac{\dot{a}}{a} - n_\gamma \nabla \cdot \mathbf{v}_\gamma$$

$$\left(\frac{\delta n_\gamma}{n_\gamma} \right)^\cdot = -\nabla \cdot \mathbf{v}_\gamma$$

Continuity Equation

- Number density $n_\gamma \propto T^3$ so define temperature fluctuation Θ

$$\frac{\delta n_\gamma}{n_\gamma} = 3 \frac{\delta T}{T} \equiv 3\Theta$$

- Real space continuity equation

$$\dot{\Theta} = -\frac{1}{3} \nabla \cdot \mathbf{v}_\gamma$$

- Fourier space

$$\dot{\Theta} = -\frac{1}{3} i\mathbf{k} \cdot \mathbf{v}_\gamma$$

Momentum Conservation

- No expansion: $\dot{\mathbf{q}} = \mathbf{F}$
- De Broglie wavelength stretches with the expansion

$$\dot{\mathbf{q}} + \frac{\dot{a}}{a}\mathbf{q} = \mathbf{F}$$

for photons this the **redshift**, for non-relativistic particles **expansion drag** on peculiar velocities

- Collection of particles: momentum \rightarrow **momentum density**
 $(\rho_\gamma + p_\gamma)\mathbf{v}_\gamma$ and force \rightarrow **pressure gradient**

$$[(\rho_\gamma + p_\gamma)\mathbf{v}_\gamma]^\cdot = -4\frac{\dot{a}}{a}(\rho_\gamma + p_\gamma)\mathbf{v}_\gamma - \nabla p_\gamma$$

$$\frac{4}{3}\rho_\gamma\dot{\mathbf{v}}_\gamma = \frac{1}{3}\nabla\rho_\gamma$$

$$\dot{\mathbf{v}}_\gamma = -\nabla\Theta$$

Euler Equation

- Fourier space

$$\dot{\mathbf{v}}_\gamma = -ik\Theta$$

- Pressure gradients (any gradient of a scalar field) generates a curl-free flow
- For convenience define velocity amplitude:

$$\mathbf{v}_\gamma \equiv -iv_\gamma \hat{\mathbf{k}}$$

- Euler Equation:

$$\dot{v}_\gamma = k\Theta$$

- Continuity Equation:

$$\dot{\Theta} = -\frac{1}{3}kv_\gamma$$

Oscillator: Take One

- Combine these to form the simple harmonic oscillator equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = 0$$

where the adiabatic sound speed is defined through

$$c_s^2 \equiv \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma}$$

here $c_s^2 = 1/3$ since we are photon-dominated

- General solution:

$$\Theta(\eta) = \Theta(0) \cos(k s) + \frac{\dot{\Theta}(0)}{k c_s} \sin(k s)$$

where the sound horizon is defined as $s \equiv \int c_s d\eta$

Harmonic Extrema

- All modes are **frozen** in at recombination (denoted with a subscript *) yielding temperature perturbations of **different amplitude** for different modes. For the adiabatic (curvature mode) $\dot{\Theta}(0) = 0$

$$\Theta(\eta_*) = \Theta(0) \cos(k s_*)$$

- Modes caught in the **extrema** of their oscillation will have enhanced fluctuations

$$k_n s_* = n\pi$$

yielding a **fundamental scale** or frequency, related to the inverse **sound horizon**

$$k_A = \pi / s_*$$

and a **harmonic relationship** to the other extrema as 1 : 2 : 3...

Peak Location

- The fundamental **physical scale** is translated into a fundamental **angular scale** by simple projection according to the angular diameter distance D_A

$$\theta_A = \lambda_A / D_A$$

$$\ell_A = k_A D_A$$

- In a flat universe, the distance is simply $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$, the horizon distance, and $k_A = \pi / s_* = \sqrt{3}\pi / \eta_*$ so

$$\theta_A \approx \frac{\eta_*}{\eta_0}$$

- In a **matter-dominated** universe $\eta \propto a^{1/2}$ so $\theta_A \approx 1/30 \approx 2^\circ$ or

$$\ell_A \approx 200$$

Curvature

- In a **curved universe**, the apparent or **angular diameter distance** is no longer the conformal distance $D_A = R \sin(D/R) \neq D$
- Objects in a **closed universe** are **further** than they appear! gravitational **lensing** of the background...
- Curvature scale of the universe must be substantially **larger than current horizon**
- **Flat universe** indicates critical density and implies missing energy given local measures of the matter density “**dark energy**”
- D also depends on **dark energy density** Ω_{DE} and **equation of state** $w = p_{\text{DE}}/\rho_{\text{DE}}$.
- Expansion rate at recombination or **matter-radiation ratio** enters into calculation of k_A .

Doppler Effect

- Bulk motion of fluid changes the observed temperature via Doppler shifts

$$\left(\frac{\Delta T}{T}\right)_{\text{dop}} = \hat{\mathbf{n}} \cdot \mathbf{v}_\gamma$$

- Averaged over directions

$$\left(\frac{\Delta T}{T}\right)_{\text{rms}} = \frac{v_\gamma}{\sqrt{3}}$$

- Acoustic solution

$$\begin{aligned} \frac{v_\gamma}{\sqrt{3}} &= -\frac{\sqrt{3}}{k} \dot{\Theta} = \frac{\sqrt{3}}{k} k c_s \Theta(0) \sin(ks) \\ &= \Theta(0) \sin(ks) \end{aligned}$$

Doppler Peaks?

- Doppler effect for the photon dominated system is of equal amplitude and $\pi/2$ out of phase: extrema of temperature are turning points of velocity
- Effects add in quadrature:

$$\left(\frac{\Delta T}{T}\right)^2 = \Theta^2(0)[\cos^2(ks) + \sin^2(ks)] = \Theta^2(0)$$

- No peaks in k spectrum! However the Doppler effect carries an angular dependence that changes its projection on the sky

$$\hat{\mathbf{n}} \cdot \mathbf{v}_\gamma \propto \hat{\mathbf{n}} \cdot \hat{\mathbf{k}}$$

- Coordinates where $\hat{\mathbf{z}} \parallel \hat{\mathbf{k}}$

$$Y_{10}Y_{\ell 0} \rightarrow Y_{\ell \pm 1 0}$$

recoupling $j'_\ell Y_{\ell 0}$: no peaks in Doppler effect

Restoring Gravity

- Take a simple **photon dominated** system **with gravity**
- **Continuity** altered since a gravitational potential represents a **stretching** of the **spatial fabric** that dilutes number densities – formally a spatial **curvature perturbation**
- Think of this as a perturbation to the **scale factor** $a \rightarrow a(1 + \Phi)$ so that the cosmological redshift is generalized to

$$\frac{\dot{a}}{a} \rightarrow \frac{\dot{a}}{a} + \dot{\Phi}$$

$$(\delta n_\gamma)' = -3\delta n_\gamma \left(\frac{\dot{a}}{a} + \dot{\Phi} \right) - n_\gamma \nabla \cdot \mathbf{v}_\gamma$$

so that the **continuity equation** becomes

$$\dot{\Theta} = -\frac{1}{3}k v_\gamma - \dot{\Phi}$$

Restoring Gravity

- Gravitational force in momentum conservation $\mathbf{F} = -m\nabla\Psi$ generalized to momentum density modifies the Euler equation to

$$\dot{v}_\gamma = k(\Theta + \Psi)$$

- General relativity says that Φ and Ψ are the relativistic analogues of the Newtonian potential and that $\Phi \approx -\Psi$.
- In our matter-dominated approximation, Φ represents matter density fluctuations through the cosmological Poisson equation

$$k^2\Phi = 4\pi G a^2 \rho_m \Delta_m$$

where the difference comes from the use of comoving coordinates for k (a^2 factor), the removal of the background density into the background expansion ($\rho\Delta_m$) and finally a coordinate subtlety that enters into the definition of Δ_m

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Acoustic Dynamics

Constant Potentials

- In the matter dominated epoch potentials are constant because infall generates velocities as $v_m \sim k\eta\Psi$
- Velocity divergence generates density perturbations as $\Delta_m \sim -k\eta v_m \sim -(k\eta)^2\Psi$
- And density perturbations generate potential fluctuations as $\Phi \sim \Delta_m/(k\eta)^2 \sim -\Psi$, keeping them constant. Note that because of the expansion, density perturbations must grow to keep potentials constant.
- Here we have used the Friedman equation $H^2 = 8\pi G\rho_m/3$ and $\eta = \int d\ln a/(aH) \sim 1/(aH)$
- More generally, if stress perturbations are negligible compared with density perturbations ($\delta p \ll \delta\rho$) then potential will remain roughly constant – more specifically a variant called the Bardeen or comoving curvature ζ is constant

Oscillator: Take Two

- Combine these to form the **simple harmonic oscillator** equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - \ddot{\Phi}$$

- In a **CDM dominated** expansion $\dot{\Phi} = \dot{\Psi} = 0$. Also for **photon domination** $c_s^2 = 1/3$ so the oscillator equation becomes

$$\ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0$$

- Solution is just an **offset version** of the original

$$[\Theta + \Psi](\eta) = [\Theta + \Psi](0) \cos(ks)$$

- $\Theta + \Psi$ is also the **observed temperature fluctuation** since photons lose energy climbing out of **gravitational potentials** at recombination

Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or **effective temperature**

$$\Theta + \Psi$$

- Effective temperature oscillates around **zero** with amplitude given by the **initial conditions**
- Note: initial conditions are set when the perturbation is **outside of horizon**, need inflation or other modification to matter-radiation FRW universe.
- GR says that **initial temperature** is given by **initial potential**

Sachs-Wolfe Effect and the Magic 1/3

- A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]

$$\frac{\delta t}{t} = \Psi$$

- Convert this to a perturbation in the scale factor,

$$t = \int \frac{da}{aH} \propto \int \frac{da}{a\rho^{1/2}} \propto a^{3(1+w)/2}$$

where $w \equiv p/\rho$ so that during matter domination

$$\frac{\delta a}{a} = \frac{2}{3} \frac{\delta t}{t}$$

- CMB temperature is cooling as $T \propto a^{-1}$ so

$$\Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3}\Psi$$

Baryon Loading

- Baryons add extra **mass** to the photon-baryon fluid
- Controlling parameter is the **momentum density ratio**:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}} \right)$$

of order **unity** at recombination

- Momentum density of the **joint system** is conserved

$$\begin{aligned} (\rho_\gamma + p_\gamma)v_\gamma + (\rho_b + p_b)v_b &\approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma)v_\gamma \\ &= (1 + R)(\rho_\gamma + p_\gamma)v_{\gamma b} \end{aligned}$$

where the controlling parameter is the **momentum density ratio**:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}} \right)$$

of order **unity** at recombination

New Euler Equation

- Momentum density ratio enters as

$$\begin{aligned} [(1 + R)(\rho_\gamma + p_\gamma)\mathbf{v}_{\gamma b}]^\cdot &= -4\frac{\dot{a}}{a}(1 + R)(\rho_\gamma + p_\gamma)\mathbf{v}_{\gamma b} \\ &\quad - \nabla p_\gamma - (1 + R)(\rho_\gamma + p_\gamma)\nabla\Psi \end{aligned}$$

same as before except for $(1 + R)$ terms so

$$[(1 + R)v_{\gamma b}]^\cdot = k\Theta + (1 + R)k\Psi$$

- Photon continuity remains the same

$$\dot{\Theta} = -\frac{k}{3}v_{\gamma b} - \dot{\Phi}$$

- Modification of oscillator equation

$$[(1 + R)\dot{\Theta}]^\cdot + \frac{1}{3}k^2\Theta = -\frac{1}{3}k^2(1 + R)\Psi - [(1 + R)\dot{\Phi}]^\cdot$$

Oscillator: Take Three

- Combine these to form the not-quite-so simple harmonic oscillator equation

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})$$

where $c_s^2 \equiv \dot{p}_{\gamma b} / \dot{\rho}_{\gamma b}$

$$c_s^2 = \frac{1}{3} \frac{1}{1 + R}$$

- In a CDM dominated expansion $\dot{\Phi} = \dot{\Psi} = 0$ and the adiabatic approximation $\dot{R}/R \ll \omega = kc_s$

$$[\Theta + (1 + R)\Psi](\eta) = [\Theta + (1 + R)\Psi](0) \cos(k s)$$

Baryon Peak Phenomenology

- Photon-baryon ratio enters in **three ways**
- Overall larger **amplitude**:

$$[\Theta + (1 + R)\Psi](0) = \frac{1}{3}(1 + 3R)\Psi(0)$$

- Even-odd peak **modulation** of effective temperature

$$[\Theta + \Psi]_{\text{peaks}} = [\pm(1 + 3R) - 3R] \frac{1}{3} \Psi(0)$$

$$[\Theta + \Psi]_1 - [\Theta + \Psi]_2 = [-6R] \frac{1}{3} \Psi(0)$$

- Shifting of the **sound horizon** down or ℓ_A up

$$\ell_A \propto \sqrt{1 + R}$$

- Actual effects **smaller** since R evolves

Photon Baryon Ratio Evolution

- Oscillator equation has time evolving mass

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = 0$$

- Effective mass is $m_{\text{eff}} = 3c_s^{-2} = (1 + R)$
- Adiabatic invariant

$$\frac{E}{\omega} = \frac{1}{2} m_{\text{eff}} \omega A^2 = \frac{1}{2} 3c_s^{-2} k c_s A^2 \propto A^2 (1 + R)^{1/2} = \text{const.}$$

- Amplitude of oscillation $A \propto (1 + R)^{-1/4}$ decays adiabatically as the photon-baryon ratio changes

Oscillator: Take Three and a Half

- The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})$$

changes in the gravitational potentials alter the form of the acoustic oscillations

- If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator
- Term involving Ψ is the ordinary gravitational force
- Term involving Φ involves the $\dot{\Phi}$ term in the continuity equation as a (curvature) perturbation to the scale factor

Potential Decay

- Matter-to-radiation ratio

$$\frac{\rho_m}{\rho_r} \approx 24\Omega_m h^2 \left(\frac{a}{10^{-3}} \right)$$

of order **unity** at recombination in a low Ω_m universe

- Radiation is not stress free and so **impedes** the growth of structure

$$k^2\Phi = 4\pi G a^2 \rho_r \Delta_r$$

$\Delta_r \sim 4\Theta$ **oscillates** around a constant value, $\rho_r \propto a^{-4}$ so the Newtonian **curvature decays**.

- General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale

Radiation Driving

- Decay is timed precisely to **drive** the oscillator - close to fully coherent

$$\begin{aligned} [\Theta + \Psi](\eta) &= [\Theta + \Psi](0) + \Delta\Psi - \Delta\Phi \\ &= \frac{1}{3}\Psi(0) - 2\Psi(0) = \frac{5}{3}\Psi(0) \end{aligned}$$

- **5**× the amplitude of the Sachs-Wolfe effect!
- Coherent approximation is **exact** for a photon-baryon fluid but reality is reduced to ~ 4 × because of **neutrino contribution** to radiation
- Actual **initial conditions** are $\Theta + \Psi = \Psi/2$ for radiation domination but comparison to matter dominated SW correct

External Potential Approach

- Solution to homogeneous equation

$$(1 + R)^{-1/4} \cos(ks), \quad (1 + R)^{-1/4} \sin(ks)$$

- Give the general solution for an external potential by propagating impulsive forces

$$(1 + R)^{1/4} \Theta(\eta) = \Theta(0) \cos(ks) + \frac{\sqrt{3}}{k} \left[\dot{\Theta}(0) + \frac{1}{4} \dot{R}(0) \Theta(0) \right] \sin ks \\ + \frac{\sqrt{3}}{k} \int_0^\eta d\eta' (1 + R')^{3/4} \sin[ks - ks'] F(\eta')$$

where

$$F = -\ddot{\Phi} - \frac{\dot{R}}{1 + R} \dot{\Phi} - \frac{k^2}{3} \Psi$$

- Useful if general form of potential evolution is known

Damping

- Tight coupling equations assume a perfect fluid: no viscosity, no heat conduction
- Fluid imperfections are related to the mean free path of the photons in the baryons

$$\lambda_C = \dot{\tau}^{-1} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a$$

is the conformal opacity to Thompson scattering

- Dissipation is related to the diffusion length: random walk approximation

$$\lambda_D = \sqrt{N} \lambda_C = \sqrt{\eta / \lambda_C} \lambda_C = \sqrt{\eta \lambda_C}$$

the geometric mean between the horizon and mean free path

- $\lambda_D / \eta_* \sim \text{few } \%$, so expect the peaks > 3 to be affected by dissipation

Equations of Motion

- Continuity

$$\dot{\Theta} = -\frac{k}{3}v_\gamma - \dot{\Phi}, \quad \dot{\delta}_b = -kv_b - 3\dot{\Phi}$$

where the photon equation remains unchanged and the baryons follow number conservation with $\rho_b = m_b n_b$

- Euler

$$\begin{aligned}\dot{v}_\gamma &= k(\Theta + \Psi) - \frac{k}{6}\pi_\gamma - \dot{\tau}(v_\gamma - v_b) \\ \dot{v}_b &= -\frac{\dot{a}}{a}v_b + k\Psi + \dot{\tau}(v_\gamma - v_b)/R\end{aligned}$$

where the photons gain an anisotropic stress term π_γ from radiation viscosity and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation

Viscosity

- Viscosity is generated from radiation streaming from hot to cold regions
- Expect

$$\pi_\gamma \sim v_\gamma \frac{k}{\dot{\tau}}$$

generated by streaming, suppressed by scattering in a wavelength of the fluctuation. Radiative transfer says

$$\pi_\gamma \approx 2A_v v_\gamma \frac{k}{\dot{\tau}}$$

where $A_v = 16/15$

$$\dot{v}_\gamma = k(\Theta + \Psi) - \frac{k}{3} A_v \frac{k}{\dot{\tau}} v_\gamma$$

Oscillator: Penultimate Take

- Adiabatic approximation ($\omega \gg \dot{a}/a$)

$$\dot{\Theta} \approx -\frac{k}{3}v_\gamma$$

- Oscillator equation contains a $\dot{\Theta}$ damping term

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} A_v \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})$$

- Heat conduction term similar in that it is proportional to v_γ and is suppressed by scattering $k/\dot{\tau}$. Expansion of Euler equations to leading order in $k\dot{\tau}$ gives

$$A_h = \frac{R^2}{1 + R}$$

since the effects are only significant if the baryons are dynamically important

Oscillator: Final Take

- Final oscillator equation

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} [A_v + A_h] \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})$$

- Solve in the adiabatic approximation

$$\Theta \propto \exp(i \int \omega d\eta)$$

$$-\omega^2 + \frac{k^2 c_s^2}{\dot{\tau}} (A_v + A_h) i\omega + k^2 c_s^2 = 0 \quad (1)$$

Dispersion Relation

- Solve

$$\begin{aligned}\omega^2 &= k^2 c_s^2 \left[1 + i \frac{\omega}{\dot{\tau}} (A_v + A_h) \right] \\ \omega &= \pm k c_s \left[1 + \frac{i}{2} \frac{\omega}{\dot{\tau}} (A_v + A_h) \right] \\ &= \pm k c_s \left[1 \pm \frac{i}{2} \frac{k c_s}{\dot{\tau}} (A_v + A_h) \right]\end{aligned}$$

- Exponentiate

$$\begin{aligned}\exp(i \int \omega d\eta) &= e^{\pm i k s} \exp\left[-k^2 \int d\eta \frac{1}{2} \frac{c_s^2}{\dot{\tau}} (A_v + A_h)\right] \\ &= e^{\pm i k s} \exp\left[-(k/k_D)^2\right]\end{aligned}\tag{2}$$

- Damping is **exponential** under the scale k_D

Diffusion Scale

- Diffusion wavenumber

$$k_D^{-2} = \int d\eta \frac{1}{\dot{\tau}} \frac{1}{6(1+R)} \left(\frac{16}{15} + \frac{R^2}{(1+R)} \right)$$

- Limiting forms

$$\lim_{R \rightarrow 0} k_D^{-2} = \frac{1}{6} \frac{16}{15} \int d\eta \frac{1}{\dot{\tau}}$$

$$\lim_{R \rightarrow \infty} k_D^{-2} = \frac{1}{6} \int d\eta \frac{1}{\dot{\tau}}$$

- Geometric mean between horizon and mean free path as expected from a **random walk**

$$\lambda_D = \frac{2\pi}{k_D} \sim \frac{2\pi}{\sqrt{6}} (\eta \dot{\tau}^{-1})^{1/2}$$

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Polarization

Stokes Parameters

- Polarization state of radiation in direction $\hat{\mathbf{n}}$ described by the intensity matrix $\langle E_i(\hat{\mathbf{n}})E_j^*(\hat{\mathbf{n}}) \rangle$, where \mathbf{E} is the electric field vector and the brackets denote time averaging.
- As a hermitian matrix, it can be decomposed into the Pauli basis

$$\begin{aligned}\mathbf{P} &= C \langle \mathbf{E}(\hat{\mathbf{n}}) \mathbf{E}^\dagger(\hat{\mathbf{n}}) \rangle \\ &= \Theta(\hat{\mathbf{n}})\boldsymbol{\sigma}_0 + Q(\hat{\mathbf{n}})\boldsymbol{\sigma}_3 + U(\hat{\mathbf{n}})\boldsymbol{\sigma}_1 + V(\hat{\mathbf{n}})\boldsymbol{\sigma}_2,\end{aligned}$$

where

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Stokes parameters recovered as $\text{Tr}(\boldsymbol{\sigma}_i \mathbf{P})/2$

Linear Polarization

- $Q \propto \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle$, $U \propto \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle$.
- Counterclockwise rotation of axes by $\theta = 45^\circ$

$$E_1 = (E'_1 - E'_2)/\sqrt{2}, \quad E_2 = (E'_1 + E'_2)/\sqrt{2}$$

- $U \propto \langle E'_1 E'_1^* \rangle - \langle E'_2 E'_2^* \rangle$, difference of intensities at 45° or Q'
- More generally, \mathbf{P} transforms as a tensor under rotations and

$$Q' = \cos(2\theta)Q + \sin(2\theta)U$$

$$U' = -\sin(2\theta)Q + \cos(2\theta)U$$

- or

$$Q' \pm iU' = e^{\mp 2i\theta}[Q \pm iU]$$

acquires a phase under rotation and is a spin ± 2 object

Coordinate Independent Representation

- Two directions: orientation of polarization and change in amplitude, i.e. Q and U in the basis of the Fourier wavevector for small sections of sky are called E and B components

$$\begin{aligned} E(\mathbf{l}) \pm iB(\mathbf{l}) &= - \int d\hat{\mathbf{n}} [Q'(\hat{\mathbf{n}}) \pm iU'(\hat{\mathbf{n}})] e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}} \\ &= -e^{\mp 2i\phi_l} \int d\hat{\mathbf{n}} [Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})] e^{i\mathbf{l}\cdot\hat{\mathbf{n}}} \end{aligned}$$

- For the B -mode to not vanish, the polarization must point in a direction not related to the wavevector - not possible for density fluctuations in linear theory
- Generalize to all-sky: plane waves are eigenmodes of the Laplace operator on the tensor \mathbf{P} .

Spin Harmonics

- Laplace Eigenfunctions

$$\nabla^2_{\pm 2} Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1] = -[l(l+1) - 4]_{\pm 2} Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1]$$

- Spin s spherical harmonics: orthogonal and complete

$$\int d\hat{\mathbf{n}} {}_s Y_{\ell m}^*(\hat{\mathbf{n}}) {}_s Y_{\ell' m'}(\hat{\mathbf{n}}) = \delta_{\ell\ell'} \delta_{mm'}$$

$$\sum_{\ell m} {}_s Y_{\ell m}^*(\hat{\mathbf{n}}) {}_s Y_{\ell m}(\hat{\mathbf{n}}') = \delta(\phi - \phi') \delta(\cos\theta - \cos\theta')$$

where the ordinary spherical harmonics are $Y_{\ell m} = {}_0 Y_{\ell m}$

- Given in terms of the rotation matrix

$${}_s Y_{\ell m}(\beta\alpha) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} D_{-ms}^{\ell}(\alpha\beta 0)$$

Statistical Representation

- All-sky decomposition

$$[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})] = \sum_{\ell m} [E_{\ell m} \pm iB_{\ell m}]_{\pm 2} Y_{\ell m}(\hat{\mathbf{n}})$$

- Power spectra

$$\langle E_{\ell m}^* E_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_{\ell}^{EE}$$

$$\langle B_{\ell m}^* B_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_{\ell}^{BB}$$

- Cross correlation

$$\langle E_{\ell m}^* E_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_{\ell}^{\ominus E}$$

others vanish if parity is conserved

Thomson Scattering

- Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T ,$$

where $\sigma_T = 8\pi\alpha^2/3m_e$ is the Thomson cross section, $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

- Summed over angle and incoming polarization

$$\sum_{i=1,2} \int d\hat{\mathbf{n}}' \frac{d\sigma}{d\Omega} = \sigma_T$$

Polarization Generation

- Heuristic: incoming radiation shakes an electron in direction of electric field vector $\hat{\mathbf{E}}'$
- Radiates photon with polarization also in direction $\hat{\mathbf{E}}'$
- But photon cannot be longitudinally polarized so that scattering into 90° can only pass one polarization
- Linearly polarized radiation like polarization by reflection
- Unlike reflection of sunlight, incoming radiation is nearly isotropic
- Missing linear polarization supplied by scattering from direction orthogonal to original incoming direction
- Only quadrupole anisotropy generates polarization by Thomson scattering

Acoustic Polarization

- Break down of tight-coupling leads to quadrupole anisotropy of

$$\pi_\gamma \approx \frac{k}{\dot{\tau}} v_\gamma$$

- Scaling $k_D = (\dot{\tau}/\eta_*)^{1/2} \rightarrow \dot{\tau} = k_D^2 \eta_*$
- Know: $k_D s_* \approx k_D \eta_* \approx 10$
- So:

$$\pi_\gamma \approx \frac{k}{k_D} \frac{1}{10} v_\gamma$$

$$\Delta_P \approx \frac{\ell}{\ell_D} \frac{1}{10} \Delta_T$$

Acoustic Polarization

- Gradient of velocity is along direction of wavevector, so polarization is pure E -mode
- Velocity is 90° out of phase with temperature – turning points of oscillator are zero points of velocity:

$$\Theta + \Psi \propto \cos(ks); \quad v_\gamma \propto \sin(ks)$$

- Polarization peaks are at troughs of temperature power

Cross Correlation

- Cross correlation of temperature and polarization

$$(\Theta + \Psi)(v_\gamma) \propto \cos(ks) \sin(ks) \propto \sin(2ks)$$

- Oscillation at twice the frequency
- Correlation: radial or tangential around hot spots
- Partial correlation: easier to measure if polarization data is noisy, harder to measure if polarization data is high S/N or if bands do not resolve oscillations
- Good check for systematics and foregrounds
- Comparison of temperature and polarization is proof against features in initial conditions mimicking acoustic features

Reionization

- Ionization depth during reionization

$$\begin{aligned}\tau(z) &= \int d\eta n_e \sigma_T a = \int d \ln a \frac{n_e \sigma_T}{H(a)} \propto (\Omega_b h^2) (\Omega_m h^2)^{-1/2} (1+z)^{3/2} \\ &= \left(\frac{\Omega_b h^2}{0.02} \right) \left(\frac{\Omega_m h^2}{0.15} \right)^{-1/2} \left(\frac{1+z}{61} \right)^{3/2}\end{aligned}$$

- Quasars say $z_{ri} \geq 7$ so $\tau > 0.04$
- During reionization, cosmic quadrupole of $\sim 30 \mu\text{K}$ from the Sachs-Wolfe effect scatters into E -polarization
- Few percent optical depth leads to fraction of a μK signal
- Peaks at horizon scale at recombination: quadrupole source $j_2(kD_*)$ maximal at $kD_* \approx k\eta \approx 2$

Breaking degeneracies

- First objects, breaking degeneracy of initial amplitude vs optical depth in the peak heights

$$C_\ell \propto e^{-2\tau}$$

only below horizon scale at reionization

- Breaks degeneracies in angular diameter distance by removing an ambiguity for ISW-dark energy measure, helps in $\Omega_{DE} - w_{DE}$ plane

Gravitational Wave

- Gravitational waves produce a quadrupolar distortion in the temperature of the CMB like effect on a ring of test particles
- Like ISW effect, source is a metric perturbation with time dependent amplitude
- After recombination, is a source of observable temperature anisotropy – but is therefore confined to low order multipoles
- Generated during inflation by quantum fluctuations

Gravitational Wave Polarization

- In the tight coupling regime, quadrupole anisotropy suppressed by scattering

$$\pi_\gamma \approx \frac{\dot{h}}{\dot{\tau}}$$

- Since gravitational waves oscillate and decay at horizon crossing, the polarization peaks at the horizon scale at recombination not the damping scale
- More distinct signature in the B -mode polarization since symmetry of plane wave is broken by the transverse nature of gravity wave polarization

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Linear Perturbation Theory

Covariant Perturbation Theory

- Covariant = takes same form in all coordinate systems
- Invariant = takes the same value in all coordinate systems
- Fundamental equations: Einstein equations, covariant conservation of stress-energy tensor:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$
$$\nabla_{\mu} T^{\mu\nu} = 0$$

- Preserve general covariance by keeping all degrees of freedom: 10 for each symmetric 4×4 tensor

1	2	3	4
	5	6	7
		8	9
			10

Metric Tensor

- Expand the metric tensor around the general FRW metric

$$g_{00} = -a^2, \quad g_{ij} = a^2 \gamma_{ij}.$$

where the “0” component is conformal time $\eta = dt/a$ and γ_{ij} is a spatial metric of constant curvature $K = H_0^2(\Omega_{\text{tot}} - 1)$.

- Add in a general perturbation (Bardeen 1980)

$$\begin{aligned} g^{00} &= -a^{-2}(1 - 2A), \\ g^{0i} &= -a^{-2}B^i, \\ g^{ij} &= a^{-2}(\gamma^{ij} - 2H_L\gamma^{ij} - 2H_T^{ij}). \end{aligned}$$

- (1) $A \equiv$ a scalar potential; (3) B^i a vector shift, (1) H_L a perturbation to the spatial curvature; (5) H_T^{ij} a trace-free distortion to spatial metric = (10)

Matter Tensor

- Likewise expand the matter stress energy tensor around a homogeneous density ρ and pressure p :

$$T^0_0 = -\rho - \delta\rho,$$

$$T^0_i = (\rho + p)(v_i - B_i),$$

$$T_0^i = -(\rho + p)v^i,$$

$$T^i_j = (p + \delta p)\delta^i_j + p\Pi^i_j,$$

- (1) $\delta\rho$ a density perturbation; (3) v_i a vector velocity, (1) δp a pressure perturbation; (5) Π_{ij} an anisotropic stress perturbation
- So far this is fully general and applies to any type of matter or coordinate choice including non-linearities in the matter, e.g. cosmological defects.

Counting DOF's

20	Variables (10 metric; 10 matter)
-10	Einstein equations
-4	Conservation equations
+4	Bianchi identities
-4	Gauge (coordinate choice 1 time, 3 space)
<hr/>	
6	Degrees of freedom

- Without loss of generality these can be taken to be the 6 components of the matter stress tensor
- For the background, specify $p(a)$ or equivalently $w(a) \equiv p(a)/\rho(a)$ the equation of state parameter.

Scalar, Vector, Tensor

- In linear perturbation theory, perturbations may be separated by their transformation properties under rotation and translation.
- The eigenfunctions of the Laplacian operator form a complete set

$$\begin{aligned}\nabla^2 Q^{(0)} &= -k^2 Q^{(0)} && \mathbf{S}, \\ \nabla^2 Q_i^{(\pm 1)} &= -k^2 Q_i^{(\pm 1)} && \mathbf{V}, \\ \nabla^2 Q_{ij}^{(\pm 2)} &= -k^2 Q_{ij}^{(\pm 2)} && \mathbf{T},\end{aligned}$$

- Vector and tensor modes satisfy divergence-free and transverse-traceless conditions

$$\nabla^i Q_i^{(\pm 1)} = 0$$

$$\nabla^i Q_{ij}^{(\pm 2)} = 0$$

$$\gamma^{ij} Q_{ij}^{(\pm 2)} = 0$$

Vector and Tensor Modes vs. Vector and Tensor Quantities

- A scalar mode carries with it associated vector (curl-free) and tensor (longitudinal) quantities
- A vector mode carries and associated tensor (neither longitudinal or transverse) quantities
- These are built from the mode basis out of covariant derivatives and the metric

$$Q_i^{(0)} = -k^{-1} \nabla_i Q^{(0)},$$

$$Q_{ij}^{(0)} = (k^{-2} \nabla_i \nabla_j + \frac{1}{3} \gamma_{ij}) Q^{(0)},$$

$$Q_{ij}^{(\pm 1)} = -\frac{1}{2k} [\nabla_i Q_j^{(\pm 1)} + \nabla_j Q_i^{(\pm 1)}],$$

Spatially Flat Case

- For a spatially flat background metric, harmonics are related to plane waves:

$$Q^{(0)} = \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$Q_i^{(\pm 1)} = \frac{-i}{\sqrt{2}} (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$Q_{ij}^{(\pm 2)} = -\sqrt{\frac{3}{8}} (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k} \cdot \mathbf{x})$$

where $\hat{\mathbf{e}}_3 \parallel \mathbf{k}$. Chosen as spin states, c.f. polarization.

- For vectors, the harmonic points in a direction orthogonal to \mathbf{k} suitable for the **vortical component** of a vector
- For tensors, the harmonic is transverse and traceless as appropriate for the decomposition of **gravitational waves**

Perturbation k -Modes

- For the k th eigenmode, the scalar components become

$$\begin{aligned} A(\mathbf{x}) &= A(k) Q^{(0)}, & H_L(\mathbf{x}) &= H_L(k) Q^{(0)}, \\ \delta\rho(\mathbf{x}) &= \delta\rho(k) Q^{(0)}, & \delta p(\mathbf{x}) &= \delta p(k) Q^{(0)}, \end{aligned}$$

the vectors components become

$$B_i(\mathbf{x}) = \sum_{m=-1}^1 B^{(m)}(k) Q_i^{(m)}, \quad v_i(\mathbf{x}) = \sum_{m=-1}^1 v^{(m)}(k) Q_i^{(m)},$$

and the tensors components

$$H_{Tij}(\mathbf{x}) = \sum_{m=-2}^2 H_T^{(m)}(k) Q_{ij}^{(m)}, \quad \Pi_{ij}(\mathbf{x}) = \sum_{m=-2}^2 \Pi^{(m)}(k) Q_{ij}^{(m)},$$

Homogeneous Einstein Equations

- Einstein (Friedmann) equations:

$$\left(\frac{1}{a} \frac{da}{dt}\right)^2 = \frac{8\pi G}{3} \rho$$
$$\frac{1}{a} \frac{d^2 a}{dt^2} = -\frac{4\pi G}{3} (\rho + 3p)$$

so that $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot{\rho}}{\rho} = -3(1 + w) \frac{\dot{a}}{a}$$

Homogeneous Einstein Equations

- Counting exercise:

20	Variables (10 metric; 10 matter)
-17	Homogeneity and Isotropy
-2	Einstein equations
-1	Conservation equations
+1	Bianchi identities
<hr/>	
1	Degree of freedom

- without loss of generality choose ratio of homogeneous & isotropic component of the **stress tensor** to the density $w(a) = p(a)/\rho(a)$.

Acceleration Implies Negative Pressure

- Role of stresses in the background cosmology
- Homogeneous Einstein equations $G_{\mu\nu} = 8\pi GT_{\mu\nu}$ imply the two Friedman equations (flat universe, or associating curvature $\rho_K = -3K/8\pi Ga^2$)

$$\left(\frac{1}{a} \frac{da}{dt}\right)^2 = \frac{8\pi G}{3} \rho$$
$$\frac{1}{a} \frac{d^2a}{dt^2} = -\frac{4\pi G}{3} (\rho + 3p)$$

so that the total equation of state $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot{\rho}}{\rho} = -3(1 + w) \frac{\dot{a}}{a}$$

- so that ρ must scale more slowly than a^{-2}

Questions regarding Dark Energy

- **Coincidence:** given the very different scalings of matter and dark energy with a , why are they **comparable now**?
- **Stability:** why doesn't negative pressure imply **accelerated collapse**? or why doesn't the vacuum suck?
- **Answer:** stability is associated with stress (pressure) **gradients** not stress (pressure) itself.
- **Example:** the **cosmological constant** $w_\Lambda = -1$, a constant in time and space – no gradients.
- **Example:** a **scalar field** where $w = p/\rho \neq \delta p/\delta\rho = \text{sound speed}$.

Covariant Scalar Equations

- Einstein equations (suppressing 0) superscripts (Hu & Eisenstein 1999):

$$(k^2 - 3K)[H_L + \frac{1}{3}H_T + \frac{\dot{a}}{a} \frac{1}{k^2}(kB - \dot{H}_T)]$$

$$= 4\pi G a^2 \left[\delta\rho + 3\frac{\dot{a}}{a}(\rho + p)(v - B)/k \right], \quad \text{Poisson Equation}$$

$$k^2(A + H_L + \frac{1}{3}H_T) + \left(\frac{d}{d\eta} + 2\frac{\dot{a}}{a} \right) (kB - \dot{H}_T)$$

$$= 8\pi G a^2 p \Pi,$$

$$\frac{\dot{a}}{a}A - \dot{H}_L - \frac{1}{3}\dot{H}_T - \frac{K}{k^2}(kB - \dot{H}_T)$$

$$= 4\pi G a^2(\rho + p)(v - B)/k,$$

$$\left[2\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a} \frac{d}{d\eta} - \frac{k^2}{3} \right] A - \left[\frac{d}{d\eta} + \frac{\dot{a}}{a} \right] (\dot{H}_L + \frac{1}{3}kB)$$

$$= 4\pi G a^2(\delta p + \frac{1}{3}\delta\rho).$$

Covariant Scalar Equations

- Conservation equations: continuity and Navier Stokes

$$\left[\frac{d}{d\eta} + 3\frac{\dot{a}}{a} \right] \delta\rho + 3\frac{\dot{a}}{a} \delta p = -(\rho + p)(kv + 3\dot{H}_L),$$
$$\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] \left[(\rho + p) \frac{(v - B)}{k} \right] = \delta p - \frac{2}{3} \left(1 - 3\frac{K}{k^2} \right) p\Pi + (\rho + p)A,$$

- Equations are not independent since $\nabla_\mu G^{\mu\nu} = 0$ via the Bianchi identities.
- Related to the ability to choose a coordinate system or “gauge” to represent the perturbations.

Covariant Scalar Equations

- DOF counting exercise

8 Variables (4 metric; 4 matter)

−4 Einstein equations

−2 Conservation equations

+2 Bianchi identities

−2 Gauge (coordinate choice 1 time, 1 space)

2 Degrees of freedom

- without loss of generality choose scalar components of the stress tensor $\delta p, \Pi$.

Covariant Vector Equations

- Einstein equations

$$\begin{aligned}(1 - 2K/k^2)(kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) \\ &= 16\pi Ga^2(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k, \\ \left[\frac{d}{d\eta} + 2\frac{\dot{a}}{a} \right] (kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) \\ &= -8\pi Ga^2 p \Pi^{(\pm 1)}.\end{aligned}$$

- Conservation Equations

$$\begin{aligned}\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] [(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k] \\ &= -\frac{1}{2}(1 - 2K/k^2)p\Pi^{(\pm 1)},\end{aligned}$$

- Gravity provides **no source** to vorticity \rightarrow **decay**

Covariant Vector Equations

- DOF counting exercise

8 Variables (4 metric; 4 matter)

−4 Einstein equations

−2 Conservation equations

+2 Bianchi identities

−2 Gauge (coordinate choice 1 time, 1 space)

2 Degrees of freedom

- without loss of generality choose vector components of the **stress tensor** $\Pi^{(\pm 1)}$.

Covariant Tensor Equation

- Einstein equation

$$\left[\frac{d^2}{d\eta^2} + 2\frac{\dot{a}}{a} \frac{d}{d\eta} + (k^2 + 2K) \right] H_T^{(\pm 2)} = 8\pi G a^2 p \Pi^{(\pm 2)}.$$

- DOF counting exercise

4 Variables (2 metric; 2 matter)

−2 Einstein equations

−0 Conservation equations

+0 Bianchi identities

−0 Gauge (coordinate choice 1 time, 1 space)

2 Degrees of freedom

- wlog choose tensor components of the **stress tensor** $\Pi^{(\pm 2)}$.

Arbitrary Dark Components

- Total stress energy tensor can be broken up into **individual pieces**
- **Dark components** interact only through gravity and so satisfy **separate conservation equations**
- Einstein equation source remains the sum of components.
- To specify an arbitrary dark component, give the behavior of the **stress tensor: 6 components: $\delta p, \Pi^{(i)}$** , where $i = -2, \dots, 2$.
- Many types of dark components (dark matter, scalar fields, massive neutrinos,..) have **simple forms** for their stress tensor in terms of the energy density, i.e. described by **equations of state**.
- An equation of state for the background $w = p/\rho$ is **not sufficient** to determine the behavior of the perturbations.

Gauge

- Metric and matter fluctuations take on **different values** in different coordinate system
- No such thing as a “gauge invariant” density perturbation!
- General **coordinate transformation**:

$$\begin{aligned}\tilde{\eta} &= \eta + T \\ \tilde{x}^i &= x^i + L^i\end{aligned}$$

free to choose (T, L^i) to simplify equations or physics.

Decompose these into scalar and vector harmonics.

- $G_{\mu\nu}$ and $T_{\mu\nu}$ transform as **tensors**, so components in different frames can be related

Gauge Transformation

- Scalar Metric:

$$\begin{aligned}\tilde{A} &= A - \dot{T} - \frac{\dot{a}}{a}T, \\ \tilde{B} &= B + \dot{L} + kT, \\ \tilde{H}_L &= H_L - \frac{k}{3}L - \frac{\dot{a}}{a}T, \\ \tilde{H}_T &= H_T + kL,\end{aligned}$$

- Scalar Matter (J th component):

$$\begin{aligned}\delta\tilde{\rho}_J &= \delta\rho_J - \dot{\rho}_J T, \\ \delta\tilde{p}_J &= \delta p_J - \dot{p}_J T, \\ \tilde{v}_J &= v_J + \dot{L},\end{aligned}$$

- Vector:

$$\tilde{B}^{(\pm 1)} = B^{(\pm 1)} + \dot{L}^{(\pm 1)}, \quad \tilde{H}_T^{(\pm 1)} = H_T^{(\pm 1)} + kL^{(\pm 1)}, \quad \tilde{v}_J^{(\pm 1)} = v_J^{(\pm 1)} + \dot{L}^{(\pm 1)},$$

Common Scalar Gauge Choices

- A coordinate system is **fully specified** if there is an explicit prescription for (T, L^i) or for scalars (T, L)
- Newtonian:

$$\tilde{B} = \tilde{H}_T = 0$$

$$\Psi \equiv \tilde{A} \quad (\text{Newtonian potential})$$

$$\Phi \equiv \tilde{H}_L \quad (\text{Newtonian curvature})$$

$$L = -H_T/k$$

$$T = -B/k + \dot{H}_T/k^2$$

Good: intuitive Newtonian like gravity; matter and metric algebraically related; commonly chosen for **analytic CMB** and **lensing** work

Bad: numerically **unstable**

Example: Newtonian Reduction

- In the general equations, set $B = H_T = 0$:

$$\begin{aligned}(k^2 - 3K)\Phi &= 4\pi G a^2 \left[\delta\rho + 3\frac{\dot{a}}{a}(\rho + p)v/k \right] \\ k^2(\Psi + \Phi) &= 8\pi G a^2 p\Pi\end{aligned}$$

so $\Psi = -\Phi$ if anisotropic stress $\Pi = 0$ and

$$\begin{aligned}\left[\frac{d}{d\eta} + 3\frac{\dot{a}}{a} \right] \delta\rho + 3\frac{\dot{a}}{a} \delta p &= -(\rho + p)(kv + 3\dot{\Phi}), \\ \left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] (\rho + p)v &= k\delta p - \frac{2}{3} \left(1 - 3\frac{K}{k^2} \right) p k\Pi + (\rho + p) k\Psi,\end{aligned}$$

- Competition between **stress** (pressure and viscosity) and **potential gradients**

Common Scalar Gauge Choices

- Comoving:

$$\tilde{B} = \tilde{v} \quad (T_i^0 = 0)$$

$$H_T = 0$$

$$\xi = \tilde{A}$$

$$\zeta = \tilde{H}_L \quad (\text{Bardeen curvature})$$

$$\Delta = \tilde{\delta} \quad (\text{comoving density pert})$$

$$T = (v - B)/k$$

$$L = -H_T/k$$

Good: Algebraic relations between matter and metric

- Euler equation becomes an algebraic relation between stress and potential

$$(\rho + p)\xi = -\delta p + \frac{2}{3} \left(1 - \frac{3K}{k} \right) p\Pi$$

Common Scalar Gauge Choices

- Einstein equation lacks momentum density source

$$\frac{\dot{a}}{a}\xi - \dot{\zeta} - \frac{K}{k^2}kv = 0$$

- Combine: ζ is conserved if stress fluctuations negligible, e.g. above the horizon if $|K| \ll H^2$

$$\dot{\zeta} + Kv/k = \frac{\dot{a}}{a} \left[-\frac{\delta p}{\rho + p} + \frac{2}{3} \left(1 - \frac{3K}{k^2} \right) \frac{p}{\rho + p} \Pi \right] \rightarrow 0$$

Bad: explicitly relativistic choice

Common Scalar Gauge Choices

- Synchronous:

$$\tilde{A} = \tilde{B} = 0$$

$$\eta_L \equiv -\tilde{H}_L - \frac{1}{3}\tilde{H}_T$$

$$h_T = \tilde{H}_T \quad \text{or} \quad h = 6H_L$$

$$T = a^{-1} \int d\eta a A + c_1 a^{-1}$$

$$L = - \int d\eta (B + kT) + c_2$$

Good: stable, the choice of numerical codes

Bad: residual **gauge freedom** in constants c_1, c_2 must be specified as an initial condition, intrinsically relativistic.

Common Scalar Gauge Choices

- Spatially Unperturbed:

$$\tilde{H}_L = \tilde{H}_T = 0$$

$$L = -H_T/k$$

$$\tilde{A}, \tilde{B} = \text{metric perturbations}$$

$$T = \left(\frac{\dot{a}}{a}\right)^{-1} \left(H_L + \frac{1}{3}H_T\right)$$

Good: eliminates spatial metric in evolution equations; useful in inflationary calculations ([Mukhanov et al](#))

Bad: intrinsically relativistic.

- **Caution:** perturbation evolution is governed by the behavior of stress fluctuations and an isotropic stress fluctuation δp is gauge dependent.

Hybrid “Gauge Invariant” Approach

- With the gauge transformation relations, express variables of **one gauge** in terms of those in **another** – allows a mixture in the equations of motion
- **Example:** Newtonian curvature and comoving density

$$(k^2 - 3K)\Phi = 4\pi G a^2 \rho \Delta$$

ordinary Poisson equation then implies Φ approximately constant if stresses negligible.

- **Example:** Exact Newtonian curvature above the horizon derived through Bardeen curvature conservation

Gauge transformation

$$\Phi = \zeta + \frac{\dot{a} v}{a k}$$

Hybrid “Gauge Invariant” Approach

Einstein equation to eliminate velocity

$$\frac{\dot{a}}{a}\Psi - \dot{\Phi} = 4\pi G a^2 (\rho + p)v/k$$

Friedman equation with no spatial curvature

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} a^2 \rho$$

With $\dot{\Phi} = 0$ and $\Psi \approx -\Phi$

$$\frac{\dot{a} v}{a k} = -\frac{2}{3(1+w)}\Phi$$

Hybrid “Gauge Invariant” Approach

Combining gauge transformation with velocity relation

$$\Phi = \frac{3 + 3w}{5 + 3w} \zeta$$

Usage: calculate ζ from inflation determines Φ for any choice of matter content or causal evolution.

- **Example:** Scalar field (“quintessence” dark energy) equations in comoving gauge imply a **sound speed** $\delta p / \delta \rho = 1$ independent of potential $V(\phi)$. Solve in synchronous gauge (Hu 1998).

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Inflationary Perturbations

Scalar Fields

- Stress-energy tensor of a scalar field

$$T^{\mu}_{\nu} = \nabla^{\mu}\varphi \nabla_{\nu}\varphi - \frac{1}{2}(\nabla^{\alpha}\varphi \nabla_{\alpha}\varphi + 2V)\delta^{\mu}_{\nu}.$$

- For the background $\langle\phi\rangle \equiv \phi_0$

$$\rho_{\phi} = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + V \quad p_{\phi} = \frac{1}{2}a^{-2}\dot{\phi}_0^2 - V$$

- So for kinetic dominated $w_{\phi} = p_{\phi}/\rho_{\phi} \rightarrow 1$
- And potential dominated $w_{\phi} = p_{\phi}/\rho_{\phi} \rightarrow -1$
- A slowly rolling (potential dominated) scalar field can accelerate the expansion and so solve the horizon problem or act as a dark energy candidate

Equation of Motion

- Can use general equations of motion dictated by stress energy conservation

$$\dot{\rho}_\phi = -3(\rho_\phi + p_\phi)\frac{\dot{a}}{a},$$

to obtain the equation of motion of the background field ϕ

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2V' = 0,$$

- Likewise for the perturbations $\phi = \phi_0 + \phi_1$

$$\delta\rho_\phi = a^{-2}(\dot{\phi}_0\dot{\phi}_1 - \dot{\phi}_0^2A) + V'\phi_1,$$

$$\delta p_\phi = a^{-2}(\dot{\phi}_0\dot{\phi}_1 - \dot{\phi}_0^2A) - V'\phi_1,$$

$$(\rho_\phi + p_\phi)(v_\phi - B) = a^{-2}k\dot{\phi}_0\phi_1,$$

$$p_\phi\pi_\phi = 0,$$

Equation of Motion

- The stress of the perturbations is defined through

$$\delta p_\phi = \delta \rho_\phi + 3(\rho_\phi + p_\phi) \frac{v_\phi - B}{k} \frac{\dot{a}}{a} (1 - c_\phi^2)$$

where $c_\phi^2 \equiv \dot{p}_\phi / \dot{\rho}_\phi$ is the “adiabatic” sound speed

- So for the comoving gauge where $v_\phi = B$, $\delta p_\phi = \delta \rho_\phi$ so the sound speed relevant for stability is $\delta p_\phi / \delta \rho_\phi = 1$. Very useful for solving system since in this gauge everything is specified by $w(a)$
- Scalar field fluctuations are stable inside the horizon and are a good candidate for the smooth dark energy
- More generally, continuity and Euler equations imply

$$\ddot{\phi}_1 = -2 \frac{\dot{a}}{a} \dot{\phi}_1 - (k^2 + a^2 V'') \phi_1 + (\dot{A} - 3\dot{H}_L - kB) \dot{\phi}_0 - 2Aa^2 V'.$$

Inflationary Perturbations

- Classical equations of motion for scalar field inflaton determine the evolution of scalar field fluctuations generated by quantum fluctuations
- Since the Bardeen or comoving curvature ζ is conserved in the absence of stress fluctuations (i.e. outside the apparent horizon, calculate this and we're done no matter what happens in between inflation and the late universe (reheating etc.)
- But in the comoving gauge $\phi_1 = 0!$ no scalar-field perturbation
- Solution: solve the scalar field equation in the dual gauge where the curvature $H_L = 0$ (and $H_T = 0$ to fix the gauge completely, as the “spatially unperturbed” or “spatially flat” gauge) and transform the result to the comoving gauge

Transformation to Comoving Gauge

- Scalar field transforms as scalar field

$$\tilde{\phi}_1 = \phi_1 - \dot{\phi}_0 T$$

- To get to comoving frame $\tilde{\phi}_1 = 0$, $T = \phi_1 / \dot{\phi}_0$, and $\tilde{H}_T = H_T + kL$ so

$$\begin{aligned}\zeta &= H_L - \frac{k}{3}L - \frac{\dot{a}}{a}T, \\ &= H_L + \frac{H_T}{3} - \frac{\dot{a}}{a} \frac{\phi_1}{\dot{\phi}_0}\end{aligned}$$

- Transformation particularly simple from a gauge with $H_T = H_L = 0$, i.e. spatially unperturbed metric

$$\zeta = -\frac{\dot{a}}{a} \frac{\phi_1}{\dot{\phi}_0}$$

Scalar Field Eqn of Motion

- Scalar field perturbation in spatially unperturbed gauge is simply proportional to resulting Bardeen curvature with the proportionality constant as the expansion rate over roll rate – enhanced
- Scalar field fluctuation satisfies classical equation of motion

$$\ddot{\phi}_1 = -2\frac{\dot{a}}{a}\dot{\phi}_1 - (k^2 + a^2V'')\phi_1 + (\dot{A} - kB)\dot{\phi}_0 - 2Aa^2V' .$$

- Metric terms may be eliminated through Einstein equations

$$\begin{aligned} A &= 4\pi G a^2 \left(\frac{\dot{a}}{a}\right)^{-1} (\rho_\phi + p_\phi)(v_\phi - B)/k \\ &= 4\pi G \left(\frac{\dot{a}}{a}\right)^{-1} \dot{\phi}_0 \phi_1 \end{aligned}$$

Scalar Field Eqn of Motion

- And

$$\begin{aligned} kB &= 4\pi G a^2 \left[\left(\frac{\dot{a}}{a} \right)^{-1} \delta\rho_\phi + 3 \frac{\dot{a}}{a} (\rho_\phi + p_\phi) (v_\phi - B)/k \right] \\ &= 4\pi G \left[\left(\frac{\dot{a}}{a} \right)^{-1} (\dot{\phi}_0 \dot{\phi}_1 + a^2 V' \phi_1) - \left(\frac{\dot{a}}{a} \right)^{-2} (4\pi G \dot{\phi}_0)^2 \dot{\phi}_0 \phi_1 + 3 \dot{\phi}_0 \phi_1 \right] \end{aligned}$$

- So $\dot{A} - kB \propto \phi_1$ with proportionality that depends only on the background evolution – Einstein & scalar field equations reduce to a single second order diff eq!
- Equation resembles a damped oscillator equation with a particular dispersion relation

$$\ddot{\phi}_1 + 2 \frac{\dot{a}}{a} \dot{\phi}_1 + [k^2 + f(\eta)] \phi_1$$

Exact Equation

- Rewrite equations of motion in terms of slow roll parameters but do not require them to be small or constant.
- Deviation from de Sitter expansion

$$\begin{aligned}\epsilon &\equiv \frac{3}{2}(1 + w_\phi) \\ &= \frac{\frac{3}{2}\dot{\phi}_0^2/a^2V}{1 + \frac{1}{2}\dot{\phi}_0^2/a^2V}\end{aligned}$$

- Deviation from overdamped limit of $d^2\phi_0/dt^2 = 0$

$$\delta \equiv \frac{\ddot{\phi}_0}{\dot{\phi}_0} \left(\frac{\dot{a}}{a} \right)^{-1} - 1$$

Exact Equation

- Friedman equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = 4\pi G\dot{\phi}_0^2\epsilon^{-1}$$

$$\frac{d}{d\eta}\left(\frac{\dot{a}}{a}\right) = \left(\frac{\dot{a}}{a}\right)^2(1 - \epsilon)$$

- Homogenous scalar field equation

$$\dot{\phi}_0\frac{\dot{a}}{a}(3 + \delta) = -a^2V'$$

- Combination

$$\dot{\epsilon} = 2\epsilon(\delta + \epsilon)\frac{\dot{a}}{a}$$

Exact equation

- Rewrite in $u \equiv a\phi$ to remove expansion damping

$$\ddot{u} + [k^2 + g(\eta)]u = 0$$

where Mukhanov

$$\begin{aligned} g(\eta) &\equiv f(\eta) + \epsilon - 2 \\ &= - \left(\frac{\dot{a}}{a} \right)^2 [2 + 3\delta + 2\epsilon + (\delta + \epsilon)(\delta + 2\epsilon)] - \frac{\dot{a}}{a} \dot{\delta} \\ &= - \frac{\ddot{z}}{z} \end{aligned}$$

and

$$z \equiv a \left(\frac{\dot{a}}{a} \right)^{-1} \dot{\phi}_0$$

Slow Roll Limit

- Slow roll $\epsilon \ll 1$, $\delta \ll 1$, $\dot{\delta} \ll \frac{\dot{a}}{a}$

$$\ddot{u} + \left[k^2 - 2 \left(\frac{\dot{a}}{a} \right)^2 \right] u = 0$$

- or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\text{end}}$$

$$\tilde{\eta} = \int_{a_{\text{end}}}^a \frac{da}{H a^2} \approx -\frac{1}{aH}$$

- Compact, slow-roll equation:

$$\ddot{u} + \left[k^2 - \frac{2}{\tilde{\eta}^2} \right] u = 0$$

Slow Roll Limit

- Slow roll equation has the exact solution:

$$u = A\left(k \pm \frac{i}{\tilde{\eta}}\right)e^{\mp ik\tilde{\eta}}$$

- For $|k\tilde{\eta}| \gg 1$ (early times, inside Hubble length) behaves as free oscillator

$$\lim_{|k\tilde{\eta}| \rightarrow \infty} u = Ake^{\mp ik\tilde{\eta}}$$

- Normalization A will be set by origin in quantum fluctuations of free field

Slow Roll Limit

- For $|k\tilde{\eta}| \ll 1$ (late times, \gg Hubble length) fluctuation freezes in

$$\lim_{|k\tilde{\eta}| \rightarrow 0} u = \pm \frac{i}{\tilde{\eta}} A = \pm i H a A$$

$$\phi_1 = \pm i H A$$

$$\zeta = \mp i H A \left(\frac{\dot{a}}{a} \right) \frac{1}{\dot{\phi}_0}$$

- Slow roll replacement

$$\left(\frac{\dot{a}}{a} \right)^2 \frac{1}{\dot{\phi}_0^2} = \frac{8\pi G a^2 V}{3} \frac{3}{2a^2 V \epsilon} = 4\pi G = \frac{4\pi}{m_{\text{pl}}^2}$$

- Bardeen curvature power spectrum

$$\Delta_{\zeta}^2 \equiv \frac{k^3 |\zeta|^2}{2\pi^2} = \frac{2k^3}{\pi} \frac{H^2}{\epsilon m_{\text{pl}}^2} A^2$$

Quantum Fluctuations

- Simple harmonic oscillator \ll Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k, \eta) \hat{a} + u^*(k, \eta) \hat{a}^\dagger$$

where $u(k, \eta)$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \quad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\eta] = i$

$$u(k, \eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

Quantum Fluctuations

- Zero point fluctuations of ground state

$$\begin{aligned}\langle u^2 \rangle &= \langle 0 | u^\dagger u | 0 \rangle \\ &= \langle 0 | (u^* \hat{a}^\dagger + u \hat{a}) (u \hat{a} + u^* \hat{a}^\dagger) | 0 \rangle \\ &= \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle |u(k, \tilde{\eta})|^2 \\ &= \langle 0 | [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} | 0 \rangle |u(k, \tilde{\eta})|^2 \\ &= |u(k, \tilde{\eta})|^2 = \frac{1}{2k}\end{aligned}$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in. Slow roll equation
- So $A = (2k^3)^{1/2}$ and curvature power spectrum

$$\Delta_\zeta^2 \equiv \frac{1}{\pi} \frac{H^2}{\epsilon m_{\text{pl}}^2}$$

Tilt

- Curvature power spectrum is scale invariant to the extent that H is constant
- Scalar spectral index

$$\begin{aligned}\frac{d \ln \Delta_{\zeta}^2}{d \ln k} &\equiv n_S - 1 \\ &= 2 \frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon}{d \ln k}\end{aligned}$$

- Evaluate at horizon crossing where fluctuation freezes

$$\begin{aligned}\frac{d \ln H}{d \ln k} \Big|_{-k\tilde{\eta}=1} &= \frac{k}{H} \frac{dH}{d\tilde{\eta}} \Big|_{-k\tilde{\eta}=1} \frac{d\tilde{\eta}}{dk} \Big|_{-k\tilde{\eta}=1} \\ &= \frac{k}{H} (-aH^2\epsilon) \Big|_{-k\tilde{\eta}=1} \frac{1}{k^2} = -\epsilon\end{aligned}$$

where $aH = -1/\tilde{\eta} = k$

Tilt

- Evolution of ϵ

$$\frac{d \ln \epsilon}{d \ln k} = -\frac{d \ln \epsilon}{d \ln \tilde{\eta}} = -2(\delta + \epsilon) \frac{\dot{a}}{a} \tilde{\eta} = 2(\delta + \epsilon)$$

- Tilt in the slow-roll approximation

$$n_S = 1 - 4\epsilon - 2\delta$$

Relationship to Potential

- To leading order in slow roll parameters

$$\begin{aligned}\epsilon &= \frac{\frac{3}{2}\dot{\phi}_0^2/a^2V}{1 + \frac{1}{2}\dot{\phi}_0^2/a^2V} \\ &\approx \frac{3}{2}\dot{\phi}_0^2/a^2V \\ &\approx \frac{3}{a^2V} \frac{a^4V'^2}{9(\dot{a}/a)^2}, \quad \left(3\dot{\phi}_0 \frac{\dot{a}}{a} = -a^2V'\right) \\ &\approx \frac{1}{6} \frac{3}{8\pi G} \left(\frac{V'}{V}\right)^2, \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}a^2V \\ &\approx \frac{1}{16\pi G} \left(\frac{V'}{V}\right)^2\end{aligned}$$

so $\epsilon \ll 1$ is related to the first derivative of potential being small

Relationship to Potential

- And

$$\delta = \frac{\ddot{\phi}_0}{\dot{\phi}_0} \left(\frac{\dot{a}}{a} \right)^{-1} - 1$$

$$\left(\dot{\phi}_0 \approx -a^2 \left(\frac{\dot{a}}{a} \right)^{-1} \frac{V'}{3} \right)$$

$$\left(\ddot{\phi}_0 \approx -\frac{a^2 V'}{3} (1 + \epsilon) + a^4 \left(\frac{\dot{a}}{a} \right)^{-2} \frac{V' V''}{9} \right)$$

$$\approx -\frac{1}{a^2 V' / 3} \left(-\frac{a^2 V'}{3} (1 + \epsilon) + \frac{a^2}{9} \frac{3}{8\pi G} \frac{V' V''}{V} \right) - 1 \approx \epsilon - \frac{1}{8\pi G} \frac{V''}{V}$$

so δ is related to second derivative of potential being small. Very flat potential.

Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation ($K = 0$), same as scalar field

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0.$$

- Acquires quantum fluctuations in same manner as ϕ . Lagrangian sets the normalization

$$\phi_1 \rightarrow H_T^{(\pm 2)} \sqrt{\frac{3}{16\pi G}}$$

- Scale-invariant gravitational wave amplitude (each component: NB more traditional notation $H_T^{(\pm 2)} = (h_+ \pm ih_\times)/\sqrt{6}$)

$$\Delta_H^2 = \frac{16\pi G}{3 \cdot 2\pi^2} \frac{H^2}{2} = \frac{4}{3\pi} \frac{H^2}{m_{\text{pl}}^2}$$

Gravitational Waves

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation
- Tensor tilt:

$$\frac{d \ln \Delta_H^2}{d \ln k} \equiv n_T = 2 \frac{d \ln H}{d \ln k} = -2\epsilon$$

- Consistency relation between tensor-scalar ratio and tensor tilt

$$\frac{\Delta_H^2}{\Delta_\zeta^2} = \frac{4}{3}\epsilon = -\frac{2}{3}\epsilon$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparison of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself

Gravitational Wave Phenomenology

- Equation of motion

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0.$$

- has solutions

$$H_T^{(\pm 2)} = C_1 H_1(k\eta) + C_2 H_2(k\eta)$$

$$H_1 \propto x^{-m} j_m(x)$$

$$H_2 \propto x^{-m} n_m(x)$$

where $m = (1 - 3w)/(1 + 3w)$

- If $w > -1/3$ then gravity wave is constant above horizon $x \ll 1$ and then oscillates and damps
- If $w < -1/3$ then gravity wave oscillates and freezes into some value, just like scalar field

Gravitational Wave Phenomenology

- A gravitational wave makes a quadrupolar (transverse-traceless) distortion to metric
- Just like the scale factor or spatial curvature, a temporal variation in its amplitude leaves a residual temperature variation in CMB photons – here anisotropic
- Before recombination, anisotropic variation is eliminated by scattering
- Gravitational wave temperature effect drops sharply at the horizon scale at recombination
- Source to polarization goes as $\dot{\tau}/\dot{H}_T$ and peaks at the horizon not damping scale
- B modes formed as photons propagate – the spatial variation in the plane waves modulate the signal: described by Boltzmann eqn.

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Boltzmann Formalism

Boltzmann Equation

- CMB radiation is generally described by the phase space distribution function for each polarization state $f_a(\mathbf{x}, \mathbf{q}, \eta)$, where \mathbf{x} is the comoving position and \mathbf{q} is the photon momentum
- Boltzmann equation describes the evolution of the distribution function under gravity and collisions
- Low order moments of the Boltzmann equation are simply the covariant conservation equations
- Higher moments provide the closure condition to the conservation law (specification of stress tensor) and the CMB observable – fine scale anisotropy
- Higher moments mainly describe the simple geometry of source projection

Liouville Equation

- In absence of scattering, the phase space distribution of photons is conserved along the propagation path
- Rewrite variables in terms of the photon propagation direction $\mathbf{q} = q\hat{\mathbf{n}}$, so $f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta)$ and

$$\begin{aligned}\frac{d}{d\eta} f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta) &= 0 \\ &= \left(\frac{\partial}{\partial \eta} + \frac{d\mathbf{x}}{d\eta} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{d\hat{\mathbf{n}}}{d\eta} \cdot \frac{\partial}{\partial \hat{\mathbf{n}}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a\end{aligned}$$

- For simplicity, assume spatially flat universe $K = 0$ then $d\hat{\mathbf{n}}/d\eta = 0$ and $d\mathbf{x} = \hat{\mathbf{n}}d\eta$

$$\dot{f}_a + \hat{\mathbf{n}} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0$$

Correspondence to Einstein Eqn.

- Geodesic equation gives the redshifting term

$$\frac{\dot{q}}{q} = -\frac{\dot{a}}{a} - \frac{1}{2}n^i n^j \dot{H}_{Tij} - \dot{H}_L + n^i \dot{B}_i - \hat{\mathbf{n}} \cdot \nabla A$$

- which is incorporated in the conservation and gauge transformation equations
- Stress energy tensor involves integrals over the distribution function the two polarization states

$$T^{\mu\nu} = \int \frac{d^3q}{(2\pi)^3} \frac{q^\mu q^\nu}{E} (f_a + f_b)$$

- Components are simply the low order angular moments of the distribution function

Angular Moments

- Define the angularly dependent temperature perturbation

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \frac{1}{4\rho_\gamma} \int \frac{q^3 dq}{2\pi^2} (f_a + f_b) - 1$$

and likewise for the linear polarization states Q and U

- Decompose into normal modes: plane waves for spatial part and spherical harmonics for angular part

$$G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$\pm_2 G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} \pm_2 Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$

- In a spatially curved universe generalize the plane wave part

Normal Modes

- Temperature and polarization fields

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} \Theta_{\ell}^{(m)} G_{\ell}^m$$

$$[Q \pm iU](\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} [E_{\ell}^{(m)} \pm iB_{\ell}^{(m)}]_{\pm 2} G_{\ell}^m$$

- For each \mathbf{k} mode, work in coordinates where $\mathbf{k} \parallel \mathbf{z}$ and so $m = 0$ represents scalar modes, $m = \pm 1$ vector modes, $m = \pm 2$ tensor modes, $|m| > 2$ vanishes. Since modes add incoherently and $Q \pm iU$ is invariant up to a phase, rotation back to a fixed coordinate system is trivial.

Scalar, Vector, Tensor

- Normalization of modes is chosen so that the lowest angular mode for scalars, vectors and tensors are normalized in the same way as the mode function

$$\begin{aligned}G_0^0 &= Q^{(0)} & G_1^0 &= n^i Q_i^{(0)} & G_2^0 &\propto n^i n^j Q_{ij}^{(0)} \\G_1^{\pm 1} &= n^i Q_i^{(\pm 1)} & G_2^{\pm 1} &\propto n^i n^j Q_{ij}^{(\pm 1)} \\G_2^{\pm 2} &= n^i n^j Q_{ij}^{(\pm 2)}\end{aligned}$$

where recall

$$\begin{aligned}Q^{(0)} &= \exp(i\mathbf{k} \cdot \mathbf{x}) \\Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}} (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k} \cdot \mathbf{x}) \\Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}} (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k} \cdot \mathbf{x})\end{aligned}$$

Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies
- Spatial gradient term hits plane wave:

$$\hat{\mathbf{n}} \cdot \nabla e^{i\mathbf{k} \cdot \mathbf{x}} = i\hat{\mathbf{n}} \cdot \mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x}} = i\sqrt{\frac{4\pi}{3}} k Y_1^0(\hat{\mathbf{n}}) e^{i\mathbf{k} \cdot \mathbf{x}}$$

- Dipole term adds to angular dependence through the addition of angular momentum

$$\sqrt{\frac{4\pi}{3}} Y_1^0 Y_\ell^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell+1)(2\ell-1)}} Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell+1)(2\ell+3)}} Y_{\ell+1}^m$$

where $\kappa_\ell^m = \sqrt{\ell^2 - m^2}$ is given by Clebsch-Gordon coefficients.

Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

$$\dot{\Theta}_\ell^{(m)} = k \left[\frac{\kappa_\ell^m}{2\ell + 1} \Theta_{\ell-1}^{(m)} - \frac{\kappa_{\ell+1}^m}{2\ell + 3} \Theta_{\ell+1}^{(m)} \right] - \dot{\tau} \Theta_\ell^{(m)} + S_\ell^{(m)}$$

where $S_\ell^{(m)}$ are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic $\ell = 0$ temperature perturbation will eventually become a high order anisotropy by “free streaming” or simple projection
- Original CMB codes solved the full hierarchy equations out to the ℓ of interest.

Integral Solution

- Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface
- In general, the solution describes the decomposition of the source $S_\ell^{(m)}$ with its local angular dependence as seen at a distance $\mathbf{x} = D\hat{\mathbf{n}}$.
- Proceed by decomposing the angular dependence of the plane wave

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\ell} (-i)^{\ell} \sqrt{4\pi(2\ell + 1)} j_{\ell}(kD) Y_{\ell}^0(\hat{\mathbf{n}})$$

- Recouple to the local angular dependence of G_{ℓ}^m

$$G_{\ell_s}^m = \sum_{\ell} (-i)^{\ell} \sqrt{4\pi(2\ell + 1)} \alpha_{\ell_s \ell}^{(m)}(kD) Y_{\ell}^m(\hat{\mathbf{n}})$$

Integral Solution

- Projection kernels:

$$\ell_s = 0, \quad m = 0 \qquad \alpha_{0\ell}^{(0)} \equiv j_\ell$$

$$\ell_s = 1, \quad m = 0 \qquad \alpha_{1\ell}^{(0)} \equiv j'_\ell$$

- Integral solution:

$$\frac{\Theta_\ell^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S_{\ell_s}^{(m)} \alpha_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

- Power spectrum:

$$C_\ell = \frac{2}{\pi} \int \frac{dk}{k} \sum_m \frac{k^3 \langle \Theta_\ell^{(m)*} \Theta_\ell^{(m)} \rangle}{(2\ell + 1)^2}$$

- Solving for C_ℓ reduces to solving for the behavior of a handful of sources

Polarization Hierarchy

- In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hierarchy:

$$\dot{E}_\ell^{(m)} = k \left[\frac{{}_2\kappa_\ell^m}{2\ell - 1} E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell + 1)} B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell + 3} \right] - \dot{\tau} E_\ell^{(m)} + \mathcal{E}_\ell^{(m)}$$

$$\dot{B}_\ell^{(m)} = k \left[\frac{{}_2\kappa_\ell^m}{2\ell - 1} B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell + 1)} B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell + 3} \right] - \dot{\tau} E_\ell^{(m)} + \mathcal{B}_\ell^{(m)}$$

where ${}_2\kappa_\ell^m = \sqrt{(\ell^2 - m^2)(\ell^2 - 4)}/\ell^2$ is given by the Clebsch-Gordon coefficients and \mathcal{E} , \mathcal{B} are the sources (scattering only).

- Note that for vectors and tensors $|m| > 0$ and B modes may be generated from E modes by projection. Cosmologically $\mathcal{B}_\ell^{(m)} = 0$

Polarization Integral Solution

- Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

$$\frac{E_\ell^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \epsilon_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

$$\frac{B_\ell^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \beta_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

- The only source to the polarization is from the quadrupole anisotropy so we only need $\ell_s = 2$, e.g. for scalars

$$\epsilon_{2\ell}^{(0)}(x) = \sqrt{\frac{3(\ell + 2)!}{8(\ell - 2)!}} \frac{j_\ell(x)}{x^2} \quad \beta_{2\ell}^{(0)} = 0$$

Truncated Hierarchy

- CMBFast uses the integral solution and relies on a fast j_ℓ generator
- However sources are not external to system and are defined through the Boltzmann hierarchy itself
- Solution: recall that we used this technique in the tight coupling regime by applying a closure condition from tight coupling
- CMBFast extends this idea by solving a truncated hierarchy of equations, e.g. out to $\ell = 25$ with non-reflecting boundary conditions

Thomson Collision Term

- Full Boltzmann equation

$$\frac{d}{d\eta} f_{a,b} = C[f_a, f_b]$$

- Collision term describes the scattering out of and into a phase space element
- Thomson collision based on differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T ,$$

where $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

Scattering Calculation

- Start in the electron rest frame and in a coordinate system fixed by the scattering plane, spanned by incoming and outgoing directional vectors $-\hat{\mathbf{n}}' \cdot \hat{\mathbf{n}} = \cos \beta$, where β is the scattering angle
- Θ_{\parallel} : in-plane polarization state; Θ_{\perp} : \perp -plane polarization state
- Transfer probability (constant set by $\dot{\tau}$)

$$\Theta_{\parallel} \propto \cos^2 \beta \Theta'_{\parallel}, \quad \Theta_{\perp} \propto \Theta'_{\perp}$$

- and with the 45° axes as

$$\hat{\mathbf{E}}_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\parallel} + \hat{\mathbf{E}}_{\perp}), \quad \hat{\mathbf{E}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\parallel} - \hat{\mathbf{E}}_{\perp})$$

Stokes Parameters

- Define the temperature in this basis

$$\begin{aligned}\Theta_1 &\propto |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 + |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 \\ &\propto \frac{1}{4}(\cos \beta + 1)^2 \Theta'_1 + \frac{1}{4}(\cos \beta - 1)^2 \Theta'_2\end{aligned}$$

$$\begin{aligned}\Theta_2 &\propto |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 + |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 \\ &\propto \frac{1}{4}(\cos \beta + 1)^2 \Theta'_2 + \frac{1}{4}(\cos \beta - 1)^2 \Theta'_1\end{aligned}$$

or $\Theta_1 - \Theta_2 \propto \cos \beta (\Theta'_1 - \Theta'_2)$

- Define Θ , Q , U in the scattering coordinates

$$\Theta \equiv \frac{1}{2}(\Theta_{\parallel} + \Theta_{\perp}), \quad Q \equiv \frac{1}{2}(\Theta_{\parallel} - \Theta_{\perp}), \quad U \equiv \frac{1}{2}(\Theta_1 - \Theta_2)$$

Scattering Matrix

- Transfer of Stokes states, e.g.

$$\Theta = \frac{1}{2}(\Theta_{\parallel} + \Theta_{\perp}) \propto \frac{1}{4}(\cos^2 \beta + 1)\Theta' + \frac{1}{4}(\cos^2 \beta - 1)Q'$$

- Transfer matrix of Stokes state $\mathbf{T} \equiv (\Theta, Q + iU, Q - iU)$

$$\mathbf{T} \propto \mathbf{S}(\beta)\mathbf{T}'$$

$$\mathbf{S}(\beta) = \frac{3}{4} \begin{pmatrix} \cos^2 \beta + 1 & -\frac{1}{2} \sin^2 \beta & -\frac{1}{2} \sin^2 \beta \\ -\frac{1}{2} \sin^2 \beta & \frac{1}{2}(\cos \beta + 1)^2 & \frac{1}{2}(\cos \beta - 1)^2 \\ -\frac{1}{2} \sin^2 \beta & \frac{1}{2}(\cos \beta - 1)^2 & \frac{1}{2}(\cos \beta + 1)^2 \end{pmatrix}$$

normalization factor of 3 is set by photon conservation in scattering

Scattering Matrix

- Transform to a fixed basis, by a rotation of the incoming and outgoing states $\mathbf{T} = \mathbf{R}(\psi)\mathbf{T}$ where

$$\mathbf{R}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & e^{2i\psi} \end{pmatrix}$$

giving the scattering matrix

$$\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha) =$$

$$\frac{1}{2}\sqrt{\frac{4\pi}{5}} \begin{pmatrix} Y_2^0(\beta, \alpha) + 2\sqrt{5}Y_0^0(\beta, \alpha) & -\sqrt{\frac{3}{2}}Y_2^{-2}(\beta, \alpha) & -\sqrt{\frac{3}{2}}Y_2^2(\beta, \alpha) \\ -\sqrt{6}{}_2Y_2^0(\beta, \alpha)e^{2i\gamma} & 3{}_2Y_2^{-2}(\beta, \alpha)e^{2i\gamma} & 3{}_2Y_2^2(\beta, \alpha)e^{2i\gamma} \\ -\sqrt{6}{}_{-2}Y_2^0(\beta, \alpha)e^{-2i\gamma} & 3{}_{-2}Y_2^{-2}(\beta, \alpha)e^{-2i\gamma} & 3{}_{-2}Y_2^2(\beta, \alpha)e^{-2i\gamma} \end{pmatrix}$$

Addition Theorem for Spin Harmonics

- Spin harmonics are related to rotation matrices as

$${}_s Y_\ell^m(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} \mathcal{D}_{-ms}^\ell(\phi, \theta, 0)$$

Note: for explicit evaluation sign convention differs from usual (e.g. Jackson) by $(-1)^m$

- Multiplication of rotations

$$\sum_{m''} \mathcal{D}_{mm''}^\ell(\alpha_2, \beta_2, \gamma_2) \mathcal{D}_{m''m}^\ell(\alpha_1, \beta_1, \gamma_1) = \mathcal{D}_{mm'}^\ell(\alpha, \beta, \gamma)$$

- Implies

$$\sum_m {}_{s_1} Y_\ell^{m*}(\theta', \phi') {}_{s_2} Y_\ell^m(\theta, \phi) = (-1)^{s_1 - s_2} \sqrt{\frac{2\ell + 1}{4\pi}} {}_{s_2} Y_\ell^{-s_1}(\beta, \alpha) e^{is_2\gamma}$$

Sky Basis

- Scattering into the state (rest frame)

$$\begin{aligned}
 C_{\text{in}}[\mathbf{T}] &= \dot{\tau} \int \frac{d\hat{\mathbf{n}}'}{4\pi} \mathbf{R}(-\gamma) \mathbf{S}(\beta) \mathbf{R}(\alpha) \mathbf{T}(\hat{\mathbf{n}}'), \\
 &= \dot{\tau} \int \frac{d\hat{\mathbf{n}}'}{4\pi} (\Theta', 0, 0) + \frac{1}{10} \dot{\tau} \int d\hat{\mathbf{n}}' \sum_{m=-2}^2 \mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') \mathbf{T}(\hat{\mathbf{n}}').
 \end{aligned}$$

where the quadrupole coupling term is $\mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') =$

$$\begin{pmatrix}
 Y_2^{m*}(\hat{\mathbf{n}}') Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}} {}_2Y_2^{m*}(\hat{\mathbf{n}}') Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}} {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}') Y_2^m(\hat{\mathbf{n}}) \\
 -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}') {}_2Y_2^m(\hat{\mathbf{n}}) & 3 {}_2Y_2^{m*}(\hat{\mathbf{n}}') {}_2Y_2^m(\hat{\mathbf{n}}) & 3 {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}') {}_2Y_2^m(\hat{\mathbf{n}}) \\
 -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}') {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3 {}_2Y_2^{m*}(\hat{\mathbf{n}}') {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3 {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}') {}_{-2}Y_2^m(\hat{\mathbf{n}})
 \end{pmatrix},$$

expression uses angle addition relation above. We call this term C_Q .

Scattering Matrix

- Full scattering matrix involves difference of scattering into and out of state

$$C[\mathbf{T}] = C_{\text{in}}[\mathbf{T}] - C_{\text{out}}[\mathbf{T}]$$

- In the electron rest frame

$$C[\mathbf{T}] = \dot{\tau} \int \frac{d\hat{\mathbf{n}}'}{4\pi} (\Theta', 0, 0) - \dot{\tau} \mathbf{T} + C_Q[\mathbf{T}]$$

which describes isotropization in the rest frame. All moments have $e^{-\tau}$ suppression except for isotropic temperature Θ_0 .

Transformation into the background frame simply induces a dipole term

$$C[\mathbf{T}] = \dot{\tau} \left(\hat{\mathbf{n}} \cdot \mathbf{v}_b + \int \frac{d\hat{\mathbf{n}}'}{4\pi} \Theta', 0, 0 \right) - \dot{\tau} \mathbf{T} + C_Q[\mathbf{T}]$$

Source Terms

- Temperature source terms $S_l^{(m)}$ (rows $\pm|m|$; flat assumption)

$$\begin{pmatrix} \dot{\tau}\Theta_0^{(0)} - \dot{H}_L^{(0)} & \dot{\tau}v_b^{(0)} + \dot{B}^{(0)} & \dot{\tau}P^{(0)} - \frac{2}{3}\dot{H}_T^{(0)} \\ 0 & \dot{\tau}v_b^{(\pm 1)} + \dot{B}^{(\pm 1)} & \dot{\tau}P^{(\pm 1)} - \frac{\sqrt{3}}{3}\dot{H}_T^{(\pm 1)} \\ 0 & 0 & \dot{\tau}P^{(\pm 2)} - \dot{H}_T^{(\pm 2)} \end{pmatrix}$$

where

$$P^{(m)} \equiv \frac{1}{10}(\Theta_2^{(m)} - \sqrt{6}E_2^{(m)})$$

- Polarization source term

$$\mathcal{E}_\ell^{(m)} = -\dot{\tau}\sqrt{6}P^{(m)}\delta_{\ell,2}$$

$$\mathcal{B}_\ell^{(m)} = 0$$

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Secondary Anisotropy

Secondary Anisotropy

- CMB photons traverse the large-scale structure of the universe from $z = 1000$ to the present.
- With the nearly scale-invariant adiabatic fluctuations observed in the CMB, structures form from the bottom up, i.e. small scales first, a.k.a. hierarchical structure formation.
- First objects reionize the universe between $z \sim 7 - 30$
- Main sources of secondary anisotropy
- Gravitational: Integrated Sachs-Wolfe effect (gravitational redshift) and gravitational lensing
- Scattering: peak suppression, large-angle polarization, Doppler effect(s), inverse Compton scattering

Transfer Function

- **Transfer function** transfers the initial Newtonian curvature to its value today (linear response theory)

$$T(k) = \frac{\Phi(k, a = 1)}{\Phi(k, a_{\text{init}})} \frac{\Phi(k_{\text{norm}}, a_{\text{init}})}{\Phi(k_{\text{norm}}, a = 1)}$$

- Conservation of Bardeen curvature: Newtonian curvature is a **constant** when **stress perturbations are negligible**: above the horizon during radiation and dark energy domination, on all scales during matter domination
- When stress fluctuations dominate, perturbations are stabilized by the **Jeans mechanism**
- Hybrid **Poisson equation**: Newtonian curvature, comoving density perturbation $\Delta \equiv (\delta\rho/\rho)_{\text{com}}$ implies Φ decays

$$(k^2 - 3K)\Phi = 4\pi G\rho\Delta \sim \eta^{-2}\Delta$$

Transfer Function

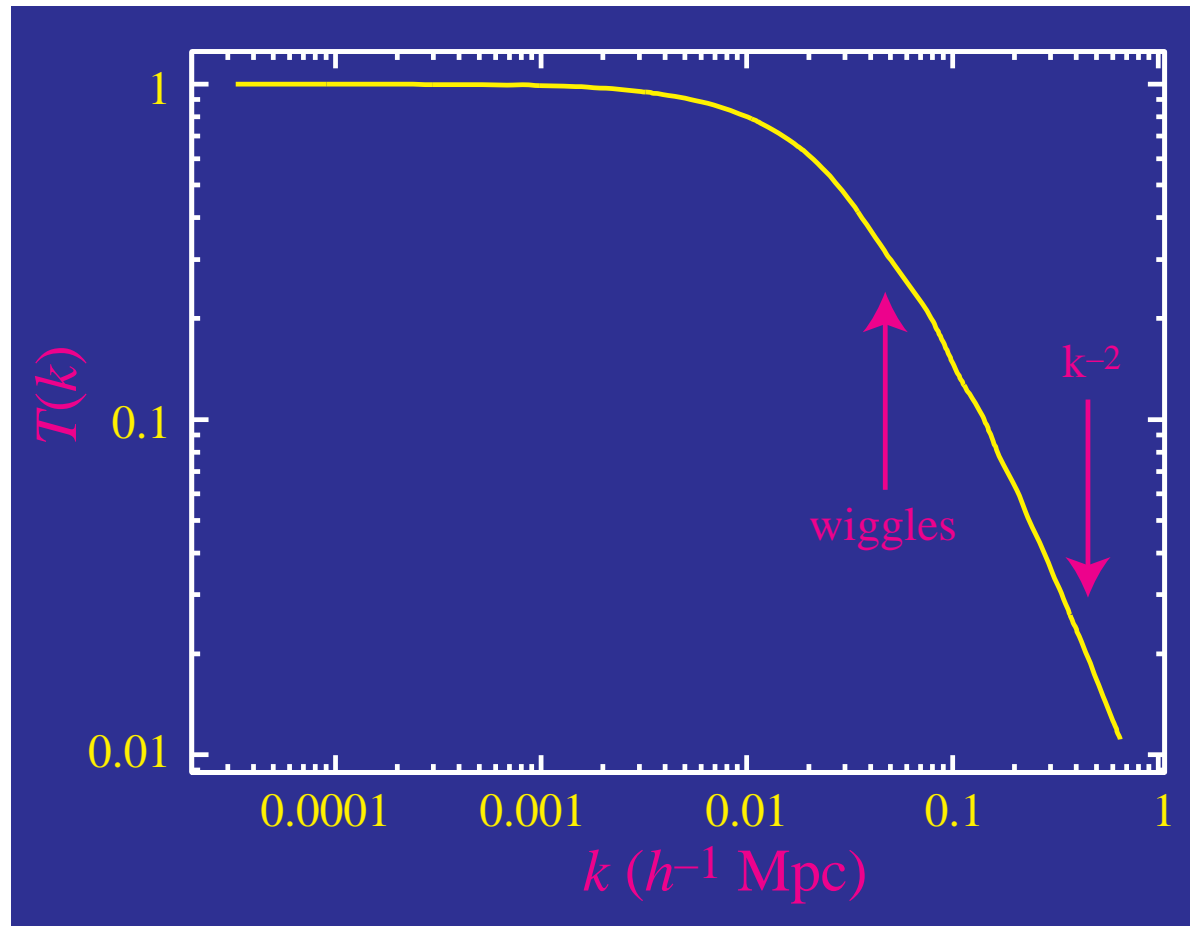
- Matter-radiation example: Jeans scale is horizon scale and Δ freezes into its value at horizon crossing $\Delta_H \approx \Phi_{\text{init}}$
- Freezing of Δ stops at η_{eq}

$$\Phi \sim (k\eta_{\text{eq}})^{-2} \Delta_H \sim (k\eta_{\text{eq}})^{-2} \Phi_{\text{init}}$$

- Conventionally k_{norm} is chosen as a scale between the horizon at matter radiation equality and dark energy domination.
- Small correction since growth with a smooth radiation component is logarithmic not frozen
- Run CMBfast to get transfer function or use fits

Transfer Function

- Transfer function has a k^{-2} fall-off beyond $k_{\text{eq}} \sim \eta_{\text{eq}}^{-1}$



- Additional baryon wiggles are due to acoustic oscillations at recombination – an interesting means of measuring distances

Growth Function

- Same physics applies to the dark energy dominated universe
- Under the dark energy sound horizon, dark energy density frozen. Potential decays at the same rate for all scales

$$g(a) = \frac{\Phi(k_{\text{norm}}, a)}{\Phi(k_{\text{norm}}, a_{\text{init}})}$$

- Pressure growth suppression: $\delta \equiv \delta\rho_m/\rho_m \propto a\phi$

$$\frac{d^2 g}{d \ln a^2} + \left[\frac{5}{2} - \frac{3}{2} w(z) \Omega_{DE}(z) \right] \frac{dg}{d \ln a} + \frac{3}{2} [1 - w(z)] \Omega_{DE}(z) g = 0,$$

where $w \equiv p_{DE}/\rho_{DE}$ and $\Omega_{DE} \equiv \rho_{DE}/(\rho_m + \rho_{DE})$ with initial conditions $g = 1, dg/d \ln a = 0$

- As $\Omega_{DE} \rightarrow 0$ $g = \text{const.}$ is a solution. The other solution is the decaying mode, eliminated by initial conditions

ISW effect

- Potential decay leads to gravitational redshifts through the integrated Sachs-Wolfe effect
- Intrinsically a large effect since $2\Delta\Phi = 6\Psi_{\text{init}}/3$
- But net redshift is integral along line of sight

$$\begin{aligned}\frac{\Theta_\ell(k, \eta_0)}{2\ell + 1} &= \int_0^{\eta_0} d\eta e^{-\tau} [2\dot{\Phi}(k, \eta)] j_\ell(k(\eta_0 - \eta)) \\ &= 2\Phi(k, \eta_{MD}) \int_0^{\eta_0} d\eta e^{-\tau} \dot{g}(D) j_\ell(kD)\end{aligned}$$

- On small scales where $k \gg \dot{g}/g$, can pull source out of the integral

$$\int_0^{\eta_0} d\eta \dot{g}(D) j_\ell(kD) \approx \dot{g}(D = \ell/k) \frac{1}{k} \sqrt{\frac{\pi}{2\ell}}$$

evaluated at peak, where we have used $\int dx j_\ell(x) = \sqrt{\pi/2\ell}$

ISW effect

- Power spectrum

$$\begin{aligned} C_\ell &= \frac{2}{\pi} \int \frac{dk}{k} \frac{k^3 \langle \Theta_\ell^*(k, \eta_0) \Theta_\ell(k, \eta_0) \rangle}{(2\ell + 1)^2} \\ &= \frac{2\pi^2}{l^3} \int d\eta D \dot{g}^2(\eta) \Delta_\Phi^2(\ell/D, \eta_{MD}) \end{aligned}$$

- Or $l^2 C_l / 2\pi \propto 1/\ell$ for scale invariant potential. This is the Limber equation in spherical coordinates. Projection of 3D power retains only the transverse piece. For a general dark energy model, add in the scale dependence of growth rate on large scales.
- Cancellation of redshifts and blueshifts as the photon traverses many crests and troughs of a small scale fluctuation during decay. Enhancement of the $\ell < 10$ multipoles. Difficult to extract from cosmic variance and galaxy. Current ideas: cross correlation with other tracers of structure

Gravitational Lensing

- Lensing is a surface brightness conserving remapping of source to image planes by the gradient of the projected potential

$$\phi(\hat{\mathbf{n}}) = 2 \int_{\eta_*}^{\eta_0} d\eta \frac{(D_* - D)}{D D_*} \Phi(D\hat{\mathbf{n}}, \eta) .$$

such that the fields are remapped as

$$x(\hat{\mathbf{n}}) \rightarrow x(\hat{\mathbf{n}} + \nabla\phi) ,$$

where $x \in \{\Theta, Q, U\}$ temperature and polarization.

- Taylor expansion leads to product of fields and Fourier mode-coupling

Flat-sky Treatment

- Taylor expand

$$\begin{aligned}\Theta(\hat{\mathbf{n}}) &= \tilde{\Theta}(\hat{\mathbf{n}} + \nabla\phi) \\ &= \tilde{\Theta}(\hat{\mathbf{n}}) + \nabla_i\phi(\hat{\mathbf{n}})\nabla^i\tilde{\Theta}(\hat{\mathbf{n}}) + \frac{1}{2}\nabla_i\phi(\hat{\mathbf{n}})\nabla_j\phi(\hat{\mathbf{n}})\nabla^i\nabla^j\tilde{\Theta}(\hat{\mathbf{n}}) + \dots\end{aligned}$$

- Fourier decomposition

$$\begin{aligned}\phi(\hat{\mathbf{n}}) &= \int \frac{d^2l}{(2\pi)^2} \phi(\mathbf{l}) e^{i\mathbf{l}\cdot\hat{\mathbf{n}}} \\ \tilde{\Theta}(\hat{\mathbf{n}}) &= \int \frac{d^2l}{(2\pi)^2} \tilde{\Theta}(\mathbf{l}) e^{i\mathbf{l}\cdot\hat{\mathbf{n}}}\end{aligned}$$

Flat-sky Treatment

- Mode coupling of harmonics

$$\begin{aligned}\Theta(\mathbf{l}) &= \int d\hat{\mathbf{n}} \Theta(\hat{\mathbf{n}}) e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}} \\ &= \tilde{\Theta}(\mathbf{l}) - \int \frac{d^2\mathbf{l}_1}{(2\pi)^2} \tilde{\Theta}(\mathbf{l}_1) L(\mathbf{l}, \mathbf{l}_1),\end{aligned}$$

where

$$\begin{aligned}L(\mathbf{l}, \mathbf{l}_1) &= \phi(\mathbf{l} - \mathbf{l}_1) (\mathbf{l} - \mathbf{l}_1) \cdot \mathbf{l}_1 \\ &+ \frac{1}{2} \int \frac{d^2\mathbf{l}_2}{(2\pi)^2} \phi(\mathbf{l}_2) \phi^*(\mathbf{l}_2 + \mathbf{l}_1 - \mathbf{l}) (\mathbf{l}_2 \cdot \mathbf{l}_1) (\mathbf{l}_2 + \mathbf{l}_1 - \mathbf{l}) \cdot \mathbf{l}_1.\end{aligned}$$

- Represents a coupling of harmonics separated by $L \approx 60$ peak of deflection power

Power Spectrum

- Power spectra

$$\langle \Theta^*(\mathbf{l})\Theta(\mathbf{l}') \rangle = (2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') C_l^{\Theta\Theta},$$

$$\langle \phi^*(\mathbf{l})\phi(\mathbf{l}') \rangle = (2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') C_l^{\phi\phi},$$

becomes

$$C_l^{\Theta\Theta} = (1 - l^2 R) \tilde{C}_l^{\Theta\Theta} + \int \frac{d^2 \mathbf{l}_1}{(2\pi)^2} \tilde{C}_{|\mathbf{l} - \mathbf{l}_1|}^{\Theta\Theta} C_{l_1}^{\phi\phi} [(\mathbf{l} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2,$$

where

$$R = \frac{1}{4\pi} \int \frac{dl}{l} l^4 C_l^{\phi\phi}. \quad (3)$$

Smoothing Power Spectrum

- If $\tilde{C}_l^{\Theta\Theta}$ slowly varying then two term cancel

$$\tilde{C}_l^{\Theta\Theta} \int \frac{d^2\mathbf{l}_1}{(2\pi)^2} C_l^{\phi\phi} (\mathbf{l} \cdot \mathbf{l}_1)^2 \approx l^2 R \tilde{C}_l^{\Theta\Theta}.$$

- So lensing acts to smooth features in the power spectrum.
Smoothing kernel is $L \sim 60$ the peak of deflection power spectrum
- Because acoustic feature appear on a scale $l_A \sim 300$, smoothing is a subtle effect in the power spectrum.
- Lensing generates power below the damping scale which directly reflect power in deflections on the same scale

Polarization Lensing

- Polarization field harmonics lensed similarly

$$[Q \pm iU](\hat{\mathbf{n}}) = - \int \frac{d^2l}{(2\pi)^2} [E \pm iB](\mathbf{l}) e^{\pm 2i\phi_{\mathbf{l}}} e^{\mathbf{l} \cdot \hat{\mathbf{n}}}$$

so that

$$\begin{aligned} [Q \pm iU](\hat{\mathbf{n}}) &= [\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}} + \nabla\phi) \\ &\approx [\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}}) + \nabla_i\phi(\hat{\mathbf{n}})\nabla^i[\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}}) \\ &\quad + \frac{1}{2}\nabla_i\phi(\hat{\mathbf{n}})\nabla_j\phi(\hat{\mathbf{n}})\nabla^i\nabla^j[\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}}) \end{aligned}$$

Polarization Power Spectra

- Carrying through the algebra

$$C_l^{EE} = (1 - l^2 R) \tilde{C}_l^{EE} + \frac{1}{2} \int \frac{d^2 \mathbf{l}_1}{(2\pi)^2} [(\mathbf{1} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2 C_{|\mathbf{1}-\mathbf{l}_1|}^{\phi\phi} \\ \times [(\tilde{C}_{l_1}^{EE} + \tilde{C}_{l_1}^{BB}) + \cos(4\varphi_{l_1})(\tilde{C}_{l_1}^{EE} - \tilde{C}_{l_1}^{BB})],$$

$$C_l^{BB} = (1 - l^2 R) \tilde{C}_l^{BB} + \frac{1}{2} \int \frac{d^2 \mathbf{l}_1}{(2\pi)^2} [(\mathbf{1} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2 C_{|\mathbf{1}-\mathbf{l}_1|}^{\phi\phi} \\ \times [(\tilde{C}_{l_1}^{EE} + \tilde{C}_{l_1}^{BB}) - \cos(4\varphi_{l_1})(\tilde{C}_{l_1}^{EE} - \tilde{C}_{l_1}^{BB})],$$

$$C_l^{\Theta E} = (1 - l^2 R) \tilde{C}_l^{\Theta E} + \int \frac{d^2 \mathbf{l}_1}{(2\pi)^2} [(\mathbf{1} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2 C_{|\mathbf{1}-\mathbf{l}_1|}^{\phi\phi} \\ \times \tilde{C}_{l_1}^{\Theta E} \cos(2\varphi_{l_1}),$$

- Lensing generates B -modes out of the acoustic polarization E -modes contaminates gravitational wave signature if $E_i < 10^{16} \text{ GeV}$.

Reconstruction from the CMB

- Correlation between Fourier moments reflect lensing potential

$$\langle x(\mathbf{l})x'(\mathbf{l}') \rangle_{\text{CMB}} = f_\alpha(\mathbf{l}, \mathbf{l}')\phi(\mathbf{l} + \mathbf{l}'),$$

where $x \in$ temperature, polarization fields and f_α is a fixed weight that reflects geometry

- Each pair forms a **noisy estimate** of the potential or projected mass
- just like a pair of galaxy shears
- **Minimum variance weight** all pairs to form an estimator of the lensing mass

Scattering Secondaries

- Optical depth during reionization

$$\tau \approx 0.066 \left(\frac{\Omega_b h^2}{0.02} \right) \left(\frac{\Omega_m h^2}{0.15} \right)^{-1/2} \left(\frac{1+z}{10} \right)^{3/2}$$

- Anisotropy suppressed as $e^{-\tau}$. Integral solution

$$\frac{\Theta_\ell(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} S_0^{(0)} j_\ell(k(\eta_0 - \eta)) + \dots$$

- Isotropic (large scale) fluctuations not suppressed since suppression represents isotropization by scattering
- Quadrupole from the Sachs-Wolfe effect scatters into a large angle polarization bump

Doppler Effects

- Velocity fields of 10^{-3} and optical depths of 10^{-2} would imply large Doppler effect due to reionization
- Limber approximation says only fluctuations transverse to line of sight survive
- In linear theory, transverse fluctuations have no line of sight velocity and so Doppler effect is highly suppressed.
- Beyond linear theory: modulate the optical depth in the transverse direction using density fluctuations or ionization fraction fluctuations. Generate a modulated Doppler effect
- Linear fluctuations: Vishniac effect; Clusters: kinetic SZ effect; ionization patches: inhomogeneous reionization effect

Thermal SZ Effect

- Thermal velocities also lead to Doppler effect but first order contribution cancels because of random directions
- Residual effect is of order $v^2\tau \approx T_e/m_e\tau$ and can reach a sizeable level for clusters with $T_e \approx 10\text{keV}$.
- Raleigh-Jeans decrement and Wien enhancement described by second order collision term in Boltzmann equation: Kompaneets equation
- Clusters are rare objects so contribution to power spectrum suppressed, but may have been detected by CBI/BIMA: extremely sensitive to power spectrum normalization σ_8
- White noise on large-scales ($l < 2000$), turnover as cluster profile is resolved

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Data Pipeline

Gaussian Statistics

- Statistical isotropy says two-point correlation depends only on the power spectrum

$$\Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m}(\hat{\mathbf{n}})$$

$$\langle \Theta_{\ell m}^* \Theta_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_{\ell}^{\Theta\Theta}$$

- Reality of field says $\Theta_{\ell m} = (-1)^m \Theta_{\ell(-m)}$
- For a Gaussian random field, power spectrum defines all higher order statistics, e.g.

$$\langle \Theta_{\ell_1 m_1} \Theta_{\ell_2 m_2} \Theta_{\ell_3 m_3} \Theta_{\ell_4 m_4} \rangle$$

$$= (-1)^{m_1+m_2} \delta_{\ell_1 \ell_3} \delta_{m_1(-m_3)} \delta_{\ell_2 \ell_4} \delta_{m_2(-m_4)} C_{\ell_1}^{\Theta\Theta} C_{\ell_2}^{\Theta\Theta} + \text{all pairs}$$

Idealized Statistical Errors

- Take a noisy estimator of the multipoles in the map

$$\hat{\Theta}_{\ell m} = \Theta_{\ell m} + N_{\ell m}$$

and take the noise to be statistically isotropic

$$\langle N_{\ell m}^* N_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_{\ell}^{NN}$$

- Construct an unbiased estimator of the power spectrum

$$\langle \hat{C}_{\ell}^{\Theta\Theta} \rangle = C_{\ell}^{\Theta\Theta}$$

$$\hat{C}_{\ell}^{\Theta\Theta} = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} \hat{\Theta}_{\ell m}^* \hat{\Theta}_{\ell m} - C_{\ell}^{NN}$$

- Variance in estimator

$$\langle \hat{C}_{\ell}^{\Theta\Theta} \hat{C}_{\ell}^{\Theta\Theta} \rangle - \langle \hat{C}_{\ell}^{\Theta\Theta} \rangle^2 = \frac{2}{2\ell + 1} (C_{\ell}^{\Theta\Theta} + C_{\ell}^{NN})^2$$

Cosmic and Noise Variance

- RMS in estimator is simply the total power spectrum reduced by $\sqrt{2/N_{\text{modes}}}$ where N_{modes} is the number of m -mode measurements
- Even a perfect experiment where $C_\ell^{NN} = 0$ has statistical variance due to the Gaussian random realizations of the field. This cosmic variance is the result of having only one realization to measure.
- Noise variance is often approximated as white detector noise.
Removing the beam to place the measurement on the sky

$$N_\ell^{\Theta\Theta} = \left(\frac{T}{d_T}\right)^2 e^{\ell(\ell+1)\sigma^2} = \left(\frac{T}{d_T}\right)^2 e^{\ell(\ell+1)\text{FWHM}^2/8\ln 2}$$

where d_T can be thought of as a noise level per steradian of the temperature measurement, σ is the Gaussian beam width, FWHM is the full width at half maximum of the beam

Idealized Parameter Forecasts

- A crude propagation of errors is often useful for estimation purposes.
- Suppose $C_{\alpha\beta}$ describes the covariance matrix of the estimators for a given parameter set π_α .
- Define $\mathbf{F} = \mathbf{C}^{-1}$ [formalized as the Fisher matrix later]. Making an infinitesimal transformation to a new set of parameters p_μ

$$F_{\mu\nu} = \sum_{\alpha\beta} \frac{\partial\pi_\alpha}{\partial p_\mu} F_{\alpha\beta} \frac{\partial\pi_\beta}{\partial p_\nu}$$

- In our case π_α are the C_ℓ the covariance is diagonal and p_μ are cosmological parameters

$$F_{\mu\nu} = \sum_{\ell} \frac{2\ell + 1}{2(C_\ell^{\Theta\Theta} + C_\ell^{NN})^2} \frac{\partial C_\ell^{\Theta\Theta}}{\partial p_\mu} \frac{\partial C_\ell^{\Theta\Theta}}{\partial p_\nu}$$

Idealized Parameter Forecasts

- Polarization handled in same way (requires covariance)
- Fisher matrix represents a local approximation to the transformation of the covariance and hence is only accurate for well constrained directions in parameter space
- Derivatives evaluated by finite difference
- Fisher matrix identifies parameter degeneracies but only the local direction – i.e. all errors are ellipses not bananas

Beyond Idealizations: Time Ordered Data

- For the data analyst the starting point is a string of “time ordered” data coming out of the instrument (post removal of systematic errors!)
- Begin with a model of the time ordered data as

$$d_t = P_{ti}\Theta_i + n_t$$

where i denotes pixelized positions indexed by i , d_t is the data in a time ordered stream indexed by t . Number of time ordered data will be of the order 10^{10} for a satellite! number of pixels $10^6 - 10^7$.

- The noise n_t is drawn from a distribution with a known power spectrum

$$\langle n_t n_{t'} \rangle = C_{d,tt'}$$

Pointing Matrix

- The pointing matrix \mathbf{P} is the mapping between pixel space and the time ordered data
- Simplest incarnation: row with all zeros except one column which just says what point in the sky the telescope is pointing at that time

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 1 & \dots & 0 \end{pmatrix}$$

- More generally incorporates differencing, beam, rotation (for polarization)

Maximum Likelihood Mapmaking

- What is the best estimator of the underlying map Θ_i
- Likelihood function: the probability of getting the data given the theory $\mathcal{L} \equiv P[\text{data}|\text{theory}]$. In this case, the *theory* is the set of parameters Θ_i .

$$\mathcal{L}_{\Theta}(d_t) = \frac{1}{(2\pi)^{N_t/2} \sqrt{\det \mathbf{C}_d}} \exp \left[-\frac{1}{2} (d_t - P_{ti}\Theta_i) C_{d,tt'}^{-1} (d_{t'} - P_{t'j}\Theta_j) \right].$$

- Bayes theorem says that $P[\Theta_i|d_t]$, the probability that the temperatures are equal to Θ_i given the data, is proportional to the likelihood function times a *prior* $P(\Theta_i)$, taken to be uniform

$$P[\Theta_i|d_t] \propto P[d_t|\Theta_i] \equiv \mathcal{L}_{\Theta}(d_t)$$

Maximum Likelihood Mapping

- Maximizing the likelihood of Θ_i is simple since the log-likelihood is quadratic.
- Differentiating the argument of the exponential with respect to Θ_i and setting to zero leads immediately to the estimator

$$\hat{\Theta}_i = C_{N,ij} P_{jt} C_{d,tt'}^{-1} d_{t'} ,$$

where $C_N \equiv (\mathbf{P}^{\text{tr}} \mathbf{C}_d^{-1} \mathbf{P})^{-1}$ is the covariance of the estimator

- Given the large dimension of the time ordered data, direct matrix manipulation is unfeasible. A key simplifying assumption is the stationarity of the noise, that $C_{d,tt'}$ depends only on $t - t'$ (temporal statistical homogeneity)

Power Spectrum

- The next step in the chain of inference is the power spectrum extraction. Here the correlation between pixels is modelled through the power spectrum

$$C_{S,ij} \equiv \langle \Theta_i \Theta_j \rangle = \sum_{\ell} \Delta_{T,\ell}^2 W_{\ell,ij}$$

- Here W_{ℓ} , the window function, is derived by writing down the expansion of $\Theta(\hat{\mathbf{n}})$ in harmonic space, including smoothing by the beam and pixelization
- For example in the simple case of a gaussian beam of width σ it is proportional to the Legendre polynomial $P_{\ell}(\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j)$ for the pixel separation multiplied by $b_{\ell}^2 \propto e^{-\ell(\ell+1)\sigma^2}$

Band Powers

- In principle the underlying theory to extract from maximum likelihood is the power spectrum at every ℓ
- However with a finite patch of sky, it is not possible to extract multipoles separated by $\Delta\ell < 2\pi/L$ where L is the dimension of the survey
- So consider instead a theory parameterization of $\Delta_{T,\ell}^2$ constant in bands of $\Delta\ell$ chosen to match the survey forming a set of band powers B_a
- The likelihood of the bandpowers given the pixelized data is

$$\mathcal{L}_B(\Theta_i) = \frac{1}{(2\pi)^{N_p/2} \sqrt{\det \mathbf{C}_\Theta}} \exp \left(-\frac{1}{2} \Theta_i C_{\Theta,ij}^{-1} \Theta_j \right)$$

where $\mathbf{C}_\Theta = \mathbf{C}_S + \mathbf{C}_N$ and N_p is the number of pixels in the map.

Band Power Estimation

- As before, \mathcal{L}_B is Gaussian in the anisotropies Θ_i , but in this case Θ_i are *not* the parameters to be determined; the theoretical parameters are the B_a , upon which the covariance matrix depends.
- The likelihood function is not Gaussian in the parameters, and there is no simple, analytic way to find the maximum likelihood bandpowers
- Iterative approach to maximizing the likelihood: take a trial point $B_a^{(0)}$ and improve estimate based a Newton-Rhapson approach to finding zeros

$$\begin{aligned}\hat{B}_a &= \hat{B}_a^{(0)} + F_{B,ab} \frac{\partial \ln \mathcal{L}_B}{\partial B_b} \\ &= \hat{B}_a^{(0)} + \frac{1}{2} F_{B,ab}^{-1} \left(\Theta_i C_{\Theta,ij}^{-1} \frac{\partial C_{\Theta,jk}}{\partial B_b} C_{\Theta,kl}^{-1} \Theta_l - C_{\Theta,ij}^{-1} \frac{\partial C_{\Theta,ji}}{\partial B_b} \right),\end{aligned}$$

Fisher Matrix

- The expectation value of the local curvature is the Fisher matrix

$$\begin{aligned} F_{B,ab} &\equiv \left\langle -\frac{\partial^2 \ln \mathcal{L}_\Theta}{\partial B_a \partial B_b} \right\rangle \\ &= \frac{1}{2} C_{\Theta,ij}^{-1} \frac{\partial C_{\Theta,jk}}{\partial B_a} C_{\Theta,kl}^{-1} \frac{\partial C_{\Theta,li}}{\partial B_b}. \end{aligned}$$

- This is a general statement: for a gaussian distribution the Fisher matrix

$$F_{ab} = \frac{1}{2} \text{Tr}[\mathbf{C}^{-1} \mathbf{C}_{,a} \mathbf{C}^{-1} \mathbf{C}_{,b}]$$

- Kramer-Rao identity says that the best possible covariance matrix on a set of parameters is $\mathbf{C} = \mathbf{F}^{-1}$
- Thus, the iteration returns an estimate of the covariance matrix of the estimators \mathbf{C}_B

Cosmological Parameters

- The probability distribution of the bandpowers given the cosmological parameters c_i is not Gaussian but it is often an adequate approximation

$$\mathcal{L}_c(\hat{B}_a) \approx \frac{1}{(2\pi)^{N_c/2} \sqrt{\det \mathbf{C}_B}} \exp \left[-\frac{1}{2} (\hat{B}_a - B_a) C_{B,ab}^{-1} (\hat{B}_b - B_b) \right]$$

- Grid based approaches evaluate the likelihood in cosmological parameter space and maximize
- Faster approaches monte carlo the exploration of the likelihood space intelligently (“Monte Carlo Markov Chains”)
- Since the number of cosmological parameters in the working model is $N_c \sim 10$ this represents a final radical compression of information in the original timestream which recall has up to $N_t \sim 10^{10}$ data points.

Parameter Forecasts

- The Fisher matrix of the cosmological parameters becomes

$$F_{c,ij} = \frac{\partial B_a}{\partial c_i} C_{B,ab}^{-1} \frac{\partial B_b}{\partial c_j} .$$

which is the error propagation formula discussed above

- The Fisher matrix can be more accurately defined for an experiment by taking the pixel covariance and using the general formula for the Fisher matrix of gaussian data
- Corrects for edge effects with the approximate effect of

$$F_{\mu\nu} = \sum_{\ell} \frac{(2\ell + 1) f_{\text{sky}}}{2(C_{\ell}^{\Theta\Theta} + C_{\ell}^{NN})^2} \frac{\partial C_{\ell}^{\Theta\Theta}}{\partial p_{\mu}} \frac{\partial C_{\ell}^{\Theta\Theta}}{\partial p_{\nu}}$$

where the sky fraction f_{sky} quantifies the loss of independent modes due to the sky cut