Astro 448

Lecture Notes Set 3

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Covariant Perturbation Theory

- Covariant = takes same form in all coordinate systems
- Invariant = takes the same value in all coordinate systems
- Fundamental equations: Einstein equations, covariant conservation of stress-energy tensor:

$$\begin{aligned} G_{\mu\nu} &= 8\pi G T_{\mu\nu} \\ \nabla_\mu T^{\mu\nu} &= 0 \end{aligned}$$

- Preserve general covariance by keeping all degrees of freedom: 10 for each symmetric 4×4 tensor

1	2	3	4
	5	6	7
		8	9
			10

Metric Tensor

- Expand the metric tensor around the general FRW metric

$$g_{00} = -a^2, \qquad g_{ij} = a^2\gamma_{ij} \, .$$

  where the "0" component is conformal time $\eta = dt/a$ and γ_{ij} is a spatial metric of constant curvature $K = H_0^2(\Omega_{\rm tot} - 1)$.

- Add in a general perturbation (Bardeen 1980)

$$\begin{aligned} g^{00} &= -a^{-2}(1 - 2A) \, , \\ g^{0i} &= -a^{-2}B^i \, , \\ g^{ij} &= a^{-2}(\gamma^{ij} - 2H_L\gamma^{ij} - 2H_T^{ij}) \, . \end{aligned}$$

- (1) $A \equiv$ a scalar potential; (3) B^i a vector shift, (1) H_L a perturbation to the spatial curvature; (5) H_T^{ij} a trace-free distortion to spatial metric = (10)

Matter Tensor

- Likewise expand the matter stress energy tensor around a homogeneous density ρ and pressure p:

$$
\begin{aligned}
T^0{}_0 &= -\rho - \delta\rho\,, \\
T^0{}_i &= (\rho + p)(v_i - B_i)\,, \\
T_0{}^i &= -(\rho + p)v^i\,, \\
T^i{}_j &= (p + \delta p)\delta^i{}_j + p\Pi^i{}_j\,,
\end{aligned}
$$

- (1) $\delta\rho$ a density perturbation; (3) v_i a vector velocity, (1) δp a pressure perturbation; (5) Π_{ij} an anisotropic stress perturbation
- So far this is fully general and applies to any type of matter or coordinate choice including non-linearities in the matter, e.g. cosmological defects.

Counting DOF's

20	Variables (10 metric; 10 matter)
-10	Einstein equations
-4	Conservation equations
$+4$	Bianchi identities
-4	Gauge (coordinate choice 1 time, 3 space)
6	Degrees of freedom

- Without loss of generality these can be taken to be the 6 components of the matter stress tensor
- For the background, specify $p(a)$ or equivalently $w(a) \equiv p(a)/\rho(a)$ the equation of state parameter.

Scalar, Vector, Tensor

- In linear perturbation theory, perturbations may be separated by their transformation properties under rotation and translation.
- The eigenfunctions of the Laplacian operator form a complete set

$$
\begin{aligned}
\nabla^2 Q^{(0)} &= -k^2 Q^{(0)} & \mathrm{S}\,, \\
\nabla^2 Q_i^{(\pm 1)} &= -k^2 Q_i^{(\pm 1)} & \mathrm{V}\,, \\
\nabla^2 Q_{ij}^{(\pm 2)} &= -k^2 Q_{ij}^{(\pm 2)} & \mathrm{T}\,,
\end{aligned}
$$

- Vector and tensor modes satisfy divergence-free and transverse-traceless conditions

$$
\begin{aligned}
\nabla^i Q_i^{(\pm 1)} &= 0 \\
\nabla^i Q_{ij}^{(\pm 2)} &= 0 \\
\gamma^{ij} Q_{ij}^{(\pm 2)} &= 0
\end{aligned}
$$

Vector and Tensor Modes vs. Vector and Tensor Quantities

- A scalar mode carries with it associated vector (curl-free) and tensor (longitudinal) quantities
- A vector mode carries and associated tensor (neither longitudinal or transverse) quantities
- These are built from the mode basis out of covariant derivatives and the metric

$$
\begin{aligned}
Q_i^{(0)} &= -k^{-1}\nabla_i Q^{(0)} \,, \\
Q_{ij}^{(0)} &= (k^{-2}\nabla_i\nabla_j + \frac{1}{3}\gamma_{ij})Q^{(0)} \,, \\
Q_{ij}^{(\pm 1)} &= -\frac{1}{2k}[\nabla_i Q_j^{(\pm 1)} + \nabla_j Q_i^{(\pm 1)}] \,,
\end{aligned}
$$

Spatially Flat Case

- For a spatially flat background metric, harmonics are related to plane waves:

$$
\begin{aligned}
Q^{(0)} &= \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k}\cdot\mathbf{x})
\end{aligned}
$$

  where $\hat{\mathbf{e}}_3 \parallel \mathbf{k}$. Chosen as spin states, c.f. polarization.

- For vectors, the harmonic points in a direction orthogonal to $\mathbf{k}$ suitable for the vortical component of a vector

- For tensors, the harmonic is transverse and traceless as appropriate for the decompositon of gravitational waves

Perturbation k-Modes

- For the kth eigenmode, the scalar components become

$$
\begin{aligned}
A(\mathbf{x}) &= A(k)\, Q^{(0)}\,, & H_L(\mathbf{x}) &= H_L(k)\, Q^{(0)}\,, \\
\delta\rho(\mathbf{x}) &= \delta\rho(k)\, Q^{(0)}\,, & \delta p(\mathbf{x}) &= \delta p(k)\, Q^{(0)}\,,
\end{aligned}
$$

the vectors components become

$$
B_i(\mathbf{x}) = \sum_{m=-1}^{1} B^{(m)}(k)\, Q_i^{(m)}\,, \qquad v_i(\mathbf{x}) = \sum_{m=-1}^{1} v^{(m)}(k)\, Q_i^{(m)}\,,
$$

and the tensors components

$$
H_{Tij}(\mathbf{x}) = \sum_{m=-2}^{2} H_T^{(m)}(k)\, Q_{ij}^{(m)}\,, \quad \Pi_{ij}(\mathbf{x}) = \sum_{m=-2}^{2} \Pi^{(m)}(k)\, Q_{ij}^{(m)}\,,
$$

Homogeneous Einstein Equations

- Einstein (Friedmann) equations:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho$$

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p)$$

so that $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

Homogeneous Einstein Equations

- Counting exercise:

20	Variables (10 metric; 10 matter)
-17	Homogeneity and Isotropy
-2	Einstein equations
-1	Conservation equations
$+1$	Bianchi identities
1	Degree of freedom

- without loss of generality choose ratio of homogeneous & isotropic component of the stress tensor to the density $w(a) = p(a)/\rho(a)$.

Acceleration Implies Negative Pressure

- Role of stresses in the background cosmology
- Homogeneous Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ imply the two Friedman equations (flat universe, or associating curvature $\rho_K = -3K/8\pi G a^2$)

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho$$
$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p)$$

so that the total equation of state $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}$$

- so that ρ must scale more slowly than a^{-2}

Questions regarding Dark Energy

- Coincidence: given the very different scalings of matter and dark energy with a, why are they comparable now?
- Stability: why doesn't negative pressure imply accelerated collapse? or why doesn't the vacuum suck?
- Answer: stability is associated with stress (pressure) gradients not stress (pressure) itself.
- Example: the cosmological constant $w_\Lambda = -1$, a constant in time and space – no gradients.
- Example: a scalar field where $w = p/\rho \neq \delta p/\delta\rho$ = sound speed.

Covariant Scalar Equations

- Einstein equations (suppressing $0)$ superscripts (Hu & Eisenstein 1999):

$$(k^2 - 3K)[H_L + \frac{1}{3}H_T + \frac{\dot{a}}{a}\frac{1}{k^2}(kB - \dot{H}_T)]$$
$$= 4\pi G a^2 \left[\delta\rho + 3\frac{\dot{a}}{a}(\rho + p)(v - B)/k\right], \quad \text{Poisson Equation}$$

$$k^2(A + H_L + \frac{1}{3}H_T) + \left(\frac{d}{d\eta} + 2\frac{\dot{a}}{a}\right)(kB - \dot{H}_T)$$
$$= 8\pi G a^2 p\Pi\,,$$

$$\frac{\dot{a}}{a}A - \dot{H}_L - \frac{1}{3}\dot{H}_T - \frac{K}{k^2}(kB - \dot{H}_T)$$
$$= 4\pi G a^2(\rho + p)(v - B)/k\,,$$

$$\left[2\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\frac{d}{d\eta} - \frac{k^2}{3}\right]A - \left[\frac{d}{d\eta} + \frac{\dot{a}}{a}\right](\dot{H}_L + \frac{1}{3}kB)$$
$$= 4\pi G a^2(\delta p + \frac{1}{3}\delta\rho)\,.$$

Covariant Scalar Equations

- Conservation equations: continuity and Navier Stokes

$$\begin{aligned}
\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho + 3\frac{\dot a}{a}\delta p &= -(\rho + p)(kv + 3\dot H_L)\,, \\
\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right]\left[(\rho + p)\frac{(v - B)}{k}\right] &= \delta p - \frac{2}{3}(1 - 3\frac{K}{k^2})p\Pi + (\rho + p)A\,,
\end{aligned}$$

- Equations are not independent since $\nabla_\mu G^{\mu\nu} = 0$ via the Bianchi identities.

- Related to the ability to choose a coordinate system or "gauge" to represent the perturbations.

Covariant Scalar Equations

- DOF counting exercise

8	Variables (4 metric; 4 matter)
-4	Einstein equations
-2	Conservation equations
$+2$	Bianchi identities
-2	Gauge (coordinate choice 1 time, 1 space)
2	Degrees of freedom

- without loss of generality choose scalar components of the stress tensor δp, Π .

Covariant Vector Equations

- Einstein equations

$$(1 - 2K/k^2)(kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) = 16\pi G a^2(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k\,,$$

$$\left[\frac{d}{d\eta} + 2\frac{\dot{a}}{a}\right](kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) = -8\pi G a^2 p \Pi^{(\pm 1)}\,.$$

- Conservation Equations

$$\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a}\right][(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k] = -\frac{1}{2}(1 - 2K/k^2)p\Pi^{(\pm 1)}\,,$$

- Gravity provides no source to vorticity $\rightarrow$ decay

Covariant Vector Equations

- DOF counting exercise

8	Variables (4 metric; 4 matter)
-4	Einstein equations
-2	Conservation equations
$+2$	Bianchi identities
-2	Gauge (coordinate choice 1 time, 1 space)
2	Degrees of freedom

- without loss of generality choose vector components of the stress tensor $\Pi^{(\pm 1)}$.

Covariant Tensor Equation

- Einstein equation

$$\left[\frac{d^2}{d\eta^2} + 2\frac{\dot{a}}{a}\frac{d}{d\eta} + (k^2 + 2K)\right] H_T^{(\pm 2)} = 8\pi G a^2 p \Pi^{(\pm 2)} \, .$$

- DOF counting exercise

4	Variables (2 metric; 2 matter)
-2	Einstein equations
-0	Conservation equations
$+0$	Bianchi identities
-0	Gauge (coordinate choice 1 time, 1 space)
2	Degrees of freedom

- wlog choose tensor components of the stress tensor $\Pi^{(\pm 2)}$.

Arbitrary Dark Components

- Total stress energy tensor can be broken up into individual pieces
- Dark components interact only through gravity and so satisfy separate conservation equations
- Einstein equation source remains the sum of components.
- To specify an arbitrary dark component, give the behavior of the stress tensor: 6 components: δp, $\Pi^{(i)}$, where $i = -2, ..., 2$.
- Many types of dark components (dark matter, scalar fields, massive neutrinos,..) have simple forms for their stress tensor in terms of the energy density, i.e. described by equations of state.
- An equation of state for the background $w = p/\rho$ is *not* sufficient to determine the behavior of the perturbations.

Gauge

- Metric and matter fluctuations take on different values in different coordinate system
- No such thing as a "gauge invariant" density perturbation!
- General coordinate transformation:

$$
\begin{aligned}
\tilde{\eta} &= \eta + T \\
\tilde{x}^i &= x^i + L^i
\end{aligned}
$$

  free to choose (T, L^i) to simplify equations or physics. Decompose these into scalar and vector harmonics.
- $G_{\mu\nu}$ and $T_{\mu\nu}$ transform as tensors, so components in different frames can be related

Gauge Transformation

- Scalar Metric:

$$
\begin{aligned}
\tilde{A} &= A - \dot{T} - \frac{\dot{a}}{a} T\,, \\
\tilde{B} &= B + \dot{L} + kT\,, \\
\tilde{H}_L &= H_L - \frac{k}{3} L - \frac{\dot{a}}{a} T\,, \\
\tilde{H}_T &= H_T + kL\,,
\end{aligned}
$$

- Scalar Matter (Jth component):

$$
\begin{aligned}
\delta\tilde{\rho}_J &= \delta\rho_J - \dot{\rho}_J T, \\
\delta\tilde{p}_J &= \delta p_J - \dot{p}_J T, \\
\tilde{v}_J &= v_J + \dot{L},
\end{aligned}
$$

- Vector:

$$
\tilde{B}^{(\pm1)} = B^{(\pm1)} + \dot{L}^{(\pm1)},\ \tilde{H}_T^{(\pm1)} = H_T^{(\pm1)} + kL^{(\pm1)},\ \tilde{v}_J^{(\pm1)} = v_J^{(\pm1)} + \dot{L}^{(\pm1)},
$$

Common Scalar Gauge Choices

- A coordinate system is fully specified if there is an explicit prescription for (T, L^i) or for scalars (T, L)
- Newtonian:

$$\begin{aligned}
\tilde{B} &= \tilde{H}_T = 0 \\
\Psi &\equiv \tilde{A} \quad \text{(Newtonian potential)} \\
\Phi &\equiv \tilde{H}_L \quad \text{(Newtonian curvature)} \\
L &= -H_T/k \\
T &= -B/k + \dot{H}_T/k^2
\end{aligned}$$

Good: intuitive Newtonian like gravity; matter and metric algebraically related; commonly chosen for analytic CMB and lensing work

Bad: numerically unstable

Example: Newtonian Reduction

- In the general equations, set $B = H_T = 0$:

$$
\begin{aligned}
(k^2 - 3K)\Phi &= 4\pi G a^2 \left[\delta\rho + 3\frac{\dot{a}}{a}(\rho + p)v/k\right] \\
k^2(\Psi + \Phi) &= 8\pi G a^2 p\Pi
\end{aligned}
$$

so $\Psi = -\Phi$ if anisotropic stress $\Pi = 0$ and

$$
\begin{aligned}
\left[\frac{d}{d\eta} + 3\frac{\dot{a}}{a}\right]\delta\rho + 3\frac{\dot{a}}{a}\delta p &= -(\rho + p)(kv + 3\dot{\Phi})\,, \\
\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a}\right](\rho + p)v &= k\delta p - \frac{2}{3}(1 - 3\frac{K}{k^2})p\,k\Pi + (\rho + p)\,k\Psi\,,
\end{aligned}
$$

- Competition between stress (pressure and viscosity) and potential gradients

Common Scalar Gauge Choices

- Comoving:

$$
\begin{aligned}
\tilde{B} &= \tilde{v} \quad (T^0_i = 0) \\
H_T &= 0 \\
\xi &= \tilde{A} \\
\zeta &= \tilde{H}_L \quad \text{(Bardeen curvature)} \\
\Delta &= \tilde{\delta} \quad \text{(comoving density pert)} \\
T &= (v - B)/k \\
L &= -H_T/k
\end{aligned}
$$

Good: Algebraic relations between matter and metric

- Euler equation becomes an algebraic relation between stress and potential

$$(\rho + p)\xi = -\delta p + \frac{2}{3}\left(1 - \frac{3K}{k}\right) p\Pi$$

Common Scalar Gauge Choices

- Einstein equation lacks momentum density source

$$\frac{\dot{a}}{a}\xi - \dot{\zeta} - \frac{K}{k^2}kv = 0$$

- Combine: ζ is conserved if stress fluctuations negligible, e.g. above the horizon if $|K| \ll H^2$

$$\dot{\zeta} + Kv/k = \frac{\dot{a}}{a}\left[-\frac{\delta p}{\rho + p} + \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\frac{p}{\rho + p}\Pi\right] \to 0$$

Bad: explicitly relativistic choice

Common Scalar Gauge Choices

- Synchronous:

$$
\begin{aligned}
\tilde{A} &= \tilde{B} = 0 \\
\eta_L &\equiv -\tilde{H}_L - \frac{1}{3}\tilde{H}_T \\
h_T &= \tilde{H}_T \quad \text{or} \quad h = 6H_L \\
T &= a^{-1}\int d\eta a A + c_1 a^{-1} \\
L &= -\int d\eta (B + kT) + c_2
\end{aligned}
$$

Good: stable, the choice of numerical codes

Bad: residual gauge freedom in constants c_1, c_2 must be specified as an initial condition, intrinsically relativistic.

Common Scalar Gauge Choices

- Spatially Unperturbed:

$$\begin{aligned}
\tilde{H}_L &= \tilde{H}_T = 0 \\
L &= -H_T/k \\
\tilde{A}\,,\tilde{B} &= \quad \text{metric perturbations} \\
T &= \left(\frac{\dot{a}}{a}\right)^{-1}\left(H_L + \frac{1}{3}H_T\right)
\end{aligned}$$

Good: eliminates spatial metric in evolution equations; useful in inflationary calculations (Mukhanov et al)

Bad: intrinsically relativistic.

- Caution: perturbation evolution is governed by the behavior of stress fluctuations and an isotropic stress fluctuation δp is gauge dependent.

Hybrid "Gauge Invariant" Approach

- With the gauge transformation relations, express variables of one gauge in terms of those in another – allows a mixture in the equations of motion
- Example: Newtonian curvature and comoving density

$$(k^2 - 3K)\Phi = 4\pi G a^2 \rho \Delta$$

ordinary Poisson equation then implies Φ approximately constant if stresses negligible.

- Example: Exact Newtonian curvature above the horizon derived through Bardeen curvature conservation

Gauge transformation

$$\Phi = \zeta + \frac{\dot{a}}{a}\frac{v}{k}$$

Hybrid "Gauge Invariant" Approach

Einstein equation to eliminate velocity

$$\frac{\dot{a}}{a}\Psi - \dot{\Phi} = 4\pi G a^2(\rho + p)v/k$$

Friedman equation with no spatial curvature

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}a^2\rho$$

With $\dot{\Phi} = 0$ and $\Psi \approx -\Phi$

$$\frac{\dot{a}}{a}\frac{v}{k} = -\frac{2}{3(1+w)}\Phi$$

Hybrid "Gauge Invariant" Approach

Combining gauge transformation with velocity relation

$$\Phi = \frac{3 + 3w}{5 + 3w}\zeta$$

Usage: calculate ζ from inflation determines Φ for any choice of matter content or causal evolution.

- Example: Scalar field ("quintessence" dark energy) equations in comoving gauge imply a sound speed $\delta p / \delta \rho = 1$ independent of potential $V(\phi)$. Solve in synchronous gauge (Hu 1998).