Astro 448

Lecture Notes Set 4

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Scalar Fields

- Stress-energy tensor of a scalar field

$$T^{\mu}{}_{\nu} = \nabla^{\mu}\varphi\,\nabla_{\nu}\varphi - \frac{1}{2}(\nabla^{\alpha}\varphi\,\nabla_{\alpha}\varphi + 2V)\delta^{\mu}{}_{\nu}\,.$$

- For the background $\langle\phi\rangle \equiv \phi_0$

$$\rho_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + V \quad p_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + -V$$

- So for kinetic dominated $w_\phi = p_\phi/\rho_\phi \to 1$
- And potential dominated $w_\phi = p_\phi/\rho_\phi \to -1$
- A slowly rolling (potential dominated) scalar field can accelerate the expansion and so solve the horizon problem or act as a dark energy candidate

Equation of Motion

- Can use general equations of motion of dictated by stress energy conservation

$$\dot{\rho}_\phi = -3(\rho_\phi + p_\phi)\frac{\dot{a}}{a} \,,$$

to obtain the equation of motion of the background field ϕ

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2 V' = 0 \,,$$

- Likewise for the perturbations $\phi = \phi_0 + \phi_1$

$$\begin{aligned}
\delta\rho_\phi &= a^{-2}(\dot{\phi}_0\dot{\phi}_1 - \dot{\phi}_0^2 A) + V'\phi_1 \,, \\
\delta p_\phi &= a^{-2}(\dot{\phi}_0\dot{\phi}_1 - \dot{\phi}_0^2 A) - V'\phi_1 \,, \\
(\rho_\phi + p_\phi)(v_\phi - B) &= a^{-2}k\dot{\phi}_0\phi_1 \,, \\
p_\phi\pi_\phi &= 0 \,,
\end{aligned}$$

Equation of Motion

- The stress of the perturbations is defined through

$$\delta p_\phi = \delta\rho_\phi + 3(\rho_\phi + p_\phi)\frac{v_\phi - B}{k}\frac{\dot a}{a}(1 - c_\phi^2)$$

  where $c_\phi^2 \equiv \dot p_\phi/\dot\rho_\phi$ is the "adiabatic" sound speed

- So for the comoving gauge where $v_\phi = B$, $\delta p_\phi = \delta\rho_\phi$ so the sound speed relevant for stability is $\delta p_\phi/\delta\rho_\phi = 1$. Very useful for solving system since in this gauge everything is specified by $w(a)$
- Scalar field fluctuations are stable inside the horizon and are a good candidate for the smooth dark energy
- More generally, continuity and Euler equations imply

$$\ddot\phi_1 = -2\frac{\dot a}{a}\dot\phi_1 - (k^2 + a^2 V'')\phi_1 + (\dot A - 3\dot H_L - kB)\dot\phi_0 - 2Aa^2V' \,.$$

Inflationary Perturbations

- Classical equations of motion for scalar field inflaton determine the evolution of scalar field fluctuations generated by quantum fluctuations
- Since the Bardeen or comoving curvature ζ is conserved in the absence of stress fluctuations (i.e. outside the apparent horizon, calculate this and we're done no matter what happens in between inflation and the late universe (reheating etc.)
- But in the comoving gauge $\phi_1 = 0$! no scalar-field perturbation
- Solution: solve the scalar field equation in the dual gauge where the curvature $H_L = 0$ (and $H_T = 0$ to fix the gauge completely, as the "spatially unperturbed" or "spatially flat" gauge) and transform the result to the comoving gauge

Transformation to Comoving Gauge

- Scalar field transforms as scalar field

$$\tilde{\phi}_1 = \phi_1 - \dot{\phi}_0 T$$

- To get to comoving frame $\tilde{\phi}_1 = 0$, $T = \phi_1/\dot{\phi}_0$, and $\tilde{H}_T = H_T + kL$ so

$$\begin{aligned} \zeta &= H_L - \frac{k}{3}L - \frac{\dot{a}}{a}T \,, \\ &= H_L + \frac{H_T}{3} - \frac{\dot{a}}{a}\frac{\phi_1}{\dot{\phi}_0} \end{aligned}$$

- Transformation particularly simple from a gauge with $H_T = H_L = 0$, i.e. spatially unperturbed metric

$$\zeta = -\frac{\dot{a}}{a}\frac{\phi_1}{\dot{\phi}_0}$$

Scalar Field Eqn of Motion

- Scalar field perturbation in spatially unperturbed gauge is simply proportional to resulting Bardeen curvature with the proportionality constant as the expansion rate over roll rate – enhanced
- Scalar field fluctuation satisfies classical equation of motion

$$\ddot{\phi}_1 = -2\frac{\dot{a}}{a}\dot{\phi}_1 - (k^2 + a^2 V'')\phi_1 + (\dot{A} - kB)\dot{\phi}_0 - 2Aa^2 V' \,.$$

- Metric terms may be eliminated through Einstein equations

$$\begin{aligned} A &= 4\pi G a^2 \left(\frac{\dot{a}}{a}\right)^{-1} (\rho_\phi + p_\phi)(v_\phi - B)/k \\ &= 4\pi G \left(\frac{\dot{a}}{a}\right)^{-1} \dot{\phi}_0 \phi_1 \end{aligned}$$

Scalar Field Eqn of Motion

- And

$$kB = 4\pi G a^2 [\left(\frac{\dot a}{a}\right)^{-1} \delta\rho_\phi + 3\frac{\dot a}{a}(\rho_\phi + p_\phi)(v_\phi - B)/k]$$

$$= 4\pi G [\left(\frac{\dot a}{a}\right)^{-1} (\dot\phi_0 \dot\phi_1 + a^2 V' \phi_1) - \left(\frac{\dot a}{a}\right)^{-2} (4\pi G \dot\phi_0)^2 \dot\phi_0 \phi_1 + 3\dot\phi_0 \phi_1]$$

- So $\dot A - kB \propto \phi_1$ with proportionality that depends only on the background evolution – Einstein & scalar field equations reduce to a single second order diff eq!
- Equation resembles a damped oscillator equation with a particular dispersion relation

$$\ddot\phi_1 + 2\frac{\dot a}{a}\dot\phi_1 + [k^2 + f(\eta)]\phi_1$$

Exact Equation

- Rewrite equations of motion in terms of slow roll parameters but do not require them to be small or constant.
- Deviation from de Sitter expansion

$$\epsilon \equiv \frac{3}{2}(1+w_\phi)$$
$$= \frac{\frac{3}{2}\dot{\phi}_0^2/a^2V}{1+\frac{1}{2}\dot{\phi}_0^2/a^2V}$$

- Deviation from overdamped limit of $d^2\phi_0/dt^2 = 0$

$$\delta \equiv \frac{\ddot{\phi}_0}{\dot{\phi}}\left(\frac{\dot{a}}{a}\right)^{-1} - 1$$

Exact Equation

- Friedman equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = 4\pi G \dot{\phi}_0^2 \epsilon^{-1}$$

$$\frac{d}{d\eta}\left(\frac{\dot{a}}{a}\right) = \left(\frac{\dot{a}}{a}\right)^2 (1-\epsilon)$$

- Homogenous scalar field equation

$$\dot{\phi}_0 \frac{\dot{a}}{a}(3+\delta) = -a^2 V'$$

- Combination

$$\dot{\epsilon} = 2\epsilon(\delta+\epsilon)\frac{\dot{a}}{a}$$

Exact equation

- Rewrite in $u \equiv a\phi$ to remove expansion damping

$$\ddot{u} + [k^2 + g(\eta)]u = 0$$

where Mukhanov

$$\begin{aligned} g(\eta) &\equiv f(\eta) + \epsilon - 2 \\ &= -\left(\frac{\dot{a}}{a}\right)^2 [2 + 3\delta + 2\epsilon + (\delta + \epsilon)(\delta + 2\epsilon)] - \frac{\dot{a}}{a}\dot{\delta} \\ &= -\frac{\ddot{z}}{z} \end{aligned}$$

and

$$z \equiv a \left(\frac{\dot{a}}{a}\right)^{-1} \dot{\phi}_0$$

Slow Roll Limit

- Slow roll $\epsilon \ll 1$, $\delta \ll 1$, $\dot{\delta} \ll \frac{\dot{a}}{a}$

$$\ddot{u} + [k^2 - 2\left(\frac{\dot{a}}{a}\right)^2]u = 0$$

- or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\rm end}$$

$$\tilde{\eta} = \int_{a_{\rm end}}^{a} \frac{da}{Ha^2} \approx -\frac{1}{aH}$$

- Compact, slow-roll equation:

$$\ddot{u} + [k^2 - \frac{2}{\tilde{\eta}^2}]u = 0$$

Slow Roll Limit

- Slow roll equation has the exact solution:

$$u = A(k \pm \frac{i}{\tilde{\eta}})e^{\mp ik\tilde{\eta}}$$

- For $|k\tilde{\eta}| \gg 1$ (early times, inside Hubble length) behaves as free oscillator

$$\lim_{|k\tilde{\eta}|\to\infty} u = Ake^{\mp ik\tilde{\eta}}$$

- Normalization A will be set by origin in quantum fluctuations of free field

Slow Roll Limit

- For $|k\tilde{\eta}| \ll 1$ (late times, $\gg$ Hubble length) fluctuation freezes in

$$\lim_{|k\tilde{\eta}|\to 0} u = \pm \frac{i}{\tilde{\eta}} A = \pm i H a A$$

$$\phi_1 = \pm i H A$$

$$\zeta = \mp i H A \left(\frac{\dot{a}}{a}\right) \frac{1}{\dot{\phi}_0}$$

- Slow roll replacement

$$\left(\frac{\dot{a}}{a}\right)^2 \frac{1}{\dot{\phi}_0^2} = \frac{8\pi G a^2 V}{3} \frac{3}{2a^2 V \epsilon} = 4\pi G = \frac{4\pi}{m_{\rm pl}^2}$$

- Bardeen curvature power spectrum

$$\Delta_\zeta^2 \equiv \frac{k^3 |\zeta|^2}{2\pi^2} = \frac{2k^3}{\pi} \frac{H^2}{\epsilon m_{\rm pl}^2} A^2$$

Quantum Fluctuations

- Simple harmonic oscillator $\ll$ Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k,\eta)\hat{a} + u^*(k,\eta)\hat{a}^\dagger$$

where $u(k,\eta)$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \qquad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\eta] = i$

$$u(k,\eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

Quantum Fluctuations

- Zero point fluctuations of ground state

$$\begin{aligned}\langle u^2\rangle &= \langle 0|u^\dagger u|0\rangle \\ &= \langle 0|(u^*\hat{a}^\dagger + u\hat{a})(u\hat{a} + u^*\hat{a}^\dagger)|0\rangle \\ &= \langle 0|\hat{a}\hat{a}^\dagger|0\rangle|u(k,\tilde{\eta})|^2 \\ &= \langle 0|[\hat{a},\hat{a}^\dagger] + \hat{a}^\dagger\hat{a}|0\rangle|u(k,\tilde{\eta})|^2 \\ &= |u(k,\tilde{\eta})|^2 = \frac{1}{2k}\end{aligned}$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in. Slow roll equation
- So $A = (2k^3)^{1/2}$ and curvature power spectrum

$$\Delta_\zeta^2 \equiv \frac{1}{\pi}\frac{H^2}{\epsilon m_{\rm pl}^2}$$

Tilt

- Curvature power spectrum is scale invariant to the extent that H is constant
- Scalar spectral index

$$
\begin{aligned}
\frac{d\ln \Delta_\zeta^2}{d\ln k} &\equiv n_S - 1 \\
&= 2\frac{d\ln H}{d\ln k} - \frac{d\ln \epsilon}{d\ln k}
\end{aligned}
$$

- Evaluate at horizon crossing where fluctuation freezes

$$
\begin{aligned}
\frac{d\ln H}{d\ln k}\Big|_{-k\tilde\eta=1} &= \frac{k}{H}\frac{dH}{d\tilde\eta}\Big|_{-k\tilde\eta=1}\frac{d\tilde\eta}{dk}\Big|_{-k\tilde\eta=1} \\
&= \frac{k}{H}(-aH^2\epsilon)\Big|_{-k\tilde\eta=1}\frac{1}{k^2} = -\epsilon
\end{aligned}
$$

where $aH = -1/\tilde\eta = k$

Tilt

- Evolution of ϵ

$$\frac{d\ln\epsilon}{d\ln k} = -\frac{d\ln\epsilon}{d\ln\tilde{\eta}} = -2(\delta+\epsilon)\frac{\dot{a}}{a}\tilde{\eta} = 2(\delta+\epsilon)$$

- Tilt in the slow-roll approximation

$$n_S = 1 - 4\epsilon - 2\delta$$

Relationship to Potential

- To leading order in slow roll parameters

$$
\begin{aligned}
\epsilon &= \frac{\frac{3}{2}\dot{\phi}_0^2/a^2V}{1+\frac{1}{2}\dot{\phi}_0^2/a^2V} \\
&\approx \frac{3}{2}\dot{\phi}_0^2/a^2V \\
&\approx \frac{3}{a^2V}\frac{a^4V'^2}{9(\dot{a}/a)^2}, \qquad (3\dot{\phi}_0\frac{\dot{a}}{a} = -a^2V') \\
&\approx \frac{1}{6}\frac{3}{8\pi G}\left(\frac{V'}{V}\right)^2, \qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}a^2V \\
&\approx \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2
\end{aligned}
$$

so $\epsilon \ll 1$ is related to the first derivative of potential being small

Relationship to Potential

- And

$$\delta = \frac{\ddot{\phi}_0}{\dot{\phi}_0}\left(\frac{\dot{a}}{a}\right)^{-1} - 1$$

$$(\dot{\phi}_0 \approx -a^2\left(\frac{\dot{a}}{a}\right)^{-1}\frac{V'}{3})$$

$$(\ddot{\phi}_0 \approx -\frac{a^2V'}{3}(1+\epsilon) + a^4\left(\frac{\dot{a}}{a}\right)^{-2}\frac{V'V''}{9})$$

$$\approx -\frac{1}{a^2V'/3}\left(-\frac{a^2V'}{3}(1+\epsilon) + \frac{a^2}{9}\frac{3}{8\pi G}\frac{V'V''}{V}\right) - 1 \quad \approx \epsilon - \frac{1}{8\pi G}\frac{V''}{V}$$

so δ is related to second derivative of potential being small. Very flat potential.

Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation $(K = 0)$, same as scalar field

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0\,.$$

- Acquires quantum fluctuations in same manner as ϕ. Lagrangian sets the normalization

$$\phi_1 \to H_T^{(\pm 2)}\sqrt{\frac{3}{16\pi G}}$$

- Scale-invariant gravitational wave amplitude (each component: NB more traditional notation $H_T^{(\pm 2)} = (h_+ \pm i h_\times)/\sqrt{6}$)

$$\Delta_H^2 = \frac{16\pi G}{3 \cdot 2\pi^2}\frac{H^2}{2} = \frac{4}{3\pi}\frac{H^2}{m_{\rm pl}^2}$$

Gravitational Waves

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation
- Tensor tilt:

$$\frac{d\ln\Delta_H^2}{d\ln k} \equiv n_T = 2\frac{d\ln H}{d\ln k} = -2\epsilon$$

- Consistency relation between tensor-scalar ratio and tensor tilt

$$\frac{\Delta_H^2}{\Delta_\zeta^2} = \frac{4}{3}\epsilon = -\frac{2}{3}\epsilon$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparision of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself

Gravitational Wave Phenomenology

- Equation of motion

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0\,.$$

- has solutions

$$H_T^{(\pm 2)} = C_1 H_1(k\eta) + C_2 H_2(k\eta)$$
$$H_1 \propto x^{-m} j_m(x)$$
$$H_2 \propto x^{-m} n_m(x)$$

where $m = (1-3w)/(1+3w)$

- If $w > -1/3$ then gravity wave is constant above horizon $x \ll 1$ and then oscillates and damps
- If $w < -1/3$ then gravity wave oscillates and freezes into some value, just like scalar field

Gravitational Wave Phenomenology

- A gravitational wave makes a quadrupolar (transverse-traceless) distortion to metric
- Just like the scale factor or spatial curvature, a temporal variation in its amplitude leaves a residual temperature variation in CMB photons – here anisotropic
- Before recombination, anisotropic variation is eliminated by scattering
- Gravitational wave temperature effect drops sharply at the horizon scale at recombination
- Source to polarization goes as $\dot{\tau}/\dot{H}_T$ and peaks at the horizon not damping scale
- B modes formed as photons propagate – the spatial variation in the plane waves modulate the signal: described by Boltzmann eqn.