

Astro 448

Lecture Notes *Set 4*

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Scalar Fields

- Stress-energy tensor of a scalar field

$$T^{\mu}_{\nu} = \nabla^{\mu}\varphi \nabla_{\nu}\varphi - \frac{1}{2}(\nabla^{\alpha}\varphi \nabla_{\alpha}\varphi + 2V)\delta^{\mu}_{\nu}.$$

- For the background $\langle\phi\rangle \equiv \phi_0$

$$\rho_{\phi} = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + V \quad p_{\phi} = \frac{1}{2}a^{-2}\dot{\phi}_0^2 - V$$

- So for kinetic dominated $w_{\phi} = p_{\phi}/\rho_{\phi} \rightarrow 1$
- And potential dominated $w_{\phi} = p_{\phi}/\rho_{\phi} \rightarrow -1$
- A slowly rolling (potential dominated) scalar field can accelerate the expansion and so solve the horizon problem or act as a dark energy candidate

Equation of Motion

- Can use general equations of motion dictated by stress energy conservation

$$\dot{\rho}_\phi = -3(\rho_\phi + p_\phi)\frac{\dot{a}}{a},$$

to obtain the equation of motion of the background field ϕ

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2V' = 0,$$

- Likewise for the perturbations $\phi = \phi_0 + \phi_1$

$$\delta\rho_\phi = a^{-2}(\dot{\phi}_0\dot{\phi}_1 - \dot{\phi}_0^2A) + V'\phi_1,$$

$$\delta p_\phi = a^{-2}(\dot{\phi}_0\dot{\phi}_1 - \dot{\phi}_0^2A) - V'\phi_1,$$

$$(\rho_\phi + p_\phi)(v_\phi - B) = a^{-2}k\dot{\phi}_0\phi_1,$$

$$p_\phi\pi_\phi = 0,$$

Equation of Motion

- The stress of the perturbations is defined through

$$\delta p_\phi = \delta \rho_\phi + 3(\rho_\phi + p_\phi) \frac{v_\phi - B \dot{a}}{k a} (1 - c_\phi^2)$$

where $c_\phi^2 \equiv \dot{p}_\phi / \dot{\rho}_\phi$ is the “adiabatic” sound speed

- So for the comoving gauge where $v_\phi = B$, $\delta p_\phi = \delta \rho_\phi$ so the sound speed relevant for stability is $\delta p_\phi / \delta \rho_\phi = 1$. Very useful for solving system since in this gauge everything is specified by $w(a)$
- Scalar field fluctuations are stable inside the horizon and are a good candidate for the smooth dark energy
- More generally, continuity and Euler equations imply

$$\ddot{\phi}_1 = -2 \frac{\dot{a}}{a} \dot{\phi}_1 - (k^2 + a^2 V'') \phi_1 + (\dot{A} - 3\dot{H}_L - kB) \dot{\phi}_0 - 2Aa^2 V'.$$

Inflationary Perturbations

- Classical equations of motion for scalar field inflaton determine the evolution of scalar field fluctuations generated by quantum fluctuations
- Since the Bardeen or comoving curvature ζ is conserved in the absence of stress fluctuations (i.e. outside the apparent horizon, calculate this and we're done no matter what happens in between inflation and the late universe (reheating etc.)
- But in the comoving gauge $\phi_1 = 0!$ no scalar-field perturbation
- Solution: solve the scalar field equation in the dual gauge where the curvature $H_L = 0$ (and $H_T = 0$ to fix the gauge completely, as the “spatially unperturbed” or “spatially flat” gauge) and transform the result to the comoving gauge

Transformation to Comoving Gauge

- Scalar field transforms as scalar field

$$\tilde{\phi}_1 = \phi_1 - \dot{\phi}_0 T$$

- To get to comoving frame $\tilde{\phi}_1 = 0$, $T = \phi_1 / \dot{\phi}_0$, and $\tilde{H}_T = H_T + kL$ so

$$\begin{aligned}\zeta &= H_L - \frac{k}{3}L - \frac{\dot{a}}{a}T, \\ &= H_L + \frac{H_T}{3} - \frac{\dot{a}}{a} \frac{\phi_1}{\dot{\phi}_0}\end{aligned}$$

- Transformation particularly simple from a gauge with $H_T = H_L = 0$, i.e. spatially unperturbed metric

$$\zeta = -\frac{\dot{a}}{a} \frac{\phi_1}{\dot{\phi}_0}$$

Scalar Field Eqn of Motion

- Scalar field perturbation in spatially unperturbed gauge is simply proportional to resulting Bardeen curvature with the proportionality constant as the expansion rate over roll rate – enhanced
- Scalar field fluctuation satisfies classical equation of motion

$$\ddot{\phi}_1 = -2\frac{\dot{a}}{a}\dot{\phi}_1 - (k^2 + a^2V'')\phi_1 + (\dot{A} - kB)\dot{\phi}_0 - 2Aa^2V'.$$

- Metric terms may be eliminated through Einstein equations

$$\begin{aligned} A &= 4\pi G a^2 \left(\frac{\dot{a}}{a}\right)^{-1} (\rho_\phi + p_\phi)(v_\phi - B)/k \\ &= 4\pi G \left(\frac{\dot{a}}{a}\right)^{-1} \dot{\phi}_0 \phi_1 \end{aligned}$$

Scalar Field Eqn of Motion

- And

$$\begin{aligned} kB &= 4\pi G a^2 \left[\left(\frac{\dot{a}}{a} \right)^{-1} \delta\rho_\phi + 3 \frac{\dot{a}}{a} (\rho_\phi + p_\phi) (v_\phi - B) / k \right] \\ &= 4\pi G \left[\left(\frac{\dot{a}}{a} \right)^{-1} (\dot{\phi}_0 \dot{\phi}_1 + a^2 V' \phi_1) - \left(\frac{\dot{a}}{a} \right)^{-2} (4\pi G \dot{\phi}_0)^2 \dot{\phi}_0 \phi_1 + 3 \dot{\phi}_0 \phi_1 \right] \end{aligned}$$

- So $\dot{A} - kB \propto \phi_1$ with proportionality that depends only on the background evolution – Einstein & scalar field equations reduce to a single second order diff eq!
- Equation resembles a damped oscillator equation with a particular dispersion relation

$$\ddot{\phi}_1 + 2 \frac{\dot{a}}{a} \dot{\phi}_1 + [k^2 + f(\eta)] \phi_1$$

Exact Equation

- Rewrite equations of motion in terms of slow roll parameters but do not require them to be small or constant.
- Deviation from de Sitter expansion

$$\begin{aligned}\epsilon &\equiv \frac{3}{2}(1 + w_\phi) \\ &= \frac{\frac{3}{2}\dot{\phi}_0^2/a^2V}{1 + \frac{1}{2}\dot{\phi}_0^2/a^2V}\end{aligned}$$

- Deviation from overdamped limit of $d^2\phi_0/dt^2 = 0$

$$\delta \equiv \frac{\ddot{\phi}_0}{\dot{\phi}_0} \left(\frac{\dot{a}}{a} \right)^{-1} - 1$$

Exact Equation

- Friedman equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = 4\pi G\dot{\phi}_0^2\epsilon^{-1}$$

$$\frac{d}{d\eta}\left(\frac{\dot{a}}{a}\right) = \left(\frac{\dot{a}}{a}\right)^2(1 - \epsilon)$$

- Homogenous scalar field equation

$$\dot{\phi}_0\frac{\dot{a}}{a}(3 + \delta) = -a^2V'$$

- Combination

$$\dot{\epsilon} = 2\epsilon(\delta + \epsilon)\frac{\dot{a}}{a}$$

Exact equation

- Rewrite in $u \equiv a\phi$ to remove expansion damping

$$\ddot{u} + [k^2 + g(\eta)]u = 0$$

where Mukhanov

$$\begin{aligned} g(\eta) &\equiv f(\eta) + \epsilon - 2 \\ &= - \left(\frac{\dot{a}}{a} \right)^2 [2 + 3\delta + 2\epsilon + (\delta + \epsilon)(\delta + 2\epsilon)] - \frac{\dot{a}}{a} \dot{\delta} \\ &= - \frac{\ddot{z}}{z} \end{aligned}$$

and

$$z \equiv a \left(\frac{\dot{a}}{a} \right)^{-1} \dot{\phi}_0$$

Slow Roll Limit

- Slow roll $\epsilon \ll 1$, $\delta \ll 1$, $\dot{\delta} \ll \frac{\dot{a}}{a}$

$$\ddot{u} + \left[k^2 - 2 \left(\frac{\dot{a}}{a} \right)^2 \right] u = 0$$

- or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\text{end}}$$

$$\tilde{\eta} = \int_{a_{\text{end}}}^a \frac{da}{H a^2} \approx -\frac{1}{aH}$$

- Compact, slow-roll equation:

$$\ddot{u} + \left[k^2 - \frac{2}{\tilde{\eta}^2} \right] u = 0$$

Slow Roll Limit

- Slow roll equation has the exact solution:

$$u = A\left(k \pm \frac{i}{\tilde{\eta}}\right)e^{\mp ik\tilde{\eta}}$$

- For $|k\tilde{\eta}| \gg 1$ (early times, inside Hubble length) behaves as free oscillator

$$\lim_{|k\tilde{\eta}| \rightarrow \infty} u = Ake^{\mp ik\tilde{\eta}}$$

- Normalization A will be set by origin in quantum fluctuations of free field

Slow Roll Limit

- For $|k\tilde{\eta}| \ll 1$ (late times, \gg Hubble length) fluctuation freezes in

$$\lim_{|k\tilde{\eta}| \rightarrow 0} u = \pm \frac{i}{\tilde{\eta}} A = \pm i H a A$$

$$\phi_1 = \pm i H A$$

$$\zeta = \mp i H A \left(\frac{\dot{a}}{a} \right) \frac{1}{\dot{\phi}_0}$$

- Slow roll replacement

$$\left(\frac{\dot{a}}{a} \right)^2 \frac{1}{\dot{\phi}_0^2} = \frac{8\pi G a^2 V}{3} \frac{3}{2a^2 V \epsilon} = 4\pi G = \frac{4\pi}{m_{\text{pl}}^2}$$

- Bardeen curvature power spectrum

$$\Delta_{\zeta}^2 \equiv \frac{k^3 |\zeta|^2}{2\pi^2} = \frac{2k^3}{\pi} \frac{H^2}{\epsilon m_{\text{pl}}^2} A^2$$

Quantum Fluctuations

- Simple harmonic oscillator \ll Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k, \eta) \hat{a} + u^*(k, \eta) \hat{a}^\dagger$$

where $u(k, \eta)$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \quad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\eta] = i$

$$u(k, \eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

Quantum Fluctuations

- Zero point fluctuations of ground state

$$\begin{aligned}\langle u^2 \rangle &= \langle 0 | u^\dagger u | 0 \rangle \\ &= \langle 0 | (u^* \hat{a}^\dagger + u \hat{a})(u \hat{a} + u^* \hat{a}^\dagger) | 0 \rangle \\ &= \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle |u(k, \tilde{\eta})|^2 \\ &= \langle 0 | [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} | 0 \rangle |u(k, \tilde{\eta})|^2 \\ &= |u(k, \tilde{\eta})|^2 = \frac{1}{2k}\end{aligned}$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in. Slow roll equation
- So $A = (2k^3)^{1/2}$ and curvature power spectrum

$$\Delta_\zeta^2 \equiv \frac{1}{\pi} \frac{H^2}{\epsilon m_{\text{pl}}^2}$$

Tilt

- Curvature power spectrum is scale invariant to the extent that H is constant
- Scalar spectral index

$$\begin{aligned}\frac{d \ln \Delta_{\zeta}^2}{d \ln k} &\equiv n_S - 1 \\ &= 2 \frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon}{d \ln k}\end{aligned}$$

- Evaluate at horizon crossing where fluctuation freezes

$$\begin{aligned}\frac{d \ln H}{d \ln k} \Big|_{-k\tilde{\eta}=1} &= \frac{k}{H} \frac{dH}{d\tilde{\eta}} \Big|_{-k\tilde{\eta}=1} \frac{d\tilde{\eta}}{dk} \Big|_{-k\tilde{\eta}=1} \\ &= \frac{k}{H} (-aH^2\epsilon) \Big|_{-k\tilde{\eta}=1} \frac{1}{k^2} = -\epsilon\end{aligned}$$

where $aH = -1/\tilde{\eta} = k$

Tilt

- Evolution of ϵ

$$\frac{d \ln \epsilon}{d \ln k} = -\frac{d \ln \epsilon}{d \ln \tilde{\eta}} = -2(\delta + \epsilon) \frac{\dot{a}}{a} \tilde{\eta} = 2(\delta + \epsilon)$$

- Tilt in the slow-roll approximation

$$n_S = 1 - 4\epsilon - 2\delta$$

Relationship to Potential

- To leading order in slow roll parameters

$$\begin{aligned}\epsilon &= \frac{\frac{3}{2}\dot{\phi}_0^2/a^2V}{1 + \frac{1}{2}\dot{\phi}_0^2/a^2V} \\ &\approx \frac{3}{2}\dot{\phi}_0^2/a^2V \\ &\approx \frac{3}{a^2V} \frac{a^4V'^2}{9(\dot{a}/a)^2}, \quad \left(3\dot{\phi}_0 \frac{\dot{a}}{a} = -a^2V'\right) \\ &\approx \frac{1}{6} \frac{3}{8\pi G} \left(\frac{V'}{V}\right)^2, \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}a^2V \\ &\approx \frac{1}{16\pi G} \left(\frac{V'}{V}\right)^2\end{aligned}$$

so $\epsilon \ll 1$ is related to the first derivative of potential being small

Relationship to Potential

- And

$$\delta = \frac{\ddot{\phi}_0}{\dot{\phi}_0} \left(\frac{\dot{a}}{a} \right)^{-1} - 1$$

$$\left(\dot{\phi}_0 \approx -a^2 \left(\frac{\dot{a}}{a} \right)^{-1} \frac{V'}{3} \right)$$

$$\left(\ddot{\phi}_0 \approx -\frac{a^2 V'}{3} (1 + \epsilon) + a^4 \left(\frac{\dot{a}}{a} \right)^{-2} \frac{V' V''}{9} \right)$$

$$\approx -\frac{1}{a^2 V' / 3} \left(-\frac{a^2 V'}{3} (1 + \epsilon) + \frac{a^2}{9} \frac{3}{8\pi G} \frac{V' V''}{V} \right) - 1 \approx \epsilon - \frac{1}{8\pi G} \frac{V''}{V}$$

so δ is related to second derivative of potential being small. Very flat potential.

Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation ($K = 0$), same as scalar field

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0.$$

- Acquires quantum fluctuations in same manner as ϕ . Lagrangian sets the normalization

$$\phi_1 \rightarrow H_T^{(\pm 2)} \sqrt{\frac{3}{16\pi G}}$$

- Scale-invariant gravitational wave amplitude (each component: NB more traditional notation $H_T^{(\pm 2)} = (h_+ \pm ih_\times)/\sqrt{6}$)

$$\Delta_H^2 = \frac{16\pi G}{3 \cdot 2\pi^2} \frac{H^2}{2} = \frac{4}{3\pi} \frac{H^2}{m_{\text{pl}}^2}$$

Gravitational Waves

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation
- Tensor tilt:

$$\frac{d \ln \Delta_H^2}{d \ln k} \equiv n_T = 2 \frac{d \ln H}{d \ln k} = -2\epsilon$$

- Consistency relation between tensor-scalar ratio and tensor tilt

$$\frac{\Delta_H^2}{\Delta_\zeta^2} = \frac{4}{3}\epsilon = -\frac{2}{3}\epsilon$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparison of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself

Gravitational Wave Phenomenology

- Equation of motion

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0.$$

- has solutions

$$H_T^{(\pm 2)} = C_1 H_1(k\eta) + C_2 H_2(k\eta)$$

$$H_1 \propto x^{-m} j_m(x)$$

$$H_2 \propto x^{-m} n_m(x)$$

where $m = (1 - 3w)/(1 + 3w)$

- If $w > -1/3$ then gravity wave is constant above horizon $x \ll 1$ and then oscillates and damps
- If $w < -1/3$ then gravity wave oscillates and freezes into some value, just like scalar field

Gravitational Wave Phenomenology

- A gravitational wave makes a quadrupolar (transverse-traceless) distortion to metric
- Just like the scale factor or spatial curvature, a temporal variation in its amplitude leaves a residual temperature variation in CMB photons – here anisotropic
- Before recombination, anisotropic variation is eliminated by scattering
- Gravitational wave temperature effect drops sharply at the horizon scale at recombination
- Source to polarization goes as $\dot{\tau}/\dot{H}_T$ and peaks at the horizon not damping scale
- B modes formed as photons propagate – the spatial variation in the plane waves modulate the signal: described by Boltzmann eqn.