

Astro 448

Lecture Notes *Set 5*

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Boltzmann Equation

- CMB radiation is generally described by the phase space distribution function for each polarization state $f_a(\mathbf{x}, \mathbf{q}, \eta)$, where \mathbf{x} is the comoving position and \mathbf{q} is the photon momentum
- Boltzmann equation describes the evolution of the distribution function under gravity and collisions
- Low order moments of the Boltzmann equation are simply the covariant conservation equations
- Higher moments provide the closure condition to the conservation law (specification of stress tensor) and the CMB observable – fine scale anisotropy
- Higher moments mainly describe the simple geometry of source projection

Liouville Equation

- In absence of scattering, the phase space distribution of photons is conserved along the propagation path
- Rewrite variables in terms of the photon propagation direction $\mathbf{q} = q\hat{\mathbf{n}}$, so $f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta)$ and

$$\begin{aligned} \frac{d}{d\eta} f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta) &= 0 \\ &= \left(\frac{\partial}{\partial \eta} + \frac{d\mathbf{x}}{d\eta} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{d\hat{\mathbf{n}}}{d\eta} \cdot \frac{\partial}{\partial \hat{\mathbf{n}}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a \end{aligned}$$

- For simplicity, assume spatially flat universe $K = 0$ then $d\hat{\mathbf{n}}/d\eta = 0$ and $d\mathbf{x} = \hat{\mathbf{n}}d\eta$

$$\dot{f}_a + \hat{\mathbf{n}} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0$$

Correspondence to Einstein Eqn.

- Geodesic equation gives the redshifting term

$$\frac{\dot{q}}{q} = -\frac{\dot{a}}{a} - \frac{1}{2}n^i n^j \dot{H}_{Tij} - \dot{H}_L + n^i \dot{B}_i - \hat{\mathbf{n}} \cdot \nabla A$$

- which is incorporated in the conservation and gauge transformation equations
- Stress energy tensor involves integrals over the distribution function the two polarization states

$$T^{\mu\nu} = \int \frac{d^3q}{(2\pi)^3} \frac{q^\mu q^\nu}{E} (f_a + f_b)$$

- Components are simply the low order angular moments of the distribution function

Angular Moments

- Define the angularly dependent temperature perturbation

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \frac{1}{4\rho_\gamma} \int \frac{q^3 dq}{2\pi^2} (f_a + f_b) - 1$$

and likewise for the linear polarization states Q and U

- Decompose into normal modes: plane waves for spatial part and spherical harmonics for angular part

$$G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$\pm_2 G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} \pm_2 Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$

- In a spatially curved universe generalize the plane wave part

Normal Modes

- Temperature and polarization fields

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} \Theta_{\ell}^{(m)} G_{\ell}^m$$

$$[Q \pm iU](\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} [E_{\ell}^{(m)} \pm iB_{\ell}^{(m)}]_{\pm 2} G_{\ell}^m$$

- For each \mathbf{k} mode, work in coordinates where $\mathbf{k} \parallel \mathbf{z}$ and so $m = 0$ represents scalar modes, $m = \pm 1$ vector modes, $m = \pm 2$ tensor modes, $|m| > 2$ vanishes. Since modes add incoherently and $Q \pm iU$ is invariant up to a phase, rotation back to a fixed coordinate system is trivial.

Scalar, Vector, Tensor

- Normalization of modes is chosen so that the lowest angular mode for scalars, vectors and tensors are normalized in the same way as the mode function

$$\begin{aligned}G_0^0 &= Q^{(0)} & G_1^0 &= n^i Q_i^{(0)} & G_2^0 &\propto n^i n^j Q_{ij}^{(0)} \\G_1^{\pm 1} &= n^i Q_i^{(\pm 1)} & G_2^{\pm 1} &\propto n^i n^j Q_{ij}^{(\pm 1)} \\G_2^{\pm 2} &= n^i n^j Q_{ij}^{(\pm 2)}\end{aligned}$$

where recall

$$\begin{aligned}Q^{(0)} &= \exp(i\mathbf{k} \cdot \mathbf{x}) \\Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}} (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k} \cdot \mathbf{x}) \\Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}} (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k} \cdot \mathbf{x})\end{aligned}$$

Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies
- Spatial gradient term hits plane wave:

$$\hat{\mathbf{n}} \cdot \nabla e^{i\mathbf{k} \cdot \mathbf{x}} = i\hat{\mathbf{n}} \cdot \mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x}} = i\sqrt{\frac{4\pi}{3}} k Y_1^0(\hat{\mathbf{n}}) e^{i\mathbf{k} \cdot \mathbf{x}}$$

- Dipole term adds to angular dependence through the addition of angular momentum

$$\sqrt{\frac{4\pi}{3}} Y_1^0 Y_\ell^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell+1)(2\ell-1)}} Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell+1)(2\ell+3)}} Y_{\ell+1}^m$$

where $\kappa_\ell^m = \sqrt{\ell^2 - m^2}$ is given by Clebsch-Gordon coefficients.

Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

$$\dot{\Theta}_\ell^{(m)} = k \left[\frac{\kappa_\ell^m}{2\ell + 1} \Theta_{\ell-1}^{(m)} - \frac{\kappa_{\ell+1}^m}{2\ell + 3} \Theta_{\ell+1}^{(m)} \right] - \dot{\tau} \Theta_\ell^{(m)} + S_\ell^{(m)}$$

where $S_\ell^{(m)}$ are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic $\ell = 0$ temperature perturbation will eventually become a high order anisotropy by “free streaming” or simple projection
- Original CMB codes solved the full hierarchy equations out to the ℓ of interest.

Integral Solution

- Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface
- In general, the solution describes the decomposition of the source $S_\ell^{(m)}$ with its local angular dependence as seen at a distance $\mathbf{x} = D\hat{\mathbf{n}}$.
- Proceed by decomposing the angular dependence of the plane wave

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\ell} (-i)^{\ell} \sqrt{4\pi(2\ell + 1)} j_{\ell}(kD) Y_{\ell}^0(\hat{\mathbf{n}})$$

- Recouple to the local angular dependence of G_{ℓ}^m

$$G_{\ell_s}^m = \sum_{\ell} (-i)^{\ell} \sqrt{4\pi(2\ell + 1)} \alpha_{\ell_s \ell}^{(m)}(kD) Y_{\ell}^m(\hat{\mathbf{n}})$$

Integral Solution

- Projection kernels:

$$\ell_s = 0, \quad m = 0 \qquad \alpha_{0\ell}^{(0)} \equiv j_\ell$$

$$\ell_s = 1, \quad m = 0 \qquad \alpha_{1\ell}^{(0)} \equiv j'_\ell$$

- Integral solution:

$$\frac{\Theta_\ell^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S_{\ell_s}^{(m)} \alpha_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

- Power spectrum:

$$C_\ell = \frac{2}{\pi} \int \frac{dk}{k} \sum_m \frac{k^3 \langle \Theta_\ell^{(m)*} \Theta_\ell^{(m)} \rangle}{(2\ell + 1)^2}$$

- Solving for C_ℓ reduces to solving for the behavior of a handful of sources

Polarization Hierarchy

- In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hierarchy:

$$\dot{E}_\ell^{(m)} = k \left[\frac{{}_2\kappa_\ell^m}{2\ell - 1} E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell + 1)} B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell + 3} \right] - \dot{\tau} E_\ell^{(m)} + \mathcal{E}_\ell^{(m)}$$

$$\dot{B}_\ell^{(m)} = k \left[\frac{{}_2\kappa_\ell^m}{2\ell - 1} B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell + 1)} B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell + 3} \right] - \dot{\tau} E_\ell^{(m)} + \mathcal{B}_\ell^{(m)}$$

where ${}_2\kappa_\ell^m = \sqrt{(\ell^2 - m^2)(\ell^2 - 4)}/\ell^2$ is given by the Clebsch-Gordon coefficients and \mathcal{E} , \mathcal{B} are the sources (scattering only).

- Note that for vectors and tensors $|m| > 0$ and B modes may be generated from E modes by projection. Cosmologically $\mathcal{B}_\ell^{(m)} = 0$

Polarization Integral Solution

- Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

$$\frac{E_\ell^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \epsilon_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

$$\frac{B_\ell^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \beta_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

- The only source to the polarization is from the quadrupole anisotropy so we only need $\ell_s = 2$, e.g. for scalars

$$\epsilon_{2\ell}^{(0)}(x) = \sqrt{\frac{3(\ell + 2)!}{8(\ell - 2)!}} \frac{j_\ell(x)}{x^2} \quad \beta_{2\ell}^{(0)} = 0$$

Truncated Hierarchy

- CMBFast uses the integral solution and relies on a fast j_ℓ generator
- However sources are not external to system and are defined through the Boltzmann hierarchy itself
- Solution: recall that we used this technique in the tight coupling regime by applying a closure condition from tight coupling
- CMBFast extends this idea by solving a truncated hierarchy of equations, e.g. out to $\ell = 25$ with non-reflecting boundary conditions

Thomson Collision Term

- Full Boltzmann equation

$$\frac{d}{d\eta} f_{a,b} = C[f_a, f_b]$$

- Collision term describes the scattering out of and into a phase space element
- Thomson collision based on differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T ,$$

where $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

Scattering Calculation

- Start in the electron rest frame and in a coordinate system fixed by the scattering plane, spanned by incoming and outgoing directional vectors $-\hat{\mathbf{n}}' \cdot \hat{\mathbf{n}} = \cos \beta$, where β is the scattering angle
- Θ_{\parallel} : in-plane polarization state; Θ_{\perp} : \perp -plane polarization state
- Transfer probability (constant set by $\dot{\tau}$)

$$\Theta_{\parallel} \propto \cos^2 \beta \Theta'_{\parallel}, \quad \Theta_{\perp} \propto \Theta'_{\perp}$$

- and with the 45° axes as

$$\hat{\mathbf{E}}_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\parallel} + \hat{\mathbf{E}}_{\perp}), \quad \hat{\mathbf{E}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\parallel} - \hat{\mathbf{E}}_{\perp})$$

Stokes Parameters

- Define the temperature in this basis

$$\begin{aligned}\Theta_1 &\propto |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 + |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 \\ &\propto \frac{1}{4}(\cos \beta + 1)^2 \Theta'_1 + \frac{1}{4}(\cos \beta - 1)^2 \Theta'_2\end{aligned}$$

$$\begin{aligned}\Theta_2 &\propto |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 + |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 \\ &\propto \frac{1}{4}(\cos \beta + 1)^2 \Theta'_2 + \frac{1}{4}(\cos \beta - 1)^2 \Theta'_1\end{aligned}$$

or $\Theta_1 - \Theta_2 \propto \cos \beta (\Theta'_1 - \Theta'_2)$

- Define Θ , Q , U in the scattering coordinates

$$\Theta \equiv \frac{1}{2}(\Theta_{\parallel} + \Theta_{\perp}), \quad Q \equiv \frac{1}{2}(\Theta_{\parallel} - \Theta_{\perp}), \quad U \equiv \frac{1}{2}(\Theta_1 - \Theta_2)$$

Scattering Matrix

- Transfer of Stokes states, e.g.

$$\Theta = \frac{1}{2}(\Theta_{\parallel} + \Theta_{\perp}) \propto \frac{1}{4}(\cos^2 \beta + 1)\Theta' + \frac{1}{4}(\cos^2 \beta - 1)Q'$$

- Transfer matrix of Stokes state $\mathbf{T} \equiv (\Theta, Q + iU, Q - iU)$

$$\mathbf{T} \propto \mathbf{S}(\beta)\mathbf{T}'$$

$$\mathbf{S}(\beta) = \frac{3}{4} \begin{pmatrix} \cos^2 \beta + 1 & -\frac{1}{2} \sin^2 \beta & -\frac{1}{2} \sin^2 \beta \\ -\frac{1}{2} \sin^2 \beta & \frac{1}{2}(\cos \beta + 1)^2 & \frac{1}{2}(\cos \beta - 1)^2 \\ -\frac{1}{2} \sin^2 \beta & \frac{1}{2}(\cos \beta - 1)^2 & \frac{1}{2}(\cos \beta + 1)^2 \end{pmatrix}$$

normalization factor of 3 is set by photon conservation in scattering

Scattering Matrix

- Transform to a fixed basis, by a rotation of the incoming and outgoing states $\mathbf{T} = \mathbf{R}(\psi)\mathbf{T}$ where

$$\mathbf{R}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & e^{2i\psi} \end{pmatrix}$$

giving the scattering matrix

$$\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha) =$$

$$\frac{1}{2}\sqrt{\frac{4\pi}{5}} \begin{pmatrix} Y_2^0(\beta, \alpha) + 2\sqrt{5}Y_0^0(\beta, \alpha) & -\sqrt{\frac{3}{2}}Y_2^{-2}(\beta, \alpha) & -\sqrt{\frac{3}{2}}Y_2^2(\beta, \alpha) \\ -\sqrt{6}{}_2Y_2^0(\beta, \alpha)e^{2i\gamma} & 3{}_2Y_2^{-2}(\beta, \alpha)e^{2i\gamma} & 3{}_2Y_2^2(\beta, \alpha)e^{2i\gamma} \\ -\sqrt{6}{}_{-2}Y_2^0(\beta, \alpha)e^{-2i\gamma} & 3{}_{-2}Y_2^{-2}(\beta, \alpha)e^{-2i\gamma} & 3{}_{-2}Y_2^2(\beta, \alpha)e^{-2i\gamma} \end{pmatrix}$$

Addition Theorem for Spin Harmonics

- Spin harmonics are related to rotation matrices as

$${}_s Y_\ell^m(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} \mathcal{D}_{-ms}^\ell(\phi, \theta, 0)$$

Note: for explicit evaluation sign convention differs from usual (e.g. Jackson) by $(-1)^m$

- Multiplication of rotations

$$\sum_{m''} \mathcal{D}_{mm''}^\ell(\alpha_2, \beta_2, \gamma_2) \mathcal{D}_{m''m}^\ell(\alpha_1, \beta_1, \gamma_1) = \mathcal{D}_{mm'}^\ell(\alpha, \beta, \gamma)$$

- Implies

$$\sum_m {}_{s_1} Y_\ell^{m*}(\theta', \phi') {}_{s_2} Y_\ell^m(\theta, \phi) = (-1)^{s_1 - s_2} \sqrt{\frac{2\ell + 1}{4\pi}} {}_{s_2} Y_\ell^{-s_1}(\beta, \alpha) e^{is_2\gamma}$$

Sky Basis

- Scattering into the state (rest frame)

$$\begin{aligned}
 C_{\text{in}}[\mathbf{T}] &= \dot{\tau} \int \frac{d\hat{\mathbf{n}}'}{4\pi} \mathbf{R}(-\gamma) \mathbf{S}(\beta) \mathbf{R}(\alpha) \mathbf{T}(\hat{\mathbf{n}}'), \\
 &= \dot{\tau} \int \frac{d\hat{\mathbf{n}}'}{4\pi} (\Theta', 0, 0) + \frac{1}{10} \dot{\tau} \int d\hat{\mathbf{n}}' \sum_{m=-2}^2 \mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') \mathbf{T}(\hat{\mathbf{n}}').
 \end{aligned}$$

where the quadrupole coupling term is $\mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') =$

$$\begin{pmatrix}
 Y_2^{m*}(\hat{\mathbf{n}}') Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}} {}_2Y_2^{m*}(\hat{\mathbf{n}}') Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}} {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}') Y_2^m(\hat{\mathbf{n}}) \\
 -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}') {}_2Y_2^m(\hat{\mathbf{n}}) & 3 {}_2Y_2^{m*}(\hat{\mathbf{n}}') {}_2Y_2^m(\hat{\mathbf{n}}) & 3 {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}') {}_2Y_2^m(\hat{\mathbf{n}}) \\
 -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}') {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3 {}_2Y_2^{m*}(\hat{\mathbf{n}}') {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3 {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}') {}_{-2}Y_2^m(\hat{\mathbf{n}})
 \end{pmatrix},$$

expression uses angle addition relation above. We call this term C_Q .

Scattering Matrix

- Full scattering matrix involves difference of scattering into and out of state

$$C[\mathbf{T}] = C_{\text{in}}[\mathbf{T}] - C_{\text{out}}[\mathbf{T}]$$

- In the electron rest frame

$$C[\mathbf{T}] = \dot{\tau} \int \frac{d\hat{\mathbf{n}}'}{4\pi} (\Theta', 0, 0) - \dot{\tau}\mathbf{T} + C_Q[\mathbf{T}]$$

which describes isotropization in the rest frame. All moments have $e^{-\tau}$ suppression except for isotropic temperature Θ_0 .

Transformation into the background frame simply induces a dipole term

$$C[\mathbf{T}] = \dot{\tau} \left(\hat{\mathbf{n}} \cdot \mathbf{v}_b + \int \frac{d\hat{\mathbf{n}}'}{4\pi} \Theta', 0, 0 \right) - \dot{\tau}\mathbf{T} + C_Q[\mathbf{T}]$$

Source Terms

- Temperature source terms $S_l^{(m)}$ (rows $\pm|m|$; flat assumption)

$$\begin{pmatrix} \dot{\tau}\Theta_0^{(0)} - \dot{H}_L^{(0)} & \dot{\tau}v_b^{(0)} + \dot{B}^{(0)} & \dot{\tau}P^{(0)} - \frac{2}{3}\dot{H}_T^{(0)} \\ 0 & \dot{\tau}v_b^{(\pm 1)} + \dot{B}^{(\pm 1)} & \dot{\tau}P^{(\pm 1)} - \frac{\sqrt{3}}{3}\dot{H}_T^{(\pm 1)} \\ 0 & 0 & \dot{\tau}P^{(\pm 2)} - \dot{H}_T^{(\pm 2)} \end{pmatrix}$$

where

$$P^{(m)} \equiv \frac{1}{10}(\Theta_2^{(m)} - \sqrt{6}E_2^{(m)})$$

- Polarization source term

$$\mathcal{E}_\ell^{(m)} = -\dot{\tau}\sqrt{6}P^{(m)}\delta_{\ell,2}$$

$$\mathcal{B}_\ell^{(m)} = 0$$