Secondary Anisotropy

- CMB photons traverse the large-scale structure of the universe from $z = 1000$ to the present.

- With the nearly scale-invariant adiabatic fluctuations observed in the CMB, structures form from the bottom up, i.e. small scales first, a.k.a. hierarchical structure formation.

- First objects reionize the universe between $z \sim 7 - 30$

- Main sources of secondary anisotropy
  - Gravitational: Integrated Sachs-Wolfe effect (gravitational redshift) and gravitational lensing
  - Scattering: peak suppression, large-angle polarization, Doppler effect(s), inverse Compton scattering
Transfer Function

- **Transfer function** transfers the initial Newtonian curvature to its value today (linear response theory)

\[
T(k) = \frac{\Phi(k, a = 1)}{\Phi(k, a_{\text{init}})} \frac{\Phi(k_{\text{norm}}, a_{\text{init}})}{\Phi(k_{\text{norm}}, a = 1)}
\]

- Conservation of Bardeen curvature: Newtonian curvature is a constant when stress perturbations are negligible: above the horizon during radiation and dark energy domination, on all scales during matter domination

- When stress fluctuations dominate, perturbations are stabilized by the Jeans mechanism

- **Hybrid Poisson equation**: Newtonian curvature, comoving density perturbation \( \Delta \equiv (\delta \rho / \rho_{\text{com}}) \) implies \( \Phi \) decays

\[
(k^2 - 3K)\Phi = 4\pi G \rho \Delta \sim \eta^{-2} \Delta
\]
Transfer Function

- Matter-radiation example: Jeans scale is horizon scale and $\Delta$ freezes into its value at horizon crossing $\Delta_H \approx \Phi_{\text{init}}$
- Freezing of $\Delta$ stops at $\eta_{\text{eq}}$

$$\Phi \sim (k\eta_{\text{eq}})^{-2} \Delta_H \sim (k\eta_{\text{eq}})^{-2} \Phi_{\text{init}}$$

- Conventionally $k_{\text{norm}}$ is chosen as a scale between the horizon at matter radiation equality and dark energy domination.
- Small correction since growth with a smooth radiation component is logarithmic not frozen
- Run CMBfast to get transfer function or use fits
Transfer Function

- Transfer function has a $k^{-2}$ fall-off beyond $k_{eq} \sim \eta_{eq}^{-1}$

- Additional baryon wiggles are due to acoustic oscillations at recombination – an interesting means of measuring distances
Growth Function

- Same physics applies to the dark energy dominated universe
- Under the dark energy sound horizon, dark energy density frozen. Potential decays at the same rate for all scales

\[ g(a) = \frac{\Phi(k_{\text{norm}}, a)}{\Phi(k_{\text{norm}}, a_{\text{init}})} \]

- Pressure growth suppression: \( \delta \equiv \delta \rho_m / \rho_m \propto a \phi \)

\[
\frac{d^2 g}{d \ln a^2} + \left[ \frac{5}{2} - \frac{3}{2} w(z) \Omega_{DE}(z) \right] \frac{dg}{d \ln a} + \frac{3}{2} \left[ 1 - w(z) \right] \Omega_{DE}(z) g = 0,
\]

where \( w \equiv p_{DE}/\rho_{DE} \) and \( \Omega_{DE} \equiv \rho_{DE}/(\rho_m + \rho_{DE}) \) with initial conditions \( g = 1, \frac{dg}{d \ln a} = 0 \)

- As \( \Omega_{DE} \to 0 \) \( g = \text{const.} \) is a solution. The other solution is the decaying mode, eliminated by initial conditions
ISW effect

- Potential decay leads to gravitational redshifts through the integrated Sachs-Wolfe effect
- Intrinsically a large effect since \(2 \Delta \Phi = \frac{6 \Psi_{\text{init}}}{3}\)
- But net redshift is integral along line of sight

\[
\frac{\Theta_\ell(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \left[ 2 \dot{\Phi}(k, \eta) j_\ell(k(\eta_0 - \eta)) \right] = 2\Phi(k, \eta_{MD}) \int_0^{\eta_0} d\eta e^{-\tau} \dot{g}(D) j_\ell(kD)
\]

- On small scales where \(k \gg \dot{g}/g\), can pull source out of the integral

\[
\int_0^{\eta_0} d\eta \dot{g}(D) j_\ell(kD) \approx \dot{g}(D = \ell/k) \frac{1}{k} \sqrt{\frac{\pi}{2\ell}}
\]

evaluated at peak, where we have used \(\int dx j_\ell(x) = \sqrt{\pi/2\ell}\)
ISW effect

- Power spectrum

\[
C_\ell = \frac{2}{\pi} \int \frac{dk}{k} \frac{k^3 \langle \Theta^*_\ell(k, \eta_0) \Theta_\ell(k, \eta_0) \rangle}{(2\ell + 1)^2} = \frac{2\pi^2}{l^3} \int d\eta D \dot{g}^2(\eta) \Delta^2_\Phi(\ell/D, \eta_{MD})
\]

- Or \(l^2C_\ell/2\pi \propto 1/\ell\) for scale invariant potential. This is the Limber equation in spherical coordinates. Projection of 3D power retains only the transverse piece.

- Cancellation of redshifts and blueshifts as the photon traverses many crests and troughs of a small scale fluctuation during decay.

- Enhancement of the \(\ell < 10\) multipoles. Difficult to extract from cosmic variance and galaxy. Current ideas: cross correlation with other tracers of structure.
Gravitational Lensing

- Lensing is a surface brightness conserving remapping of source to image planes by the gradient of the projected potential

\[ \phi(\hat{n}) = 2 \int_{\eta_*}^{\eta_0} d\eta \frac{(D_* - D)}{D D_*} \Phi(D\hat{n}, \eta). \]

such that the fields are remapped as

\[ x(\hat{n}) \rightarrow x(\hat{n} + \nabla \phi), \]

where \( x \in \{\Theta, Q, U\} \) temperature and polarization.

- Taylor expansion leads to product of fields and Fourier mode-coupling
Flat-sky Treatment

- Taylor expand

\[ \Theta(\hat{n}) = \tilde{\Theta}(\hat{n} + \nabla \phi) \]
\[ = \tilde{\Theta}(\hat{n}) + \nabla_i \phi(\hat{n}) \nabla^i \tilde{\Theta}(\hat{n}) + \frac{1}{2} \nabla_i \phi(\hat{n}) \nabla_j \phi(\hat{n}) \nabla^i \nabla^j \tilde{\Theta}(\hat{n}) + \ldots \]

- Fourier decomposition

\[
\phi(\hat{n}) = \int \frac{d^2 l}{(2\pi)^2} \phi(l) e^{il \cdot \hat{n}} \\
\tilde{\Theta}(\hat{n}) = \int \frac{d^2 l}{(2\pi)^2} \tilde{\Theta}(l) e^{il \cdot \hat{n}}
\]
Flat-sky Treatment

• Mode coupling of harmonics

\[ \Theta(l) = \int d\hat{n} \Theta(\hat{n}) e^{-il \cdot \hat{n}} \]

\[ = \tilde{\Theta}(l) - \int \frac{d^2l_1}{(2\pi)^2} \tilde{\Theta}(l_1) L(l, l_1), \]

where

\[ L(l, l_1) = \phi(l - l_1) (l - l_1) \cdot l_1 \]

\[ + \frac{1}{2} \int \frac{d^2l_2}{(2\pi)^2} \phi(l_2) \phi^*(l_2 + l_1 - l) (l_2 \cdot l_1)(l_2 + l_1 - l) \cdot l_1. \]

• Represents a coupling of harmonics separated by \( L \approx 60 \) peak of deflection power
Power Spectrum

- Power spectra

\[
\langle \Theta^* (l) \Theta (l') \rangle = (2\pi)^2 \delta(l - l') \, C_{l}^{\Theta \Theta},
\]

\[
\langle \phi^* (l) \phi (l') \rangle = (2\pi)^2 \delta(l - l') \, C_{l}^{\phi \phi},
\]

becomes

\[
C_{l}^{\Theta \Theta} = (1 - l^2 R) \tilde{C}_{l}^{\Theta \Theta} + \int \frac{d^2 l_1}{(2\pi)^2} \tilde{C}_{|l - l_1|}^{\Theta \Theta} [C_{l_1}^{\phi \phi}|(l - l_1) \cdot l_1|^2,
\]

where

\[
R = \frac{1}{4\pi} \int \frac{d l}{l} \, l^4 C_{l}^{\phi \phi}.
\]  \(1\)
Smoothing Power Spectrum

- If $\tilde{C}_l^{\Theta\Theta}$ slowly varying then two term cancel

$$\tilde{C}_l^{\Theta\Theta} \int \frac{d^2 l_1}{(2\pi)^2} C_l^{\phi\phi} (l \cdot l_1)^2 \approx l^2 R \tilde{C}_l^{\Theta\Theta}.$$ 

- So lensing acts to smooth features in the power spectrum. Smoothing kernel is $L \sim 60$ the peak of deflection power spectrum.

- Because acoustic feature appear on a scale $l_A \sim 300$, smoothing is a subtle effect in the power spectrum.

- Lensing generates power below the damping scale which directly reflect power in deflections on the same scale.
Polarization Lensing

- Polarization field harmonics lensed similarly

\[
[Q \pm iU](\hat{n}) = -\int \frac{d^2l}{(2\pi)^2} [E \pm iB](l)e^{\pm 2i\phi_1}e^{l \cdot \hat{n}}
\]

so that

\[
[Q \pm iU](\hat{n}) = [\tilde{Q} \pm i\tilde{U}](\hat{n} + \nabla \phi)
\]

\[
\approx [\tilde{Q} \pm i\tilde{U}](\hat{n}) + \nabla_i \phi(\hat{n}) \nabla^i [\tilde{Q} \pm i\tilde{U}](\hat{n})
\]

\[
+ \frac{1}{2} \nabla_i \phi(\hat{n}) \nabla_j \phi(\hat{n}) \nabla^i \nabla^j [\tilde{Q} \pm i\tilde{U}](\hat{n})
\]
Polarization Power Spectra

- Carrying through the algebra

\[
C_l^{EE} = (1 - l^2 R) \tilde{C}_l^{EE} + \frac{1}{2} \int \frac{d^2 l_1}{(2\pi)^2} [(l - l_1) \cdot l_1]^2 C_{|l-l_1|}^{\phi\phi} \times \left[ (\tilde{C}_l^{EE} + \tilde{C}_l^{BB}) + \cos(4\varphi_l)(\tilde{C}_l^{EE} - \tilde{C}_l^{BB}) \right],
\]

\[
C_l^{BB} = (1 - l^2 R) \tilde{C}_l^{BB} + \frac{1}{2} \int \frac{d^2 l_1}{(2\pi)^2} [(l - l_1) \cdot l_1]^2 C_{|l-l_1|}^{\phi\phi} \times \left[ (\tilde{C}_l^{EE} + \tilde{C}_l^{BB}) - \cos(4\varphi_l)(\tilde{C}_l^{EE} - \tilde{C}_l^{BB}) \right],
\]

\[
C_l^{\Theta E} = (1 - l^2 R) \tilde{C}_l^{\Theta E} + \int \frac{d^2 l_1}{(2\pi)^2} [(l - l_1) \cdot l_1]^2 C_{|l-l_1|}^{\phi\phi} \times \tilde{C}_l^{\Theta E} \cos(2\varphi_l),
\]

- Lensing generates $B$-modes out of the acoustic polarization $E$-modes contaminates gravitational wave signature if $E_i < 10^{16}\text{GeV}$.
Reconstruction from the CMB

- Correlation between Fourier moments reflect lensing potential

\[ \langle x(l)x'(l') \rangle_{\text{CMB}} = f_{\alpha}(l, l') \phi(l + l') , \]

where \( x \) \( \in \) temperature, polarization fields and \( f_{\alpha} \) is a fixed weight that reflects geometry

- Each pair forms a noisy estimate of the potential or projected mass - just like a pair of galaxy shears

- Minimum variance weight all pairs to form an estimator of the lensing mass
Scattering Secondaries

- Optical depth during reionization
  \[ \tau \approx 0.066 \left( \frac{\Omega_b h^2}{0.02} \right) \left( \frac{\Omega_m h^2}{0.15} \right)^{-1/2} \left( \frac{1 + z}{10} \right)^{3/2} \]

- Anisotropy suppressed as \( e^{-\tau} \). Integral solution
  \[ \frac{\Theta_\ell(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} S_0^{(0)} j_\ell(k(\eta_0 - \eta)) + \ldots \]

- Isotropic (large scale) fluctuations not suppressed since suppression represents isotropization by scattering

- Quadrupole from the Sachs-Wolfe effect scatters into a large angle polarization bump
Doppler Effects

- Velocity fields of $10^{-3}$ and optical depths of $10^{-2}$ would imply large Doppler effect due to reionization.
- Limber approximation says only fluctuations transverse to line of sight survive.
- In linear theory, transverse fluctuations have no line of sight velocity and so Doppler effect is highly suppressed.
- Beyond linear theory: modulate the optical depth in the transverse direction using density fluctuations or ionization fraction fluctuations. Generate a modulated Doppler effect.
- Linear fluctuations: Vishniac effect; Clusters: kinetic SZ effect; ionization patches: inhomogeneous reionization effect.
Thermal SZ Effect

- Thermal velocities also lead to Doppler effect but first order contribution cancels because of random directions.
- Residual effect is of order $v^2 \tau \approx T_e / m_e \tau$ and can reach a sizeable level for clusters with $T_e \approx 10$keV.
- Raleigh-Jeans decrement and Wien enhancement described by second order collision term in Boltzmann equation: Kompaneets equation.
- Clusters are rare objects so contribution to power spectrum suppressed, but may have been detected by CBI/BIMA: extremely sensitive to power spectrum normalization $\sigma_8$.
- White noise on large-scales ($l < 2000$), turnover as cluster profile is resolved.