Astro 448: Problem Set 6

Due Nov. 14

Classical Scalar Fields

Scalar fields are the basis of inflation and dark energy models. In the next two problem sets we derive the classical equations of motion for a scalar field and its perturbations.

The stress-energy tensor of a minimally coupled scalar field φ with a potential $V(\varphi)$ is given by

$$T^{\mu}{}_{\nu} = \nabla^{\mu}\varphi\,\nabla_{\nu}\varphi - \frac{1}{2}(\nabla^{\alpha}\varphi\,\nabla_{\alpha}\varphi + 2V)\delta^{\mu}{}_{\nu}\,. \tag{1}$$

We will expand the scalar field fluctuations about its background value ϕ_0 as $\varphi = \phi_0 + \phi_1$.

1 Homogeneous Case

(1) Using the FRW metric for the background and the general relation for the components of the stress energy tensor, derive the expressions for the energy density of the field $\rho_\phi(\phi_0, \dot{\phi}_0)$ and the pressure $p_\phi(\phi_0, \dot{\phi}_0)$. You will need this below so if you are unsure of the result check it in Kolb & Turner.

(2) If the energy density is dominated by the potential term what is the equation of state $w_\phi = p_\phi/\rho_\phi$? If the energy density is dominated by the kinetic term ($\dot{\phi}$) what is the equation of state?

(3) Show that the continuity equation

$$\dot{\rho}_\phi = -3(\rho_\phi + p_\phi)\frac{\dot{a}}{a}\,, \tag{2}$$

implies the homogeneous scalar field equation

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2 V' = 0\,, \tag{3}$$

primes are derivatives with respect to the argument ϕ_0, and overdots are derivatives with respect to conformal time.

2 Fluctuations

The same procedure as in (1) works for the fluctuations. Ignoring metric fluctuations show:

$$\begin{aligned}
\delta\rho_\phi &= a^{-2}(\dot{\phi}_0\dot{\phi}_1) + V'\phi_1\,,\\
\delta p_\phi &= a^{-2}(\dot{\phi}_0\dot{\phi}_1) - V'\phi_1\,,\\
(\rho_\phi + p_\phi)v_\phi &= a^{-2}k\dot{\phi}_0\phi_1\,,\\
p_\phi\pi_\phi &= 0\,,
\end{aligned} \tag{4}$$

where $V' = \partial V/\partial\phi$