Astro 448

Set 1: Descriptive Statistics

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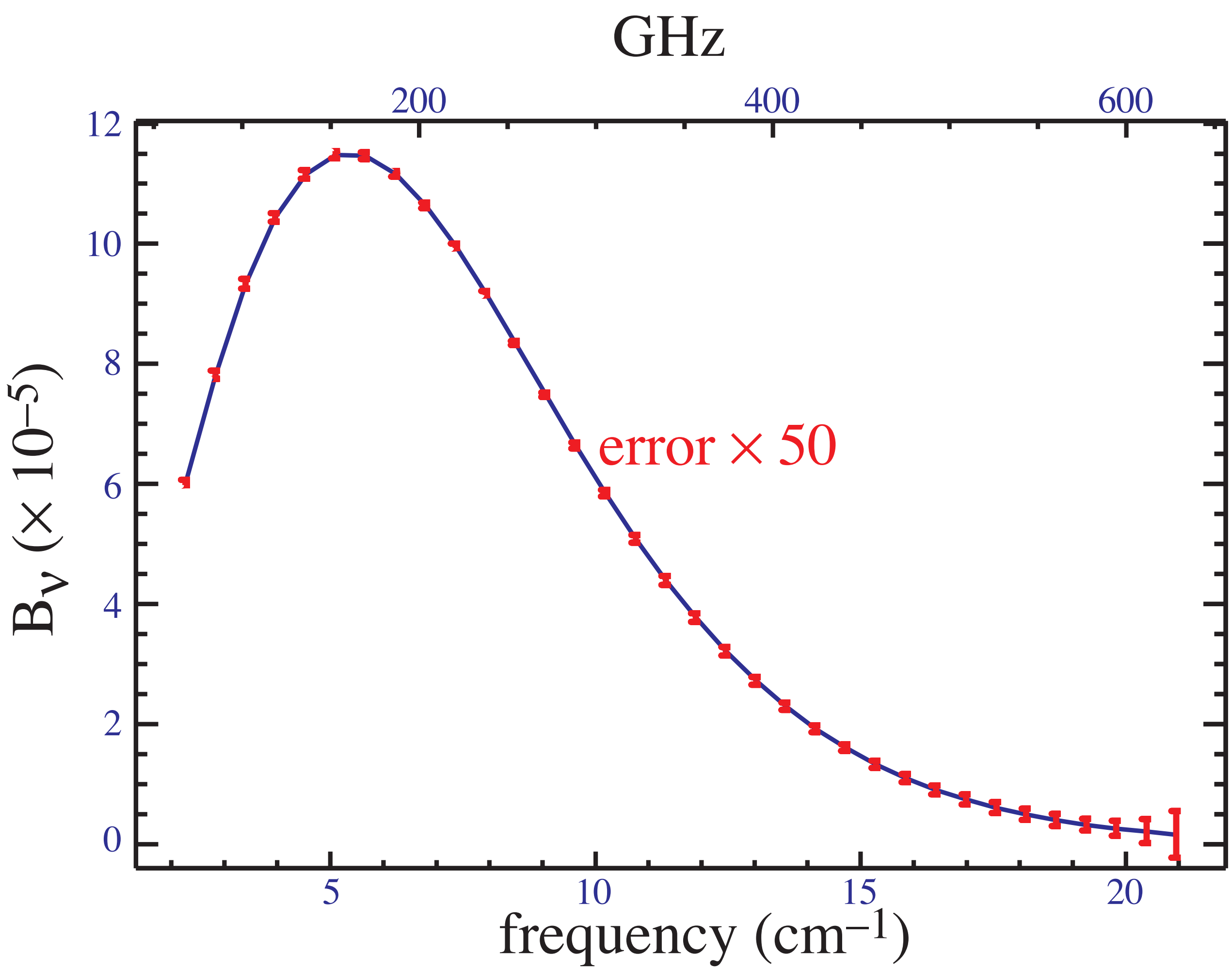
CMB Blackbody

- COBE FIRAS spectral measurement. yellBlackbody spectrum. $T = 2.725\text{K}$ giving $\Omega_\gamma h^2 = 2.471 \times 10^{-5}$



CMB Blackbody

- CMB is a (nearly) perfect blackbody characterized by a phase space distribution function

$$f = \frac{1}{e^{E/T} - 1}$$

  where the temperature $T(\mathbf{x}, \hat{\mathbf{n}}, t)$ is observed at our position $\mathbf{x} = 0$ and time t_0 to be nearly isotropic with a mean temperature of $\bar{T} = 2.725\mathrm{K}$

- Our observable then is the temperature anisotropy

$$\Theta(\hat{\mathbf{n}}) \equiv \frac{T(0, \hat{\mathbf{n}}, t_0) - \bar{T}}{\bar{T}}$$

- Given that physical processes essentially put a band limit on this function it is useful to decompose it into a complete set of harmonic coefficients

Spherical Harmonics

- Laplace Eigenfunctions

$$\nabla^2 Y_\ell^m = -[l(l+1)]Y_\ell^m$$

- Orthogonal and complete

$$\int d\hat{\mathbf{n}} Y_\ell^{m*}(\hat{\mathbf{n}}) Y_\ell^m(\hat{\mathbf{n}}) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\sum_{\ell m} Y_\ell^{m*}(\hat{\mathbf{n}}) Y_\ell^m(\hat{\mathbf{n}}') = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta')$$

Generalizable to tensors on the sphere (polarization), modes on a curved FRW metric

- Conjugation

$$Y_\ell^{m*} = (-1)^m Y_\ell^{-m}$$

Multipole Moments

- Decompose into multipole moments

$$\Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_\ell^m(\hat{\mathbf{n}})$$

- So $\Theta_{\ell m}$ is complex but $\Theta(\hat{\mathbf{n}})$ real:

$$\begin{aligned}
\Theta^*(\hat{\mathbf{n}}) &= \sum_{\ell m} \Theta^*_{\ell m} Y_\ell^{m*}(\hat{\mathbf{n}}) \\
&= \sum_{\ell m} \Theta^*_{\ell m} (-1)^m Y_\ell^{-m}(\hat{\mathbf{n}}) \\
&= \Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_\ell^m(\hat{\mathbf{n}}) = \sum_{\ell -m} \Theta_{\ell -m} Y_\ell^{-m}(\hat{\mathbf{n}})
\end{aligned}$$

so m and $-m$ are not independent

$$\Theta^*_{\ell m} = (-1)^m \Theta_{\ell\, -m}$$

N-pt correlation

- Since the fluctuations are random and zero mean we are interested in characterizing the N-point correlation

$$\langle \Theta(\hat{\mathbf{n}}_1)\ldots\Theta(\hat{\mathbf{n}}_n)\rangle = \sum_{\ell_1\ldots\ell_n}\sum_{m_1\ldots m_n}\langle \Theta_{\ell_1 m_1}\ldots\Theta_{\ell_n m_n}\rangle Y_{\ell_1}^{m_1}(\hat{\mathbf{n}}_1)\ldots Y_{\ell_n}^{m_n}(\hat{\mathbf{n}}_n)$$

- Statistical isotropy implies that we should get the same result in a rotated frame

$$R[Y_\ell^m(\hat{\mathbf{n}})] = \sum_{m'} D^\ell_{m'm}(\alpha,\beta,\gamma)Y_\ell^{m'}(\hat{\mathbf{n}})$$

where α, β and γ are the Euler angles of the rotation and D is the Wigner function (note Y_ℓ^m is a D function)

$$\langle \Theta_{\ell_1 m_1}\ldots\Theta_{\ell_n m_n}\rangle = \sum_{m_1'\ldots m_n'}\langle \Theta_{\ell_1 m_1'}\ldots\Theta_{\ell_n m_n'}\rangle D^{\ell_1}_{m_1 m_1'}\ldots D^{\ell_n}_{m_n m_n'}$$

N-pt correlation

- For any N-point function, combine rotation matrices (group multiplication; angular momentum addition) and orthogonality

$$\sum_m (-1)^{m_2-m} D^{\ell_1}_{m_1 m} D^{\ell_1}_{-m_2 -m} = \delta_{m_1 m_2}$$

- The simplest case is the 2pt function:

$$\langle \Theta_{\ell_1 m_1} \Theta_{\ell_2 m_2} \rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 -m_2} (-1)^{m_1} C_{\ell_1}$$

where C_ℓ is the power spectrum. Check

$$\begin{aligned}
&= \sum_{m_1' m_2'} \delta_{\ell_1 \ell_2} \delta_{m_1' -m_2'} (-1)^{m_1'} C_{\ell_1} D^{\ell_1}_{m_1 m_1'} D^{\ell_2}_{m_2 m_2'} \\
&= \delta_{\ell_1 \ell_2} C_{\ell_1} \sum_{m_1'} (-1)^{m_1'} D^{\ell_1}_{m_1 m_1'} D^{\ell_2}_{m_2 -m_1'} = \delta_{\ell_1 \ell_2} \delta_{m_1 -m_2} (-1)^{m_1} C_{\ell_1}
\end{aligned}$$

N-pt correlation

- Using the reality of the field

$$\langle \Theta^*_{\ell_1 m_1} \Theta_{\ell_2 m_2} \rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} C_{\ell_1} \, .$$

- If the statistics were Gaussian then all the N-point functions would be defined in terms of the products of two-point contractions, e.g.

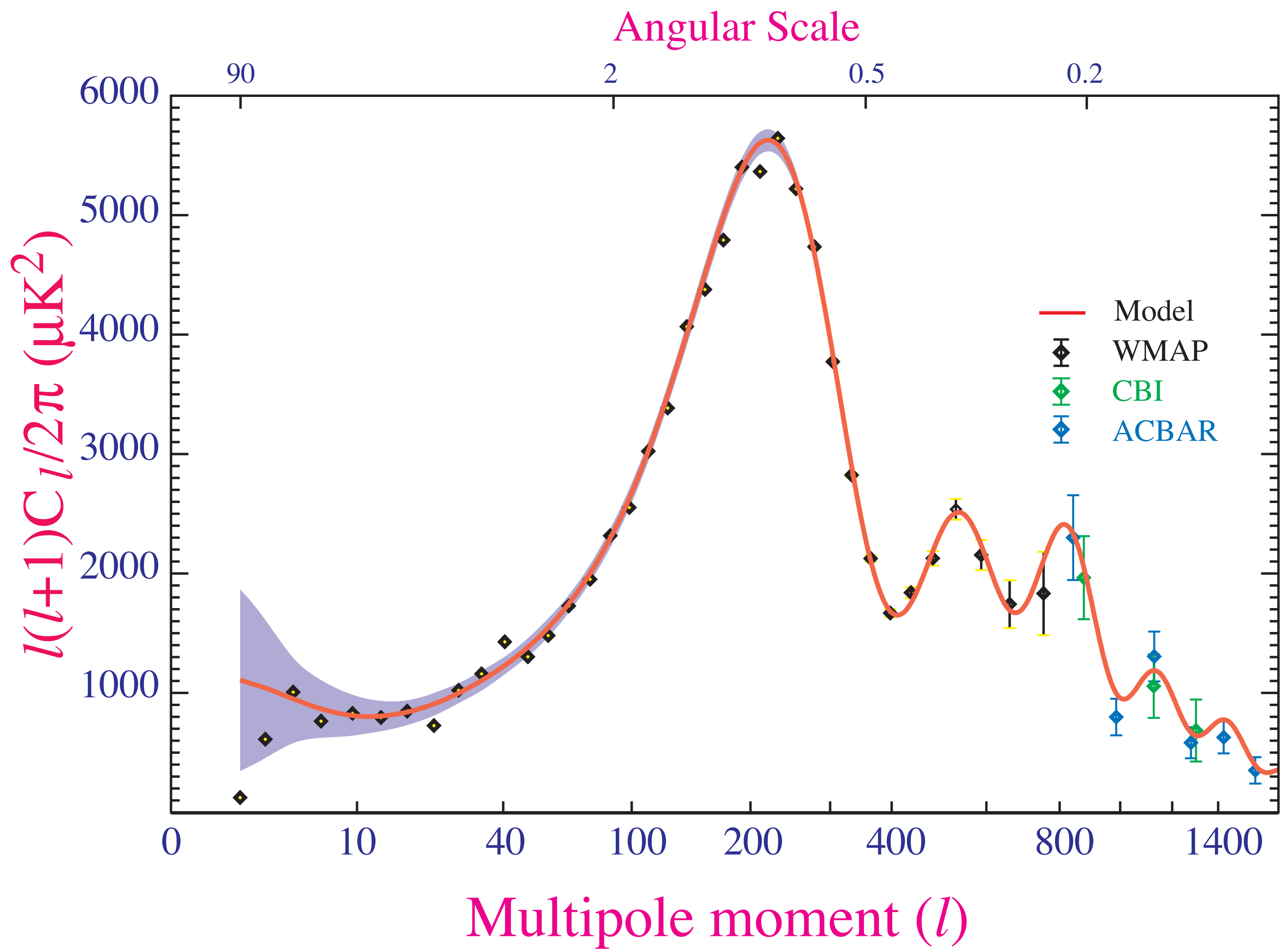
$$\langle \Theta_{\ell_1 m_1} \Theta_{\ell_2 m_2} \Theta_{\ell_3 m_3} \Theta_{\ell_4 m_4} \rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} \delta_{\ell_3 \ell_4} \delta_{m_3 m_4} C_{\ell_1} C_{\ell_3} + \text{perm.}$$

- More generally we can define the isotropy condition beyond Gaussianity, e.g. the bispectrum

$$\langle \Theta_{\ell_1 m_1} \ldots \Theta_{\ell_3 m_3} \rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1 \ell_2 \ell_3}$$

CMB Temperature Fluctuations

- Angular Power Spectrum



Why $\ell^2 C_\ell/2\pi$?

- Variance of the temperature fluctuation field

$$\begin{aligned}
\langle \Theta(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}})\rangle &= \sum_{\ell m}\sum_{\ell' m'} \langle \Theta_{\ell m}\Theta^*_{\ell' m'}\rangle Y_\ell^m(\hat{\mathbf{n}}) Y_{\ell'}^{m' *}(\hat{\mathbf{n}}) \\
&= \sum_\ell C_\ell \sum_m Y_\ell^m(\hat{\mathbf{n}}) Y_\ell^{m*}(\hat{\mathbf{n}}) \\
&= \sum_\ell \frac{2\ell+1}{4\pi} C_\ell
\end{aligned}$$

  via the angle addition formula for spherical harmonics

- For some range $\Delta\ell \approx \ell$ the contribution to the variance is

$$\langle \Theta(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}})\rangle_{\ell\pm\Delta\ell/2} \approx \Delta\ell \frac{2\ell+1}{4\pi} C_\ell \approx \frac{\ell^2}{2\pi} C_\ell$$

- Conventional to use $\ell(\ell+1)/2\pi$ for reasons below

Cosmic Variance

- We only have access to our sky, not the ensemble average
- There are $2\ell + 1$ m-modes of given ℓ mode, so average

$$\hat{C}_\ell = \frac{1}{2\ell+1} \sum_m \Theta^*_{\ell m} \Theta_{\ell m}$$

- $\langle \hat{C}_\ell \rangle = C_\ell$ but now there is a cosmic variance

$$\sigma^2_{C_\ell} = \frac{\langle (\hat{C}_\ell - C_\ell)(\hat{C}_\ell - C_\ell) \rangle}{C_\ell^2} = \frac{\langle \hat{C}_\ell \hat{C}_\ell \rangle - C_\ell^2}{C_\ell^2}$$

- For Gaussian statistics

$$\begin{aligned}
\sigma^2_{C_\ell} &= \frac{1}{(2\ell+1)^2 C_\ell^2} \langle \sum_{mm'} \Theta^*_{\ell m} \Theta_{\ell m} \Theta^*_{\ell m'} \Theta_{\ell m'} \rangle - 1 \\
&= \frac{1}{(2\ell+1)^2} \sum_{mm'} (\delta_{mm'} + \delta_{m-m'}) = \frac{2}{2\ell+1}
\end{aligned}$$

Cosmic Variance

- Note that the distribution of $\hat{C}_\ell$ is that of a sum of squares of Gaussian variates
- Distributed as a χ^2 of $2\ell + 1$ degrees of freedom
- Approaches a Gaussian for $2\ell + 1 \to \infty$ (central limit theorem)
- Anomalously low quadrupole is not that unlikely
- σ_{C_ℓ} is a useful quantification of errors at high ℓ
- Suppose C_ℓ depends on a set of cosmological parameters c_i then we can estimate errors of c_i measurements by error propagation

$$
\begin{aligned}
F_{ij} &= \mathrm{Cov}^{-1}(c_i, c_j) = \sum_{\ell\ell'} \frac{\partial C_\ell}{\partial c_i} \mathrm{Cov}^{-1}(\mathrm{C}_\ell, \mathrm{C}_{\ell'}) \frac{\partial C_{\ell'}}{\partial c_j} \\
&= \sum_\ell \frac{(2\ell+1)}{2C_\ell^2} \frac{\partial C_\ell}{\partial c_i} \frac{\partial C_\ell}{\partial c_j}
\end{aligned}
$$

Idealized Statistical Errors

- Take a noisy estimator of the multipoles in the map

$$\hat{\Theta}_{\ell m} = \Theta_{\ell m} + N_{\ell m}$$

and take the noise to be statistically isotropic

$$\langle N^*_{\ell m} N_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell^{NN}$$

- Construct an unbiased estimator of the power spectrum $\langle \hat{C}_\ell \rangle = C_\ell$

$$\hat{C}_\ell = \frac{1}{2\ell + 1} \sum_{m=-l}^{l} \hat{\Theta}^*_{\ell m} \hat{\Theta}_{\ell m} - C_\ell^{NN}$$

- Covariance in estimator

$$\mathrm{Cov}(C_\ell, C_{\ell'}) = \frac{2}{2\ell + 1} (C_\ell + C_\ell^{NN})^2 \delta_{\ell\ell'}$$

Incomplete Sky

- On a small section of sky, the number of independent modes of a given ℓ is no longer $2\ell + 1$
- As in Fourier analysis, there are two limitations: the lowest ℓ mode that can be measured is the wavelength that fits in angular patch θ

$$\ell_{\rm min} = \frac{2\pi}{\theta};$$

  modes separated by $\Delta\ell < \ell_{\rm min}$ cannot be measured independently
- Estimates of C_ℓ covary on a scale imposed by $\Delta\ell < \ell_{\rm min}$
- Crude approximation: account only for the loss of independent modes by rescaling the errors rather than introducing covariance

$$\mathrm{Cov}(C_\ell, C_{\ell'}) = \frac{2}{(2\ell+1) f_{\rm sky}} (C_\ell + C_\ell^{NN})^2 \delta_{\ell\ell'}$$