Astro 448

Set 3: Boltzmann Equation

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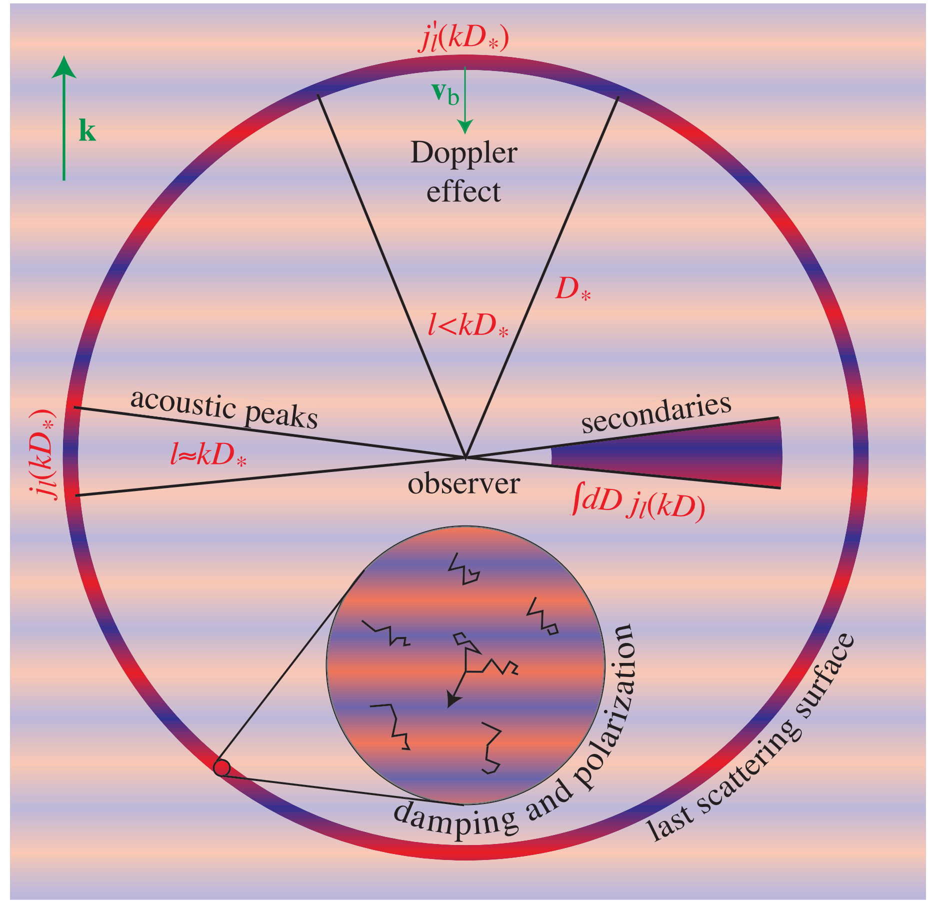
Inhomogeneity vs Anisotropy

- Θ is a function of position as well as direction but we only have access to our position
- Light travels at the speed of light so the radiation we receive in direction $\hat{\mathbf{n}}$ was $(\eta_0 - \eta)\hat{\mathbf{n}}$ at conformal time η
- Inhomogeneity at a distance appears as an anisotopy to the observer
- We need to transport the radiation from the initial conditions to the observer
- This is done with the Boltzmann or radiative transfer equation
- In the absence of scattering, emission or absorption the Boltzmann equation is simply

$$\frac{Df}{Dt} = 0$$

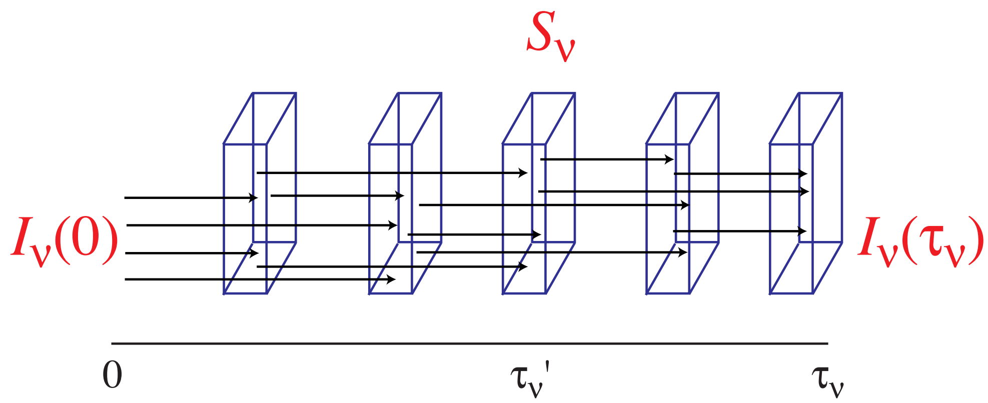
Last Scattering

- Angular distribution of radiation is the 3D temperature field projected onto a shell - surface of last scattering
- Shell radius is distance from the observer to recombination: called the last scattering surface



- Take the radiation distribution at last scattering to also be described by an isotropic temperature fluctuation field $\Theta(\mathbf{x})$

Integral Solution to Radiative Transfer



- Recall (see Ast 305): formal solution for $I_\nu = 2h\nu^3 f/c^2$

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} d\tau'_\nu S_\nu(\tau'_\nu)e^{-(\tau_\nu - \tau'_\nu)}$$

- Specific intensity I_ν attenuated by absorption and replaced by source function, attenuated by absorption from foreground matter
- Here Θ plays the role of specific intensity and $\tau_\nu - \tau'_\nu = \tau$ is optical depth to Compton scattering from $\mathbf{x} = \mathbf{0}$ to $D\hat{\mathbf{n}}$

Angular Power Spectrum

- Take recombination to be instantaneous: $d\tau e^{-\tau} = dD\delta(D - D_*)$ and the source to be the local temperature inhomogeneity

$$\Theta(\hat{\mathbf{n}}) = \int dD\, \Theta(\mathbf{x})\delta(D - D_*)$$

where D is the comoving distance and D_* denotes recombination.

- Describe the temperature field by its Fourier moments

$$\Theta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

- Note that Fourier moments $\Theta(\mathbf{k})$ have units of volume k^{-3}
- 2 point statistics of the real-space field are translationally and rotationally invariant
- Described by power spectrum

Spatial Power Spectrum

- Translational invariance

$$\langle \Theta(\mathbf{x}')\Theta(\mathbf{x})\rangle = \langle \Theta(\mathbf{x}'+\mathbf{d})\Theta(\mathbf{x}+\mathbf{d})\rangle$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\langle \Theta^*(\mathbf{k}')\Theta(\mathbf{k})\rangle e^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{x}'}$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\langle \Theta^*(\mathbf{k}')\Theta(\mathbf{k})\rangle e^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{x}'+i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{d}}$$

So two point function requires $\delta(\mathbf{k}-\mathbf{k}')$; rotational invariance says coefficient depends only on magnitude of k not it's direction

$$\langle \Theta(\mathbf{k})^*\Theta(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P_T(k)$$

Note that $\delta(\mathbf{k}-\mathbf{k}')$ has units of volume and so P_T must have units of volume

Dimensionless Power Spectrum

- Variance

$$\begin{aligned}\sigma_\Theta^2 \equiv \langle \Theta(\mathbf{x})\Theta(\mathbf{x})\rangle &= \int \frac{d^3k}{(2\pi)^3} P_T(k) \\ &= \int \frac{k^2 dk}{2\pi^2} \int \frac{d\Omega}{4\pi} P_T(k) \\ &= \int d\ln k \frac{k^3}{2\pi^2} P_T(k)\end{aligned}$$

- Define power per logarithmic interval

$$\Delta_T^2(k) \equiv \frac{k^3 P_T(k)}{2\pi^2}$$

- This quantity is dimensionless.

Angular Power Spectrum

- Temperature field

$$\Theta(\hat{\mathbf{n}}) = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) e^{i\mathbf{k}\cdot D_* \hat{\mathbf{n}}}$$

- Multipole moments $\Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m}$
- Expand out plane wave in spherical coordinates

$$e^{i\mathbf{k}D_* \cdot \hat{\mathbf{n}}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kD_*) Y^*_{\ell m}(\mathbf{k}) Y_{\ell m}(\hat{\mathbf{n}})$$

- Angular moment

$$\Theta_{\ell m} = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) 4\pi i^\ell j_\ell(kD_*) Y^*_{\ell m}(\mathbf{k})$$

Angular Power Spectrum

- Power spectrum

$$\langle \Theta^*_{\ell m} \Theta_{\ell' m'} \rangle = \int \frac{d^3k}{(2\pi)^3} (4\pi)^2 i^{\ell - \ell'} j_\ell(kD_*) j_{\ell'}(kD_*) Y_{\ell m}(\mathbf{k}) Y^*_{\ell' m'}(\mathbf{k}) P_T(k)$$

$$= \delta_{\ell\ell'} \delta_{mm'} 4\pi \int d\ln k \, j^2_\ell(kD_*) \Delta^2_T(k)$$

with $\int_0^\infty j_\ell^2(x) d\ln x = 1/(2\ell(\ell+1))$, slowly varying Δ^2_T

- Angular power spectrum:

$$C_\ell = \frac{4\pi \Delta^2_T(\ell/D_*)}{2\ell(\ell+1)} = \frac{2\pi}{\ell(\ell+1)} \Delta^2_T(\ell/D_*)$$

- Not surprisingly, a relationship between $\ell^2 C_\ell/2\pi$ and Δ^2_T at $\ell \gg 1$. By convention use $\ell(\ell+1)$ to make relationship exact

Generalized Source

- More generally, we know the Y_ℓ^m's are a complete angular basis and plane waves are complete spatial basis
- General distribution can be decomposed into

$$Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k}\cdot\mathbf{x})$$

- The observer at the origin sees this distribution in projection

$$Y_\ell^m(\hat{\mathbf{n}}) e^{i\mathbf{k}D_*\cdot\hat{\mathbf{n}}} = 4\pi \sum_{\ell' m'} i^{\ell'} j_{\ell'}(kD_*) Y_{\ell'}^{m'*}(\mathbf{k}) Y_{\ell'}^{m'}(\hat{\mathbf{n}}) Y_\ell^m(\hat{\mathbf{n}})$$

- We extract the observed multipoles by the addition of angular momentum $Y_{\ell'}^{m'}(\hat{\mathbf{n}}) Y_\ell^m(\hat{\mathbf{n}}) \to Y_L^M(\hat{\mathbf{n}})$
- Radial functions become linear sums over j_ℓ with the recoupling (Clebsch-Gordan) coefficients
- Formal integral solution to the radiative transfer equation

Boltzmann Equation

- General integral solution for radiative transfer as long as the angular distribution at emission is known
- Formalize further the evolution of angular moments in the cosmological context:

$$\frac{Df}{Dt} = \dot{f} + \dot{\mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{q}} + \dot{\mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0$$

- Momentum $\mathbf{q} = q\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a directional unit vector and in a flat universe $\dot{\mathbf{q}} = \dot{q}\hat{\mathbf{n}}$
- Particle velocity $\dot{\mathbf{x}} = \mathbf{q}/E$

$$\dot{f} + \dot{q}\frac{\partial f}{\partial q} + \frac{\mathbf{q}}{E} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0$$

Boltzmann Moments

- Multipole moments in direction correspond to integrals over $\mathbf{q}$
- First two moments are just energy and momentum conservation
- Stress energy tensor (kept general for finite mass particle)

$$T^{\mu\nu} = g \int \frac{d^3 q}{(2\pi)^3} \frac{q^\mu q^\nu}{E} f$$

- Energy density and pressure

$$\rho(\mathbf{x}, t) \equiv g \int \frac{d^3 q}{(2\pi)^3} E f, \quad p(\mathbf{x}, t) \equiv g \int \frac{d^3 q}{(2\pi)^3} \frac{|\mathbf{q}|^2}{3E} f$$

- Momentum density and anisotropic stress

$$(\rho + p)\mathbf{v} \equiv g \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} f, \quad p\delta^i{}_j + \pi^i{}_j \equiv g \int \frac{d^3 q}{(2\pi)^3} \frac{q^i q_j}{E} f$$

Gravitational Change of Momentum

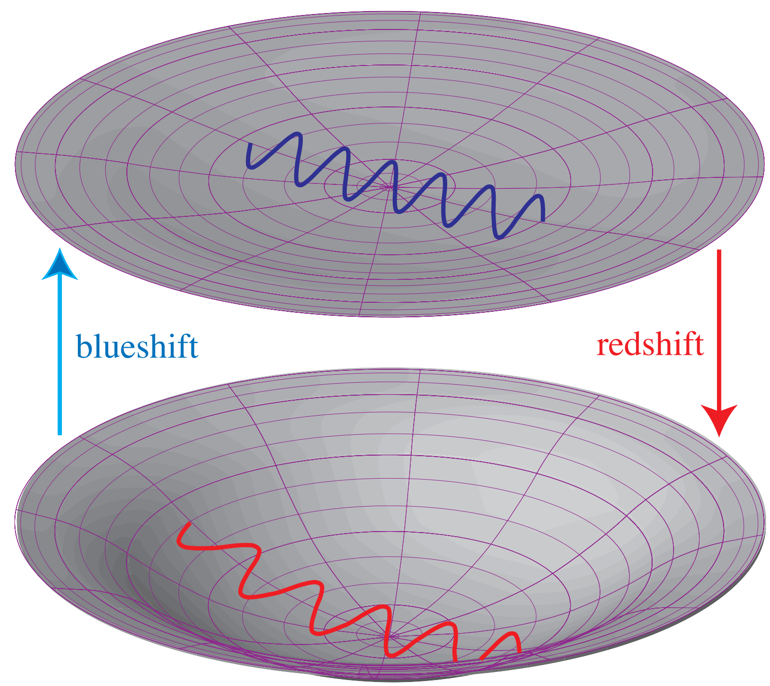
- Momentum term: carries two contributions
- Consider the perturbed FRW line element to take the form

$$d\tau^2 = a^2[(1+2\Psi)d\eta^2 - (1+2\Phi)(dD^2 + D_A^2 d\Omega)]$$

where $|\Phi| \ll 1$ and $|\Psi| \ll 1$ and $D_A = R\sin(D/R)$ is the angular diameter distance

- Just as the background scale factor changes the de Broglie wavelength of particles, a perturbation to the scale factor (or spatial curvature)

$$a(\mathbf{x}) = a(1+\Phi)$$



Gravitational Change of Momentum

- So Φ gives a time dependence to the momentum through

$$\dot{a}(\mathbf{x}) = \dot{a}(1+\Phi) + a\dot{\Phi}$$

$$\frac{\dot{a}(\mathbf{x})}{a(\mathbf{x})} \approx \frac{\dot{a}}{a} + \frac{\dot{\Phi}}{1+\Phi} \approx \frac{\dot{a}}{a} + \dot{\Phi}$$

- Contribution from the spatial metric (independent of direction)

$$\dot{q} = -\left(\frac{\dot{a}}{a} + \dot{\Phi}\right) q$$

- Second term comes from Ψ which plays the role of the gravitational potential
- Non-relativistic: gravitational force changes momentum

$$\dot{\mathbf{q}} = \mathbf{F} = -m\nabla\Psi \quad \rightarrow \quad \dot{q} = \hat{\mathbf{n}} \cdot \dot{\mathbf{q}} = -m(\hat{\mathbf{n}} \cdot \nabla\Psi)$$

Gravitational Change of Momentum

- Ultra-Relativistic: time dilation implies shift of frequency or gravitational redshift and hence momentum

$$\frac{\Delta q}{q} = -\Delta\Psi$$

Rate of change from moving through a Ψ gradient is

$$\frac{\dot{q}}{q} = -\dot{\mathbf{x}} \cdot \nabla\Psi = -\hat{\mathbf{n}} \cdot \nabla\Psi$$

- In both relativistic and non-relativistic cases

$$\dot{q} = -E(\hat{\mathbf{n}} \cdot \nabla\Psi)$$

- Combining the two momentum terms

$$\dot{q} = -\left(\frac{\dot{a}}{a} + \dot{\Phi}\right) q - (\hat{\mathbf{n}} \cdot \nabla\Psi)E$$

Energy or Continuity Equation

- Integrate Boltzmann equation over E

$$g \int \frac{d^3 q}{(2\pi)^3} E \left(\dot{f} + \dot{q} \frac{\partial f}{\partial q} + \frac{\mathbf{q}}{E} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0 \right)$$

- Time term

$$g \int \frac{d^3 q}{(2\pi)^3} E \dot{f} = \dot{\rho}$$

- Momentum terms

$$g \int \frac{d^3 q}{(2\pi)^3} \dot{q} E \frac{\partial f}{\partial q} = g \int \frac{d^3 q}{(2\pi)^3} \left[- \left(\frac{\dot{a}}{a} + \dot{\Phi} \right) q - (\hat{\mathbf{n}} \cdot \nabla \Psi) E \right] E \frac{\partial f}{\partial q}$$

second term vanishes by symmetry integrating over momenta direction

Energy or Continuity Equation

- First term remains

$$g \int \frac{d^3 q}{(2\pi)^3} \dot{q} E \frac{\partial f}{\partial q} = -\left(\frac{\dot{a}}{a} + \dot{\Phi} \right) g \int \frac{d^3 q}{(2\pi)^3} q E \frac{\partial f}{\partial q}$$

- Integrate by parts

$$g \int \frac{d^3 q}{(2\pi)^3} q E \frac{\partial f}{\partial q} = g \int \frac{d\Omega_q dq q^2}{(2\pi)^3} q E \frac{\partial f}{\partial q} = -g \int \frac{d\Omega_q dq}{(2\pi)^3} \left(\frac{d}{dq} q^3 E \right) f$$

$$= -g \int \frac{d\Omega_q dq}{(2\pi)^3} (3q^2 E + q^3 \frac{dE}{dq}) f$$

$$= -g \int \frac{d\Omega_q dq}{(2\pi)^3} (3q^2 E + \frac{q^4}{E}) f = -3(\rho + p)$$

using $d(E^2 = q^2 + m^2) \rightarrow EdE = qdq$

Energy or Continuity Equation

- So

$$g \int \frac{d^3q}{(2\pi)^3} qE \frac{\partial f}{\partial q} = \left(\frac{\dot{a}}{a} + \dot{\Phi} \right) 3(\rho + p)$$

- Position term: define average momentum as momentum density

$$\nabla \cdot g \int \frac{d^3q}{(2\pi)^3} \mathbf{q} f \equiv \nabla \cdot (\rho + p)\mathbf{v}$$

- Linearized energy/continuity equation

$$\dot{\rho} = -3 \left(\frac{\dot{a}}{a} + \dot{\Phi} \right) (\rho + p) - \nabla \cdot (\rho + p)\mathbf{v}$$

- Local energy density changes due to: global expansion, local change in expansion, flows of particles into/out of volume

Momentum or Navier-Stokes Equation

- Continuity equation is not closed since it involves the next higher moment: momentum density
- Integrate Boltzmann equation over $\mathbf{q}$

$$g \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} \left(\dot{f} + \dot{q} \frac{\partial f}{\partial q} + \frac{\mathbf{q}}{E} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0 \right)$$

- Time term

$$\frac{\partial}{\partial \eta} \to \frac{\partial}{\partial \eta} [(\rho + p)\mathbf{v}]$$

- Momentum term: de Broglie redshift

$$-[\frac{\dot{a}}{a} + \dot{\Phi}] g \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} q \frac{\partial f}{\partial q} = 4[\frac{\dot{a}}{a} + \dot{\Phi}] g \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} f$$

$$= 4 \left[\frac{\dot{a}}{a} + \dot{\Phi} \right] (\rho + p)\mathbf{v} \approx 4 \frac{\dot{a}}{a} (\rho + p)\mathbf{v}$$

Momentum Equation

- Momentum term: gravitational potential jth component

$$-\partial_i \Psi \cdot g \int \frac{d^3 q}{(2\pi)^3} q E n_j n^i \frac{\partial f}{\partial q} \approx \partial_j \Psi (\rho + p)$$

where angle averaged $\langle n^i n_j \rangle = \frac{1}{3} \delta^i{}_j$ and used relation from homogeneous energy equation

- Spatial term: recall stress tensor divided into isotropic and anisotropic pieces

$$g \int \frac{d^3 q}{(2\pi)^3} \frac{q^i q_j}{E} f \equiv p \delta^i{}_j + \pi^i{}_j$$

- Combined momentum terms

$$\frac{\partial}{\partial \eta} [(\rho + p) v^i] = -4 \frac{\dot{a}}{a} (\rho + p) v^i - \nabla^i p - \nabla^j \pi^i{}_j - (\rho + p) \nabla^i \Psi$$

Linear Perturbation Theory

- Energy (continuity) and momentum (Navier-Stokes) equations are linearized and hence Fourier modes obey

$$\frac{\partial}{\partial\eta}[(\rho+p)v^i] = -4\frac{\dot{a}}{a}(\rho+p)v^i + ikp + ik^j\pi^i{}_j + ik^i(\rho+p)\Psi$$

- If the source of perturbations is from the (scalar) gravitational potential, directional dependence of velocity and anisotropic stress follows the direction of the plane wave, so define scalar velocity and anisotropic stress as

$$\mathbf{v}(\mathbf{k}) = i\hat{\mathbf{k}}v$$

$$\pi^i{}_j(\mathbf{k}) = \left(-\hat{k}^i\hat{k}_j + \frac{1}{3}\delta^i{}_j\right)p\pi$$

Linear Perturbation Theory

- Navier-Stokes equation

$$\frac{\partial}{\partial\eta}[(\rho+p)v] = -4\frac{\dot{a}}{a}(\rho+p)v^i + kp - \frac{2}{3}kp\pi + (\rho+p)k\Psi$$

$$(w = p/\rho\,, \quad c_s^2 = \delta p/\delta\rho\,, \quad \dot{\rho}/\rho = -3(1+w)\dot{a}/a)$$

$$\dot{v} = -(1-3w)\frac{\dot{a}}{a}v - \frac{\dot{w}}{1+w}v + \frac{kc_s^2}{1+w}\delta - \frac{2}{3}\frac{w}{1+w}k\pi + k\Psi$$

- Continuity Equation

$$\dot{\rho} = -3\left[\frac{\dot{a}}{a} + \dot{\Phi}\right](\rho+p) + i\mathbf{k}\cdot(\rho+p)\mathbf{v}$$

$$\dot{\rho} = -3\left[\frac{\dot{a}}{a} + \dot{\Phi}\right](\rho+p) - k(\rho+p)v$$

$$\dot{\delta} = -3\frac{\dot{a}}{a}(c_s^2 - w)\delta - (1+w)(kv + 3\dot{\Phi})$$

Poisson Equation

- Naive expectation: $\Phi = -\Psi$ and source by the sum over all particle components

$$\nabla^2 \Phi = -4\pi G a^2 \sum_i \delta\rho_i$$
$$k^2 \Phi = 4\pi G a^2 \sum_i \rho_i \delta_i$$

where a^2 comes from physical $\to$ comoving and $\delta\rho_i$ since background densities go into scale factor evolution

Poisson Equation

- Einstein equations put in a relativistic correction (flat universe)

$$k^2\Phi = 4\pi G a^2 \sum_i \rho_i[\delta_i + 3\frac{\dot{a}}{a}(1+w_i)v_i/k]$$

$$k^2(\Phi+\Psi) = -8\pi G a^2 \sum_i p_i \pi_i$$

convenient to call combination

$$\Delta_i \equiv \delta_i + 3\frac{\dot{a}}{a}(1+w_i)v_i/k$$

Boltzmann Hierarchy

- Momentum equation is Navier-Stokes equation and requires knowledge of the second moments: stress tensor including the anisotropic stress ("viscosity")
- In general, the time derivative of a low order moment of the Boltzmann equation is given by the spatial gradient of the next higher order moment due to term

$$\frac{\mathbf{q}}{E} \cdot \frac{\partial f}{\partial \mathbf{x}}$$

- In Fourier space $\partial/\partial \mathbf{x} = i\mathbf{k}$ so in the frame of $\hat{\mathbf{k}}$ each operation consists of integrating over an additional

$$\hat{\mathbf{n}} \cdot \mathbf{k} = \cos\theta$$

Boltzmann Hierarchy

- For the CMB one adds the Compton collision term
- Isotropization from scattering closes the angular moment hierarchy at $\ell = 2$
- As medium becomes optically thin, one continues the moment hierarchy: each $\hat{\mathbf{n}} \cdot \nabla\Theta$ term brings with it a dipole coupling with the previous moment: $Y_1^0 Y_\ell^m \to Y_{\ell\pm1}^m$
- Use solution of truncated hierarchy to define the source functions for the integral solution