

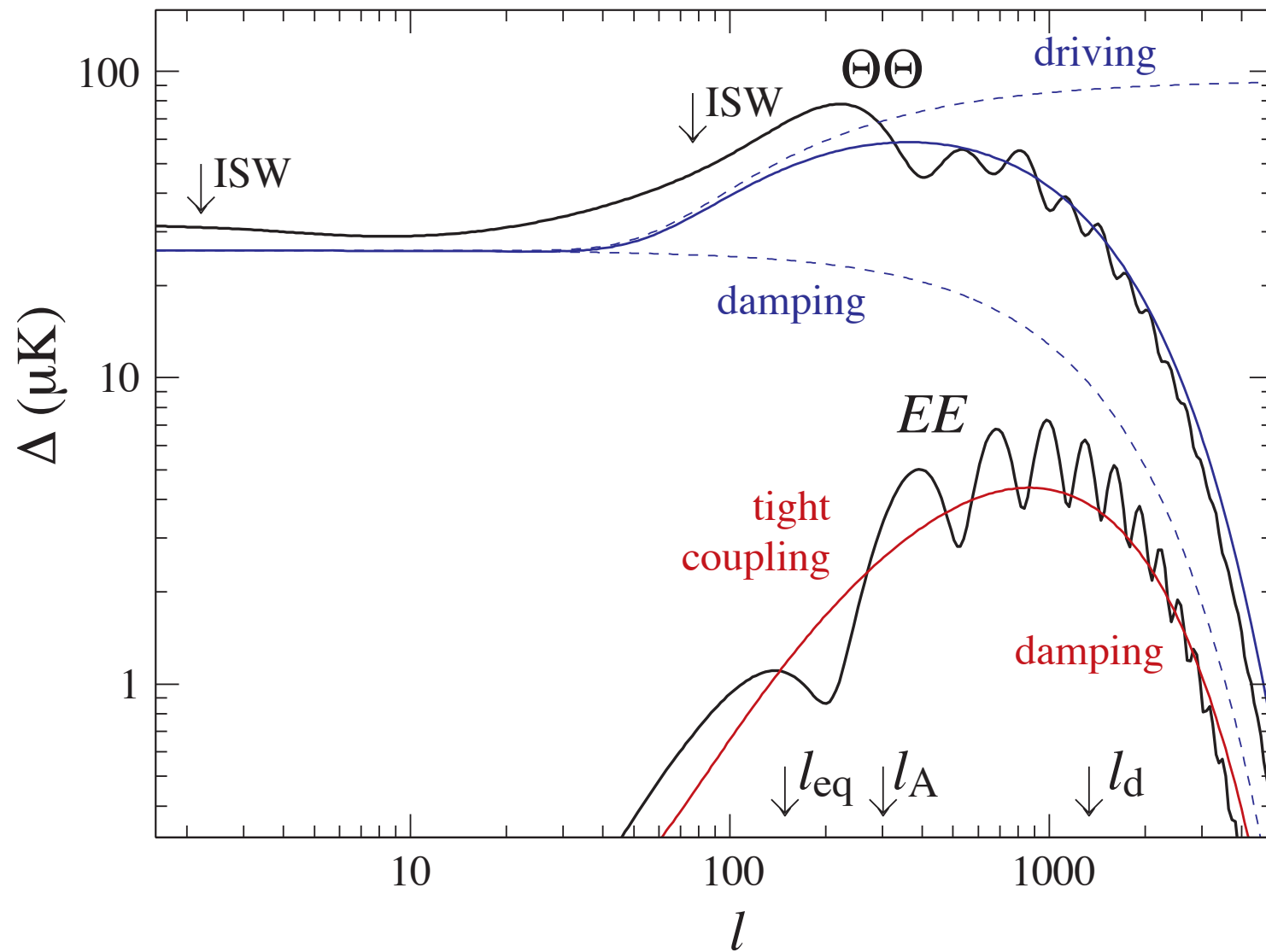
Astro 448

Set 4:  $C_\ell$  Basics

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# Schematic Outline

- Take apart features in the power spectrum



# Thomson Scattering

- Thomson scattering of photons off of free electrons is the most important CMB process with a cross section (averaged over polarization states) of

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25} \text{cm}^2$$

- Density of free electrons in a fully ionized  $x_e = 1$  universe

$$n_e = (1 - Y_p/2)x_e n_b \approx 10^{-5} \Omega_b h^2 (1 + z)^3 \text{cm}^{-3},$$

where  $Y_p \approx 0.24$  is the Helium mass fraction, creates a high (comoving) Thomson opacity

$$\dot{\tau} \equiv n_e \sigma_T a$$

where dots are conformal time  $\eta \equiv \int dt/a$  derivatives and  $\tau$  is the optical depth.

# Tight Coupling Approximation

- Near **recombination**  $z \approx 10^3$  and  $\Omega_b h^2 \approx 0.02$ , the (comoving) **mean free path** of a photon

$$\lambda_C \equiv \frac{1}{\dot{\tau}} \sim 2.5 \text{Mpc}$$

small by cosmological standards!

- On scales  $\lambda \gg \lambda_C$  photons are **tightly coupled** to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions
- Specifically, their bulk velocities are defined by a **single fluid velocity**  $v_\gamma = v_b$  and the photons carry **no anisotropy** in the rest frame of the baryons
- $\rightarrow$  No **heat conduction** or **viscosity** (anisotropic stress) in fluid

# Zeroth Order Approximation

- Momentum density of a fluid is  $(\rho + p)v$ , where  $p$  is the pressure
- Neglect the momentum density of the baryons

$$R \equiv \frac{(\rho_b + p_b)v_b}{(\rho_\gamma + p_\gamma)v_\gamma} = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}$$
$$\approx 0.6 \left( \frac{\Omega_b h^2}{0.02} \right) \left( \frac{a}{10^{-3}} \right)$$

since  $\rho_\gamma \propto T^4$  is fixed by the CMB temperature  $T = 2.73(1 + z)\text{K}$   
– OK substantially before recombination

- Neglect radiation in the expansion

$$\frac{\rho_m}{\rho_r} = 3.6 \left( \frac{\Omega_m h^2}{0.15} \right) \left( \frac{a}{10^{-3}} \right)$$

- Neglect gravity

# Fluid Equations

- Density  $\rho_\gamma \propto T^4$  so define temperature fluctuation  $\Theta$

$$\delta_\gamma = 4 \frac{\delta T}{T} \equiv 4\Theta$$

- Real space continuity equation

$$\dot{\delta}_\gamma = -(1 + w_\gamma) k v_\gamma$$

$$\dot{\Theta} = -\frac{1}{3} k v_\gamma$$

- Euler equation (neglecting gravity)

$$\dot{v}_\gamma = -(1 - 3w_\gamma) \frac{\dot{a}}{a} v_\gamma + \frac{k c_s^2}{1 + w_\gamma} \delta_\gamma$$

$$\dot{v}_\gamma = k c_s^2 \frac{3}{4} \delta_\gamma = 3 c_s^2 k \Theta$$

# Oscillator: Take One

- Combine these to form the simple harmonic oscillator equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = 0$$

where the sound speed is adiabatic

$$c_s^2 = \frac{\delta p_\gamma}{\delta \rho_\gamma} = \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma}$$

here  $c_s^2 = 1/3$  since we are photon-dominated

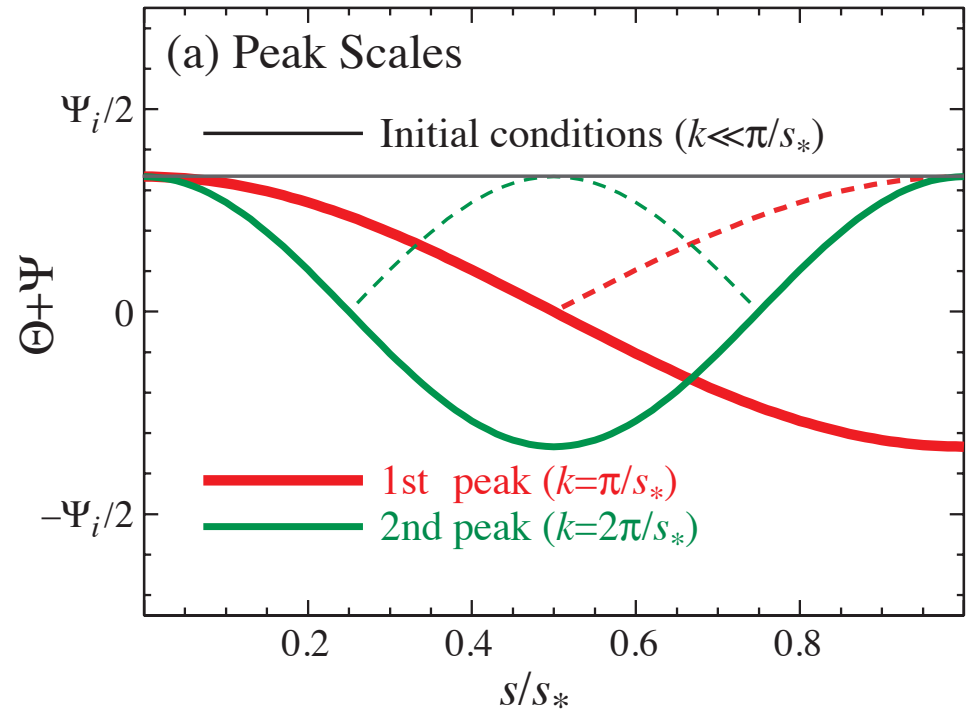
- General solution:

$$\Theta(\eta) = \Theta(0) \cos(k s) + \frac{\dot{\Theta}(0)}{k c_s} \sin(k s)$$

where the sound horizon is defined as  $s \equiv \int c_s d\eta$

# Harmonic Extrema

- All modes are **frozen** in at recombination (denoted with a subscript  $*$ )
- Temperature perturbations of **different amplitude** for different modes.
- For the adiabatic (curvature mode) initial conditions



$$\dot{\Theta}(0) = 0$$

- So solution

$$\Theta(\eta_*) = \Theta(0) \cos(k s_*)$$

# Harmonic Extrema

- Modes caught in the extrema of their oscillation will have enhanced fluctuations

$$k_n s_* = n\pi$$

yielding a fundamental scale or frequency, related to the inverse sound horizon

$$k_A = \pi / s_*$$

and a harmonic relationship to the other extrema as 1 : 2 : 3...

# Peak Location

- The fundamental **physical scale** is translated into a fundamental **angular scale** by simple projection according to the angular diameter distance  $D_A$

$$\theta_A = \lambda_A / D_A$$

$$\ell_A = k_A D_A$$

- In a flat universe, the distance is simply  $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$ , the horizon distance, and  $k_A = \pi / s_* = \sqrt{3}\pi / \eta_*$  so

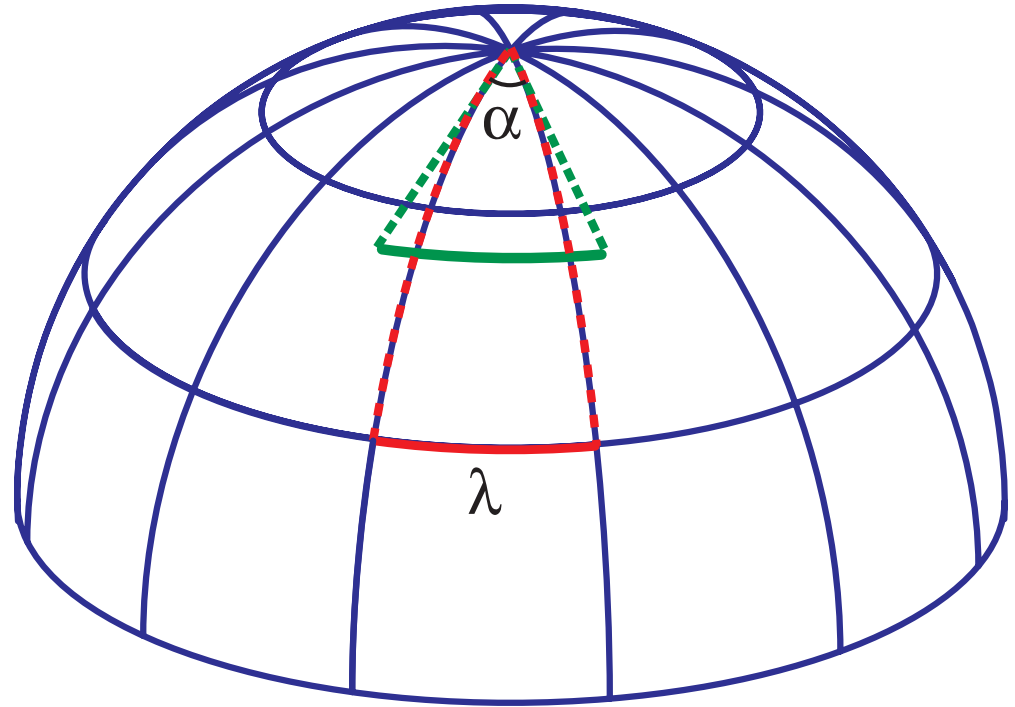
$$\theta_A \approx \frac{\eta_*}{\eta_0}$$

- In a **matter-dominated** universe  $\eta \propto a^{1/2}$  so  $\theta_A \approx 1/30 \approx 2^\circ$  or

$$\ell_A \approx 200$$

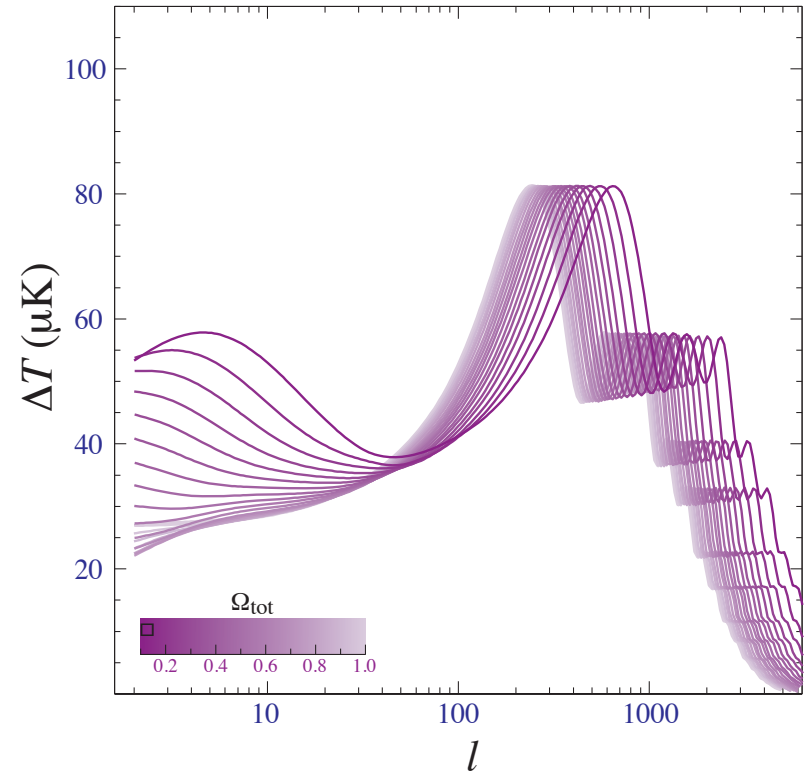
# Curvature

- In a curved universe, the apparent or angular diameter distance is no longer the conformal distance  
$$D_A = R \sin(D/R) \neq D$$
- Objects in a closed universe are further than they appear! gravitational lensing of the background...
- Curvature scale of the universe must be substantially larger than current horizon



# Curvature

- Flat universe indicates critical density and implies missing energy given local measures of the matter density “dark energy”
- $D$  also depends on dark energy density  $\Omega_{\text{DE}}$  and equation of state  $w = p_{\text{DE}}/\rho_{\text{DE}}$ .
- Expansion rate at recombination or matter-radiation ratio enters into calculation of  $k_A$ .



# Fixed Deceleration Epoch

- CMB determination of **matter density** controls all determinations in the **deceleration** (matter dominated) epoch
- **WMAP7**:  $\Omega_m h^2 = 0.133 \pm 0.006 \rightarrow 4.5\%$
- **Distance** to recombination  $D_*$  determined to  $\frac{1}{4}4.5\% \approx 1\%$
- **Expansion rate** during any redshift in the deceleration epoch determined to 4.5%
- **Distance** to **any redshift** in the deceleration epoch determined as

$$D(z) = D_* - \int_z^{z_*} \frac{dz}{H(z)}$$

- **Volumes** determined by a combination  $dV = D_A^2 d\Omega dz / H(z)$
- **Structure** also determined by growth of fluctuations from  $z_*$
- $\Omega_m h^2$  can be determined to  $\sim 1\%$  from Planck.

# Doppler Effect

- Bulk motion of fluid changes the observed temperature via Doppler shifts

$$\left(\frac{\Delta T}{T}\right)_{\text{dop}} = \hat{\mathbf{n}} \cdot \mathbf{v}_\gamma$$

- Averaged over directions

$$\left(\frac{\Delta T}{T}\right)_{\text{rms}} = \frac{v_\gamma}{\sqrt{3}}$$

- Acoustic solution

$$\begin{aligned} \frac{v_\gamma}{\sqrt{3}} &= -\frac{\sqrt{3}}{k} \dot{\Theta} = \frac{\sqrt{3}}{k} k c_s \Theta(0) \sin(ks) \\ &= \Theta(0) \sin(ks) \end{aligned}$$

# Doppler Peaks?

- Doppler effect for the photon dominated system is of equal amplitude and  $\pi/2$  out of phase: extrema of temperature are turning points of velocity
- Effects add in quadrature:

$$\left(\frac{\Delta T}{T}\right)^2 = \Theta^2(0)[\cos^2(ks) + \sin^2(ks)] = \Theta^2(0)$$

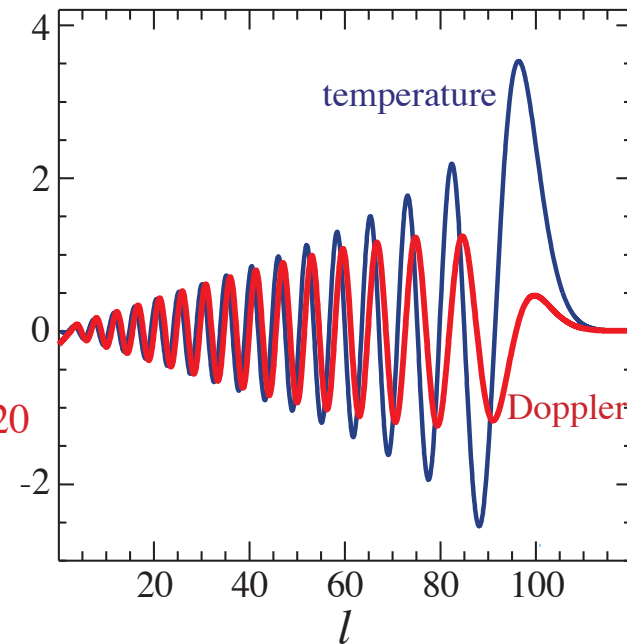
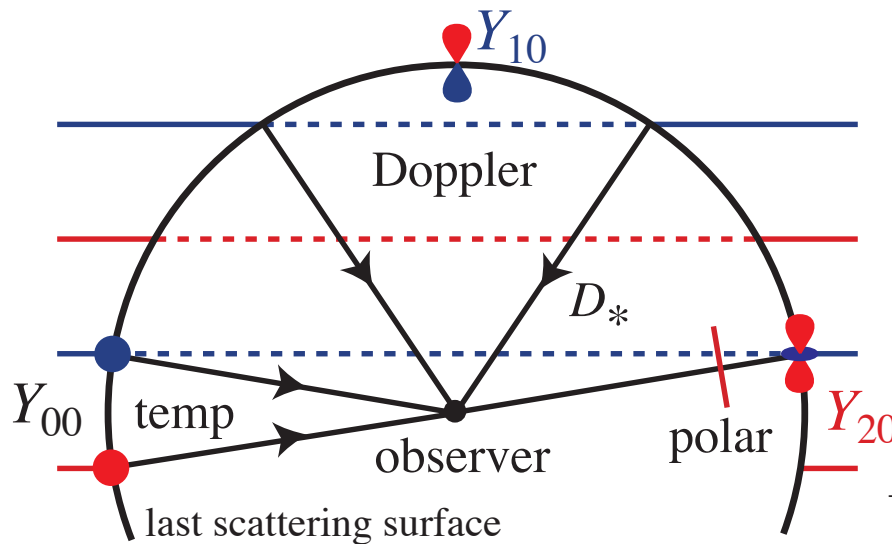
- No peaks in  $k$  spectrum! However the Doppler effect carries an angular dependence that changes its projection on the sky  
 $\hat{\mathbf{n}} \cdot \mathbf{v}_\gamma \propto \hat{\mathbf{n}} \cdot \hat{\mathbf{k}}$

# Doppler Peaks?

- Coordinates where  $\hat{\mathbf{z}} \parallel \hat{\mathbf{k}}$

$$Y_{10}Y_{\ell 0} \rightarrow Y_{\ell \pm 1 0}$$

recoupling  $j'_\ell Y_{\ell 0}$ : no peaks in Doppler effect



# Restoring Gravity

- Take a simple photon dominated system with gravity
- Continuity altered since a gravitational potential represents a stretching of the spatial fabric that dilutes number densities – formally a spatial curvature perturbation
- Think of this as a perturbation to the scale factor  $a \rightarrow a(1 + \Phi)$  so that the cosmological redshift is generalized to

$$\frac{\dot{a}}{a} \rightarrow \frac{\dot{a}}{a} + \dot{\Phi}$$

so that the continuity equation becomes

$$\dot{\Theta} = -\frac{1}{3}k v_{\gamma} - \dot{\Phi}$$

# Restoring Gravity

- Gravitational force in momentum conservation  $\mathbf{F} = -m\nabla\Psi$  generalized to momentum density modifies the Euler equation to

$$\dot{v}_\gamma = k(\Theta + \Psi)$$

- General relativity says that  $\Phi$  and  $\Psi$  are the relativistic analogues of the Newtonian potential and that  $\Phi \approx -\Psi$ .
- In our matter-dominated approximation,  $\Phi$  represents matter density fluctuations through the cosmological Poisson equation

$$k^2\Phi = 4\pi G a^2 \rho_m \Delta_m$$

where the difference comes from the use of comoving coordinates for  $k$  ( $a^2$  factor), the removal of the background density into the background expansion ( $\rho\Delta_m$ ) and finally a coordinate subtlety that enters into the definition of  $\Delta_m$

# Constant Potentials

- In the matter dominated epoch potentials are constant because infall generates velocities as  $v_m \sim k\eta\Psi$
- Velocity divergence generates density perturbations as  $\Delta_m \sim -k\eta v_m \sim -(k\eta)^2\Psi$
- And density perturbations generate potential fluctuations

$$\Phi = \frac{4\pi G a^2 \rho \Delta}{k^2} \approx \frac{3}{2} \frac{H^2 a^2}{k} \Delta \sim \frac{\Delta}{(k\eta)^2} \sim -\Psi$$

keeping them constant. Note that because of the expansion, density perturbations must **grow** to keep potentials constant.

# Constant Potentials

- More generally, if **stress perturbations** are negligible compared with **density perturbations** (  $\delta p \ll \delta \rho$  ) then potential will remain roughly constant
- More specifically a variant called the **Bardeen** or **comoving curvature** is strictly constant

$$\mathcal{R} = \text{const} \approx \frac{5 + 3w}{3 + 3w} \Phi$$

where the approximation holds when  $w \approx \text{const.}$

# Oscillator: Take Two

- Combine these to form the **simple harmonic oscillator** equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - \ddot{\Phi}$$

- In a **CDM dominated** expansion  $\dot{\Phi} = \dot{\Psi} = 0$ . Also for **photon domination**  $c_s^2 = 1/3$  so the oscillator equation becomes

$$\ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0$$

- Solution is just an **offset version** of the original

$$[\Theta + \Psi](\eta) = [\Theta + \Psi](0) \cos(ks)$$

- $\Theta + \Psi$  is also the **observed temperature fluctuation** since photons lose energy climbing out of **gravitational potentials** at recombination

# Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or effective temperature

$$\Theta + \Psi$$

- Effective temperature oscillates around zero with amplitude given by the initial conditions
- Note: initial conditions are set when the perturbation is outside of horizon, need inflation or other modification to matter-radiation FRW universe.
- GR says that initial temperature is given by initial potential

# Sachs-Wolfe Effect and the Magic 1/3

- A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]

$$\frac{\delta t}{t} = \Psi$$

- Convert this to a perturbation in the scale factor,

$$t = \int \frac{da}{aH} \propto \int \frac{da}{a\rho^{1/2}} \propto a^{3(1+w)/2}$$

where  $w \equiv p/\rho$  so that during matter domination

$$\frac{\delta a}{a} = \frac{2}{3} \frac{\delta t}{t}$$

- CMB temperature is cooling as  $T \propto a^{-1}$  so

$$\Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3}\Psi$$

# Sachs-Wolfe Normalization

- Use measurements of  $\Delta T/T \approx 10^{-5}$  in the Sachs-Wolfe effect to infer  $\Delta_{\mathcal{R}}^2$
- Recall in matter domination  $\Psi = -3\mathcal{R}/5$

$$\frac{\ell(\ell+1)C_\ell}{2\pi} \approx \Delta_T^2 \approx \frac{1}{25} \Delta_R^2$$

- So that the amplitude of initial curvature fluctuations is  $\Delta_R \approx 5 \times 10^{-5}$
- Modern usage: WMAP's measurement of 1st peak plus known radiation transfer function is used to convert  $\Delta T/T$  to  $\Delta_R$ .

# Baryon Loading

- Baryons add extra **mass** to the photon-baryon fluid
- Controlling parameter is the **momentum density ratio**:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left( \frac{a}{10^{-3}} \right)$$

of order **unity** at recombination

- Momentum density of the **joint system** is conserved

$$\begin{aligned} (\rho_\gamma + p_\gamma)v_\gamma + (\rho_b + p_b)v_b &\approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma)v_\gamma \\ &= (1 + R)(\rho_\gamma + p_\gamma)v_{\gamma b} \end{aligned}$$

where the controlling parameter is the **momentum density ratio**:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left( \frac{a}{10^{-3}} \right)$$

of order **unity** at recombination

# New Euler Equation

- Momentum density ratio enters as

$$[(1 + R)v_{\gamma b}]^{\cdot} = k\Theta + (1 + R)k\Psi$$

- Photon continuity remains the same

$$\dot{\Theta} = -\frac{k}{3}v_{\gamma b} - \dot{\Phi}$$

- Modification of oscillator equation

$$[(1 + R)\dot{\Theta}]^{\cdot} + \frac{1}{3}k^2\Theta = -\frac{1}{3}k^2(1 + R)\Psi - [(1 + R)\dot{\Phi}]^{\cdot}$$

# Oscillator: Take Three

- Combine these to form the not-quite-so simple harmonic oscillator equation

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})$$

where  $c_s^2 \equiv \dot{p}_{\gamma b} / \dot{\rho}_{\gamma b}$

$$c_s^2 = \frac{1}{3} \frac{1}{1 + R}$$

- In a CDM dominated expansion  $\dot{\Phi} = \dot{\Psi} = 0$  and the adiabatic approximation  $\dot{R}/R \ll \omega = kc_s$

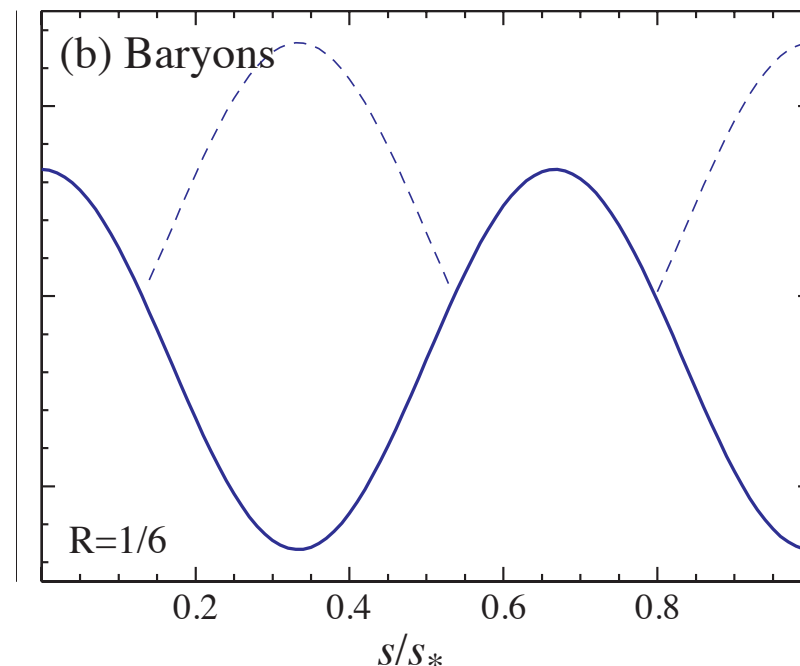
$$[\Theta + (1 + R)\Psi](\eta) = [\Theta + (1 + R)\Psi](0) \cos(k s)$$

# Baryon Peak Phenomenology

- Photon-baryon ratio enters in **three** ways
- Overall larger **amplitude**:

$$[\Theta + (1 + R)\Psi](0) = \frac{1}{3}(1 + 3R)\Psi(0)$$

- Even-odd peak **modulation** of effective temperature



$$[\Theta + \Psi]_{\text{peaks}} = [\pm(1 + 3R) - 3R] \frac{1}{3} \Psi(0)$$

$$[\Theta + \Psi]_1 - [\Theta + \Psi]_2 = [-6R] \frac{1}{3} \Psi(0)$$

- Shifting of the **sound horizon** down or  $\ell_A$  up

$$\ell_A \propto \sqrt{1 + R}$$

# Photon Baryon Ratio Evolution

- Actual effects **smaller** since  $R$  evolves
- Oscillator equation has time **evolving mass**

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = 0$$

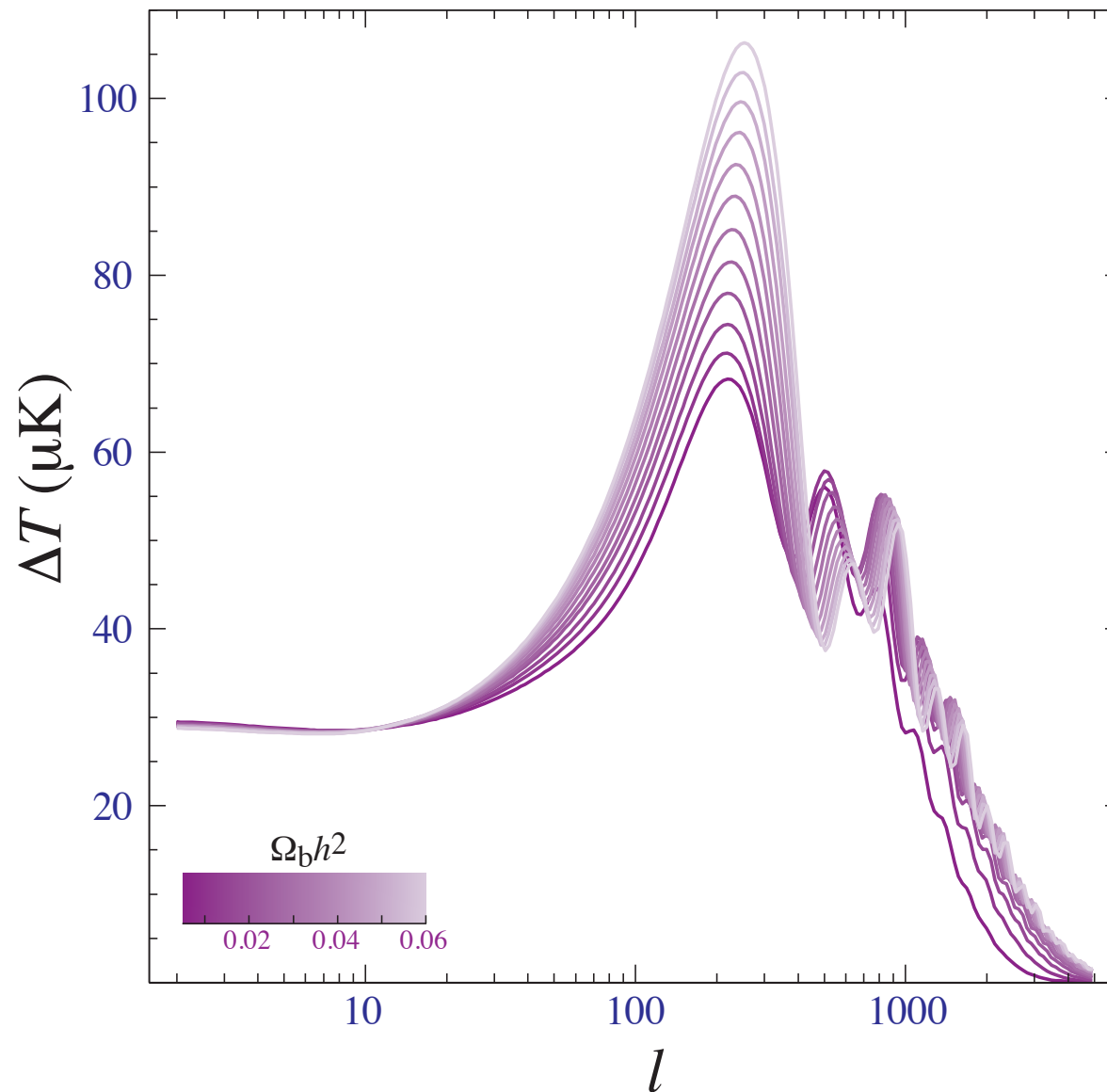
- Effective mass is  $m_{\text{eff}} = 3c_s^{-2} = (1 + R)$
- **Adiabatic invariant**

$$\frac{E}{\omega} = \frac{1}{2} m_{\text{eff}} \omega A^2 = \frac{1}{2} 3c_s^{-2} k c_s A^2 \propto A^2 (1 + R)^{1/2} = \text{const.}$$

- Amplitude of oscillation  $A \propto (1 + R)^{-1/4}$  **decays adiabatically** as the photon-baryon ratio changes

# Baryons in the Power Spectrum

- Relative heights of peaks



# Oscillator: Take Three and a Half

- The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \Phi)$$

changes in the gravitational potentials alter the form of the acoustic oscillations

- If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator
- Term involving  $\Psi$  is the ordinary gravitational force
- Term involving  $\Phi$  involves the  $\dot{\Phi}$  term in the continuity equation as a (curvature) perturbation to the scale factor

# Potential Decay

- Matter-to-radiation ratio

$$\frac{\rho_m}{\rho_r} \approx 24 \Omega_m h^2 \left( \frac{a}{10^{-3}} \right)$$

of order **unity** at recombination in a low  $\Omega_m$  universe

- Radiation is not stress free and so **impedes** the growth of structure

$$k^2 \Phi = 4\pi G a^2 \rho_r \Delta_r$$

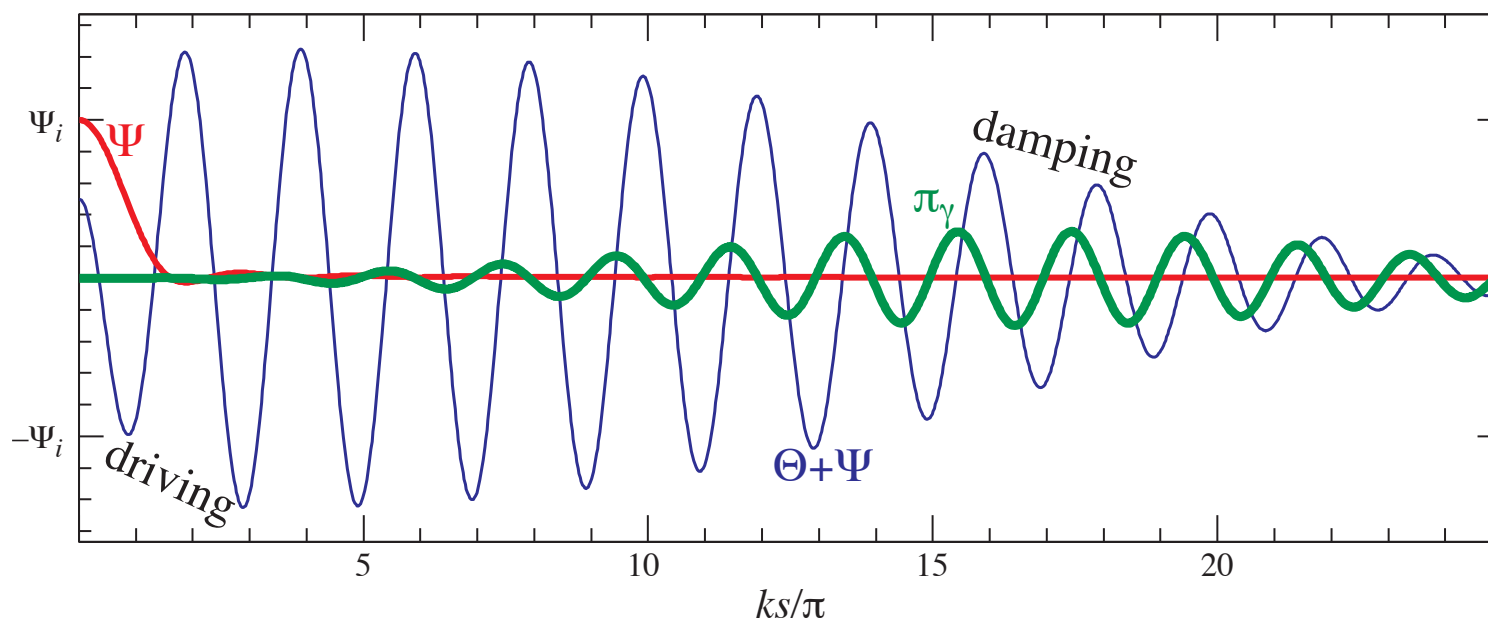
$\Delta_r \sim 4\Theta$  **oscillates** around a constant value,  $\rho_r \propto a^{-4}$  so the Newtonian **curvature decays**.

- General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale

# Radiation Driving

- Decay is timed precisely to **drive** the oscillator - close to fully coherent

$$\begin{aligned}
 |[\Theta + \Psi](\eta)| &= |[\Theta + \Psi](0) + \Delta\Psi - \Delta\Phi| \\
 &= \left| \frac{1}{3}\Psi(0) - 2\Psi(0) \right| = \left| \frac{5}{3}\Psi(0) \right|
 \end{aligned}$$



- $5\times$  the amplitude of the Sachs-Wolfe effect!

# External Potential Approach

- Solution to homogeneous equation

$$(1 + R)^{-1/4} \cos(ks), \quad (1 + R)^{-1/4} \sin(ks)$$

- Give the general solution for an external potential by propagating impulsive forces

$$(1 + R)^{1/4} \Theta(\eta) = \Theta(0) \cos(ks) + \frac{\sqrt{3}}{k} \left[ \dot{\Theta}(0) + \frac{1}{4} \dot{R}(0) \Theta(0) \right] \sin ks \\ + \frac{\sqrt{3}}{k} \int_0^\eta d\eta' (1 + R')^{3/4} \sin[ks - ks'] F(\eta')$$

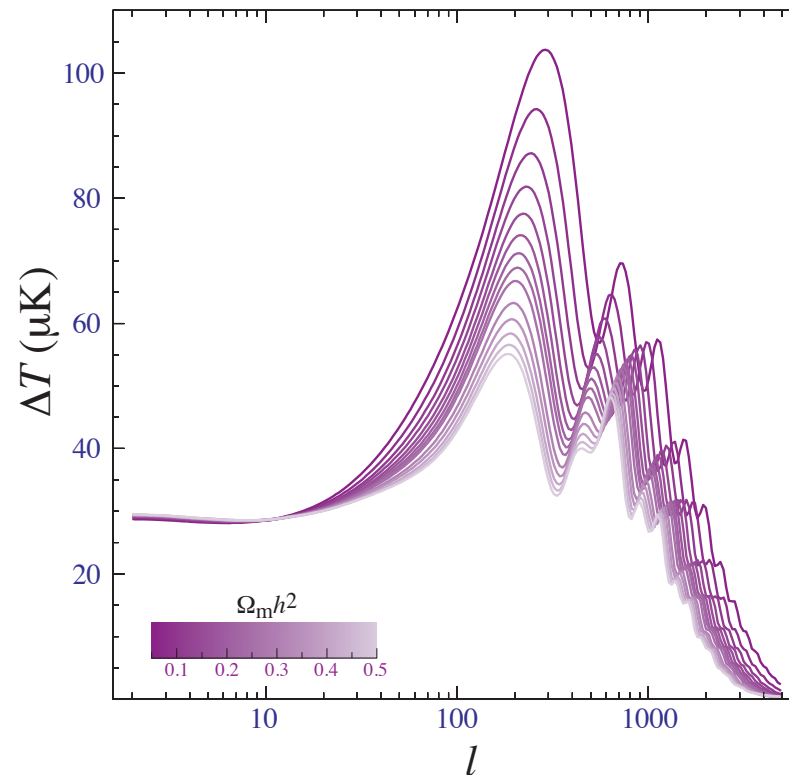
where

$$F = -\ddot{\Phi} - \frac{\dot{R}}{1 + R} \dot{\Phi} - \frac{k^2}{3} \Psi$$

- Useful if general form of potential evolution is known

# Matter-Radiation in the Power Spectrum

- Coherent approximation is **exact** for a photon-baryon fluid but reality is reduced to  $\sim 4\times$  because of **neutrino contribution** to radiation
- Actual **initial conditions** are  $\Theta + \Psi = \Psi/2$  for radiation domination but comparison to matter dominated SW correct



# Damping

- Tight coupling equations assume a perfect fluid: no viscosity, no heat conduction
- Fluid imperfections are related to the mean free path of the photons in the baryons

$$\lambda_C = \dot{\tau}^{-1} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a$$

is the conformal opacity to Thomson scattering

- Dissipation is related to the diffusion length: random walk approximation

$$\lambda_D = \sqrt{N} \lambda_C = \sqrt{\eta / \lambda_C} \lambda_C = \sqrt{\eta \lambda_C}$$

the geometric mean between the horizon and mean free path

- $\lambda_D / \eta_* \sim \text{few } \%$ , so expect the peaks  $> 3$  to be affected by dissipation

# Equations of Motion

- Continuity

$$\dot{\Theta} = -\frac{k}{3}v_\gamma - \dot{\Phi}, \quad \dot{\delta}_b = -kv_b - 3\dot{\Phi}$$

where the photon equation remains unchanged and the baryons follow number conservation with  $\rho_b = m_b n_b$

- Navier-Stokes (Euler + heat conduction, viscosity)

$$\begin{aligned}\dot{v}_\gamma &= k(\Theta + \Psi) - \frac{k}{6}\pi_\gamma - \dot{\tau}(v_\gamma - v_b) \\ \dot{v}_b &= -\frac{\dot{a}}{a}v_b + k\Psi + \dot{\tau}(v_\gamma - v_b)/R\end{aligned}$$

where the photons gain an anisotropic stress term  $\pi_\gamma$  from radiation viscosity and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation

# Viscosity

- Viscosity is generated from radiation streaming from hot to cold regions
- Expect

$$\pi_\gamma \sim v_\gamma \frac{k}{\dot{\tau}}$$

generated by streaming, suppressed by scattering in a wavelength of the fluctuation. Radiative transfer says

$$\pi_\gamma \approx 2A_v v_\gamma \frac{k}{\dot{\tau}}$$

where  $A_v = 16/15$

$$\dot{v}_\gamma = k(\Theta + \Psi) - \frac{k}{3}A_v \frac{k}{\dot{\tau}} v_\gamma$$

# Oscillator: Penultimate Take

- Adiabatic approximation ( $\omega \gg \dot{a}/a$ )

$$\dot{\Theta} \approx -\frac{k}{3}v_\gamma$$

- Oscillator equation contains a  $\dot{\Theta}$  damping term

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} A_v \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})$$

- Heat conduction term similar in that it is proportional to  $v_\gamma$  and is suppressed by scattering  $k/\dot{\tau}$ . Expansion of Euler equations to leading order in  $k\dot{\tau}$  gives

$$A_h = \frac{R^2}{1 + R}$$

since the effects are only significant if the baryons are dynamically important

# Oscillator: Final Take

- Final oscillator equation

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} [A_v + A_h] \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})$$

- Solve in the adiabatic approximation

$$\Theta \propto \exp(i \int \omega d\eta)$$

$$-\omega^2 + \frac{k^2 c_s^2}{\dot{\tau}} (A_v + A_h) i\omega + k^2 c_s^2 = 0$$

# Dispersion Relation

- Solve

$$\begin{aligned}\omega^2 &= k^2 c_s^2 \left[ 1 + i \frac{\omega}{\dot{\tau}} (A_v + A_h) \right] \\ \omega &= \pm k c_s \left[ 1 + \frac{i}{2} \frac{\omega}{\dot{\tau}} (A_v + A_h) \right] \\ &= \pm k c_s \left[ 1 \pm \frac{i}{2} \frac{k c_s}{\dot{\tau}} (A_v + A_h) \right]\end{aligned}$$

- Exponentiate

$$\begin{aligned}\exp(i \int \omega d\eta) &= e^{\pm i k s} \exp\left[-k^2 \int d\eta \frac{1}{2} \frac{c_s^2}{\dot{\tau}} (A_v + A_h)\right] \\ &= e^{\pm i k s} \exp\left[-(k/k_D)^2\right]\end{aligned}$$

- Damping is exponential under the scale  $k_D$

# Diffusion Scale

- Diffusion wavenumber

$$k_D^{-2} = \int d\eta \frac{1}{\dot{\tau}} \frac{1}{6(1+R)} \left( \frac{16}{15} + \frac{R^2}{(1+R)} \right)$$

- Limiting forms

$$\lim_{R \rightarrow 0} k_D^{-2} = \frac{1}{6} \frac{16}{15} \int d\eta \frac{1}{\dot{\tau}}$$

$$\lim_{R \rightarrow \infty} k_D^{-2} = \frac{1}{6} \int d\eta \frac{1}{\dot{\tau}}$$

- Geometric mean between horizon and mean free path as expected from a random walk

$$\lambda_D = \frac{2\pi}{k_D} \sim \frac{2\pi}{\sqrt{6}} (\eta \dot{\tau}^{-1})^{1/2}$$