Astro 321

Lecture Notes Set 6

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Transfer Function

- Transfer function transfers the initial Newtonian curvature to its value today (linear response theory)

$$T(k) = \frac{\Phi(k, a=1)}{\Phi(k, a_{\rm init})} \frac{\Phi(k_{\rm norm}, a_{\rm init})}{\Phi(k_{\rm norm}, a=1)})$$

- Conservation of Bardeen curvature: Newtonian curvature is a constant when stress perturbations are negligible: above the horizon during radiation and dark energy domination, on all scales during matter domination
- When stress fluctuations dominate, perturbations are stabilized by the Jeans mechanism
- Hybrid Poisson equation: Newtonian curvature, comoving density perturbation $\Delta \equiv (\delta\rho/\rho)_{\rm com}$ implies Φ decays

$$(k^2 - 3K)\Phi = 4\pi G\rho\Delta \sim \eta^{-2}\Delta$$

Transfer Function

- Freezing of Δ stops at $\eta_{\rm eq}$

$$\Phi \sim (k\eta_{\rm eq})^{-2}\Delta_H \sim (k\eta_{\rm eq})^{-2}\Phi_{\rm init}$$

- Transfer function has a k^{-2} fall-off beyond $k_{\rm eq} \sim \eta_{\rm eq}^{-1}$
- Small correction since growth with a smooth radiation component is logarithmic not frozen
- Transfer function is a direct output of an Einstein-Boltzmann code

Fitting Function

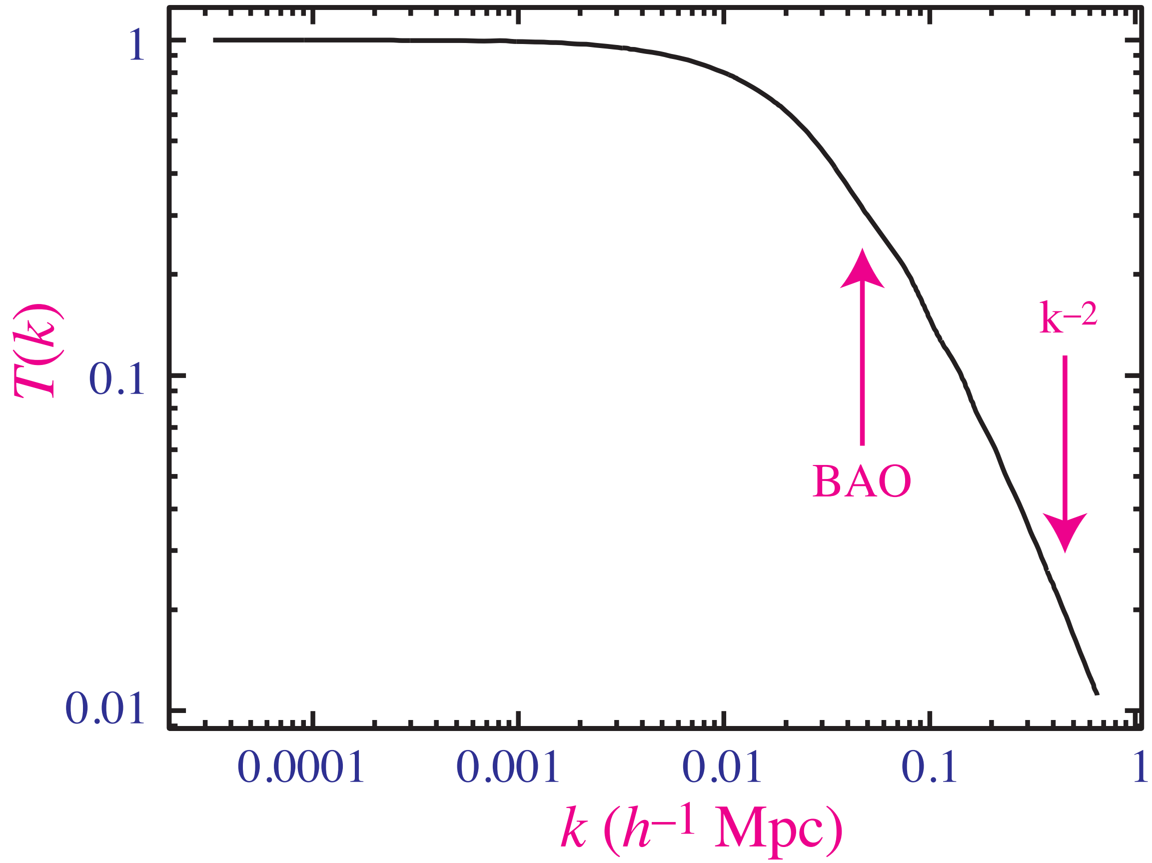
- Alternately accurate fitting formula exist, e.g. pure CDM form:

$$T(k(q)) = \frac{L(q)}{L(q) + C(q)q^2}$$
$$L(q) = \ln(e + 1.84q)$$
$$C(q) = 14.4 + \frac{325}{1 + 60.5q^{1.11}}$$
$$q = k/\Omega_m h^2 \mathrm{Mpc}^{-1} (T_{\mathrm{CMB}}/2.7K)^2$$

- In h Mpc^{-1}, the critical scale depends on $\Gamma \equiv \Omega_m h$ also known as the shape parameter

Transfer Function

- Numerical calculation



Dark Matter and the Transfer Function

- Baryons caught up in the acoustic oscillations of the CMB and impart acoustic wiggles to the transfer function. Density enhancements are produced kinematically through the continuity equation $\delta_b \sim (k\eta)v_b$ and hence are out of phase with CMB temperature peaks
- Dissipation of the acoustic oscillations eliminates both the CMB and baryon perturbations – known as Silk damping for the baryons. This suppression and the general fact that baryons are caught up with photons was one of the main arguments for CDM
- Neutrino dark matter suffers similar effects and hence cannot be the main component of dark matter in the universe

Massive Neutrinos

- Relativistic stresses of a light neutrino slow the growth of structure
- Neutrino species with cosmological abundance contribute to matter as $\Omega_\nu h^2 = \sum m_\nu/94\text{eV}$, suppressing power as $\Delta P/P \approx -8\Omega_\nu/\Omega_m$
- Current data from 2dF galaxy survey and CMB indicate $\sum m_\nu < 0.9\text{eV}$ assuming a ΛCDM model with constant tilt based on the shape of the transfer function.

Growth Function

- Same physics applies to the dark energy dominated universe
- Under the dark energy sound horizon or Jeans scale, dark energy density frozen. Potential decays at the same rate for all scales

$$g(a) = \frac{\Phi(k_{\rm norm}, a)}{\Phi(k_{\rm norm}, a_{\rm init})} \qquad {}' \equiv \frac{d}{d\ln a}$$

- Continuity + Euler + Poisson

$$g'' + \left(1 - \frac{\rho''}{\rho'} + \frac{1}{2}\frac{\rho_c'}{\rho_c}\right) g' + \left(\frac{1}{2}\frac{\rho_c' + \rho'}{\rho_c} - \frac{\rho''}{\rho'}\right) g = 0$$

where ρ is the Jeans unstable matter and ρ_c is the critical density

Dark Energy Growth Suppression

- Pressure growth suppression: $\delta \equiv \delta\rho_m/\rho_m \propto ag$

$$\frac{d^2 g}{d\ln a^2} + \left[\frac{5}{2} - \frac{3}{2}w(z)\Omega_{DE}(z)\right]\frac{dg}{d\ln a} + \frac{3}{2}[1 - w(z)]\Omega_{DE}(z)g = 0\,,$$

  where $w \equiv p_{DE}/\rho_{DE}$ and $\Omega_{DE} \equiv \rho_{DE}/(\rho_m + \rho_{DE})$ with initial conditions $g = 1$, $dg/d\ln a = 0$

- As $\Omega_{DE} \to 0$ $g =$const. is a solution. The other solution is the decaying mode, elimated by initial conditions

- As $\Omega_{DE} \to 1$ $g \propto a^{-1}$ is a solution. Corresponds to a frozen density field.

COBE Normalization

- Normalization of potential is set by observations of the CMB, aka COBE normalization
- Take the radiation distribution at last scattering to also be described by an isotropic temperature fluctuation field $\Theta(\mathbf{x})$ and recombination to be instantaneous

$$\Theta(\hat{\mathbf{n}}) = \int dD\, \Theta(\mathbf{x})\delta(D - D_*)$$

  where D is the comoving distance and D_* denotes recombination.
- Describe the temperature field by its Fourier moments

$$\Theta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

COBE Normalization

- Power spectrum

$$\langle \Theta(\mathbf{k})^* \Theta(\mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') P_T(k)$$

$$\Delta_T^2 = k^3 P_T / 2\pi^2$$

- Temperature field

$$\Theta(\hat{\mathbf{n}}) = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) e^{i\mathbf{k}\cdot D_* \hat{\mathbf{n}}}$$

- Multipole moments $\Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m}$

COBE Normalization

- Expand out plane wave in spherical coordinates

$$e^{i\mathbf{k}D_*\cdot\hat{\mathbf{n}}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kD_*) Y^*_{\ell m}(\mathbf{k}) Y_{\ell m}(\hat{\mathbf{n}})$$

$$\Theta_{\ell m} = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) 4\pi i^\ell j_\ell(kD_*) Y_{\ell m}(\mathbf{k})$$

- Power spectrum

$$\langle \Theta^*_{\ell m} \Theta_{\ell' m'} \rangle = \int \frac{d^3k}{(2\pi)^3} (4\pi)^2 (i)^{\ell-\ell'} j_\ell(kD_*) j_{\ell'}(kD_*) Y^*_{\ell m}(\mathbf{k}) Y_{\ell' m'}(\mathbf{k}) P_T(k)$$

$$= \delta_{\ell\ell'} \delta_{mm'} 4\pi \int d\ln k\, j^2_\ell(kD_*) \Delta^2_T(k)$$

with $\int_0^\infty j^2_\ell(x) d\ln x = 1/(2\ell(\ell+1))$, slowly varying Δ^2_T

COBE Normalization

- Angular power spectrum:

$$C_\ell = \frac{4\pi\Delta_T^2(\ell/D_*)}{2\ell(\ell+1)} = \frac{2\pi}{\ell(\ell+1)}\Delta_T^2(\ell/D_*)$$

- $\ell(\ell+1)C_\ell/2\pi = \Delta_T^2$ is commonly used log power
- Sachs-Wolfe effect says $\Delta_T^2 = \Delta_\Phi^2/9$
- Observed number

$$\Delta_T^2 = \left(\frac{28\mu\text{K}}{2.725\times 10^6\mu\text{K}}\right)^2$$
$$\Delta_\Phi^2 \approx 9\times 10^{-10}$$

at recombination

COBE Normalization

- Today:

$$\Delta^2_\Phi \approx 9 \times 10^{-10} g^2(a) T^2(k) \left(\frac{k}{H_0}\right)^{n-1}$$

- Density field

$$\begin{aligned} k^2\Phi &= 4\pi G a^2 \Delta\rho \\ &= \frac{3}{2} H_0^2 \Omega_m \Delta / a \\ \Delta^2_\Phi &= \frac{9}{4}\left(\frac{H_0}{k}\right)^4 \Omega_m^2 a^{-2} \Delta^2_\Delta \\ \Delta^2_\Delta &= (2 \times 10^{-5})^2 \left(\frac{k}{H_0}\right)^{n+3} \Omega_m^{-2} a^2 g^2(a) T^2(k) \end{aligned}$$

Normalization Convention

- Current density field on the horizon scale $k = H_0$

$$\delta_H = 2 \times 10^{-5} \Omega_m^{-1} g(a = 1)$$

- In a ΛCDM model, a detailed fit gives

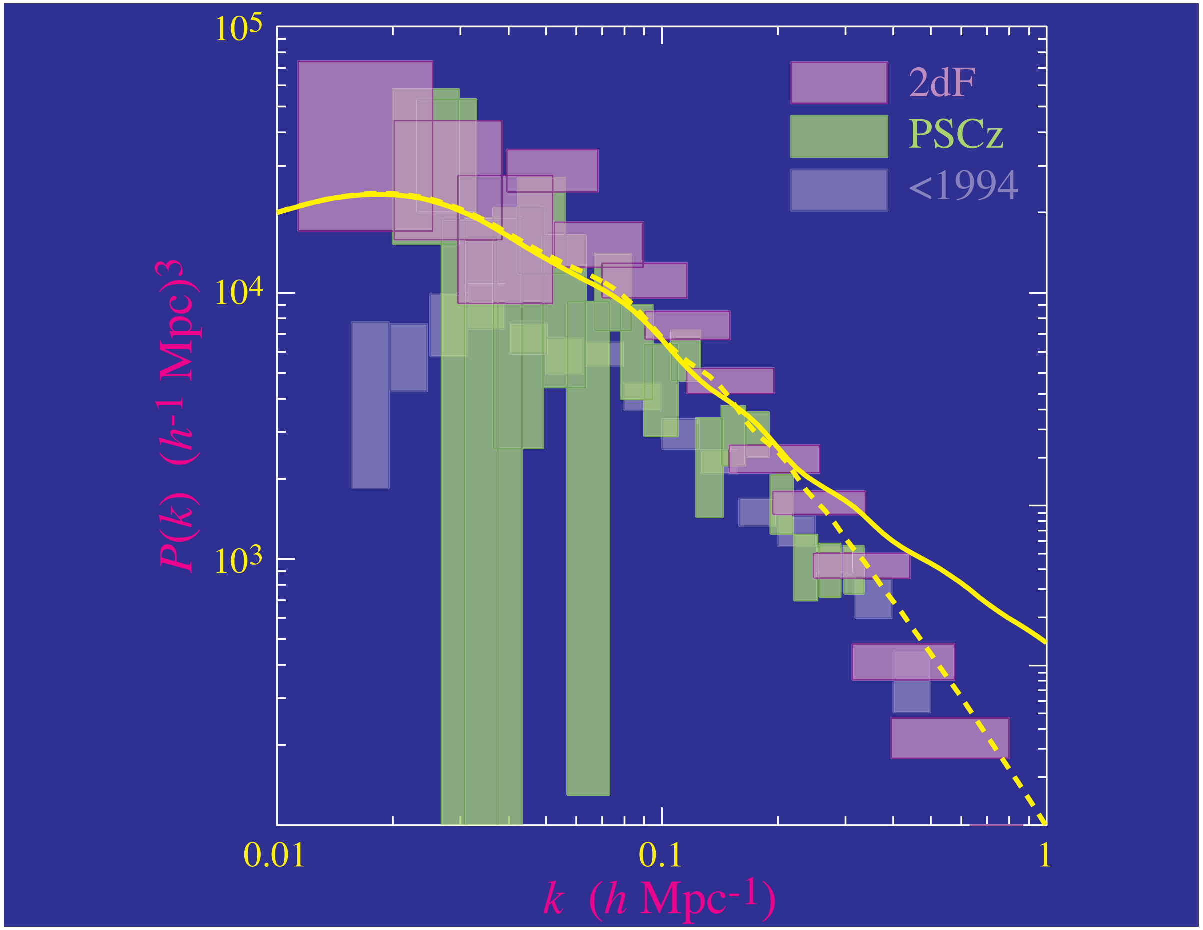
$$\delta_H = 1.94 \times 10^{-5} \Omega_m^{-0.785-0.05 \ln \Omega_m} e^{-0.95(n-1)-0.169(n-1)^2}$$

  since growth factor is smaller in a low Ω_m model and normalization scale is not exactly the horizon scale

- In the future (abount now) the COBE normalization will be superceded by CMB peak normalization

Power Spectrum

- 2dF data



- Power spectrum defines large scale structure observables: galaxy clustering, velocity field, Lyα forest clustering, cosmic shear

Velocity field

- Continuity gives the velocity from the density field as

$$v = -\dot{\Delta}/k = -\frac{aH}{k}\frac{d\Delta}{d\ln a}$$
$$= -\frac{aH}{k}\Delta\frac{d\ln(ag)}{d\ln a}$$

- In a ΛCDM model or open model $d\ln(ag)/d\ln a \approx \Omega_m^{0.6}$
- Measuring both the density field and the velocity field (through distance determination and redshift) allows a measurement of Ω_m
- Practically one measures $\beta = \Omega_m^{0.6}/b$ where b is a bias factor for the tracer of the density field, i.e. with galaxy numbers $\delta n/n = b\Delta$
- Can also measure this factor from the redshift space power spectrum - the Kaiser effect where clustering in the radial direction is apparently enhanced by gravitational infall

Lyman-α Forest

- QSO spectra absorbed by neutral hydrogen through the Lyα transition.
- Lack of complete absorption, known as the lack of a Gunn-Peterson trough indicates that the universe is nearly fully ionized out to the highest redshift quasar $z \sim 6$; recently SDSS QSO implies $z \sim 6$ is the end of the reionization epoch
- In ionization equilibrium, the Gunn-Peterson optical depth is a tracer of the underlying baryon density which itself is a tracer of the dark matter $\tau_{GP} \propto \rho_b T^{-0.7}$ with $T(\rho_b)$.
- Clustering in the Lyα forest reflects the underlying linear power spectrum as calibrated through simulations

Gravitational Lensing

- Gravitational potentials along the line of sight $\hat{\mathbf{n}}$ to some source at comoving distance D_s lens the images according to (flat universe)

$$\phi(\hat{\mathbf{n}}) = 2 \int dD \frac{D_s - D}{D D_s} \Phi(D\hat{\mathbf{n}}, \eta(D))$$

  remapping image positions as

$$\hat{\mathbf{n}}^I = \hat{\mathbf{n}}^S + \nabla_{\hat{\mathbf{n}}} \phi(\hat{\mathbf{n}})$$

- Since absolute source position is unknown, use image distortion defined by the Jacobian matrix

$$\frac{\partial n_i^I}{\partial n_j^S} = \delta_{ij} + \psi_{ij}$$

Weak Lensing

- Small image distortions described by the convergence κ and shear components (γ_1, γ_2)

$$\psi_{ij} = \begin{pmatrix} \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & \kappa + \gamma_1 \end{pmatrix}$$

where $\nabla_{\hat{\mathbf{n}}} = D\nabla$ and

$$\psi_{ij} = 2 \int dD \frac{D(D_s - D)}{D_s} \nabla_i \nabla_j \Phi(D\hat{\mathbf{n}}, \eta(D))$$

- In particular, through the Poisson equation the convergence (measured from shear) is simply the projected mass

$$\kappa = \frac{3}{2} \Omega_m H_0^2 \int dD \frac{D(D_s - D)}{D_s} \frac{\Delta(D\hat{\mathbf{n}}, \eta(D))}{a}$$