How to Compute Primordial Non-Gaussianities

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$$\text{Vacuum Fluctuations} \xrightarrow{\text{inflate}} \text{CMB Initial Conditions}$$

Gaussianity

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$$U(t, t_0) = T \exp\left(-i \int_{t_0}^{t} dt' H_{free}(t') + H_{int}(t')\right)$$

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$$U^{-1}_{free}(t,t_0)U(t,t_0) = T\exp(-i\int_{t_0}^{t} dt H^I_{int}(t))$$

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$$\sqrt{-g} \to N\sqrt{h},$$
$$R \to K_{ij}K^{ij} - K^2 +^{(3)} R,$$
$$g^{00} \to \tfrac{-1}{N^2}$$

Lagrangian in ADM form

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$$D_i[K^i_j - K\delta^i_j]$$

Example: Expanding around background $ds^2 = -dt^2 + a^2 dx^2$

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becomes the Friedmann Equation...

$3H^2 = \frac{\dot{\phi}^2}{2} + V$

Still have some gauge freedom

ADM Metric

$$\mathrm{d}s^2 = -N^2\,\mathrm{d}t^2 + h_{ij}(\mathrm{d}x^i + N_j\,\mathrm{d}t)(\mathrm{d}x^j + N_i\,\mathrm{d}t)$$

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Gauge-Fix for scalar perturbations

$$h_{ij} = a^2 e^{2\zeta}\delta_{ij}$$
$$N_i = \delta_i\psi$$

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Quadratic will give us $\zeta^I(t)$, Cubic will give $H_{\text{interaction}}(t)$

Second Order Action

$$S_2 = \int \mathrm{d}t \, \mathrm{d}^3x \; 4\frac{\dot{\phi}^2}{H^2}\left(\frac{a^3}{2}\dot{\zeta}^2 - \frac{a}{2}(\delta\zeta)^2\right)$$

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$$\frac{\dot{\phi}^2}{H^2} \equiv \epsilon$$

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$z \equiv a\sqrt{2\epsilon}$, $v \equiv \zeta z$, $\int \mathrm{d}t \to \int a\,\mathrm{d}\tau$

$$S_{MS} = \frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}^3x(v'^2 - (\delta v)^2 - \frac{z''}{z}v^2)$$

Can finally quantize the field

$$\zeta_I(\vec{k}, t) = u_k(t) a_I(\vec{k}) + u_k^*(t) a_I^\dagger(-\vec{k})$$
$$[a_I(\vec{k}), a_I^\dagger(\vec{k'})] = (2\pi)^3 \delta(\vec{k} - \vec{k}')$$

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$$u_k = zv_k$$
$$v_k'' + (k^2 - z''/z)v_k = 0,$$
$$v_k(\tau_0) = \frac{1}{\sqrt{2k}}, v_k'(\tau_0) = -i\sqrt{\frac{k}{2}}$$

So for the interaction picture field $\zeta_I(\tau)$, we have

Its time evolution
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The contraction

$\langle \zeta_I(k_1, 0)\zeta_I(k_2, 0)\rangle = \frac{H^2}{2(2\pi^3)\epsilon k_1^3}\delta(k_1 - k_2)$

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$$H_{int}(\tau) \supset \int \mathrm{d}^3x a\epsilon^2 \zeta(x,\tau)\zeta'(x,\tau)\zeta'(x,\tau)$$

$$\langle \zeta_{k_1}(t)\zeta_{k_2}(t)\zeta_{k_3}(t)\rangle =$$
$$\langle (\bar{T}\exp(-i\int_{t_0}^{t} dt H_{int}^{I}(t)))\zeta_{k_1}^{I}(t)\zeta_{k_2}^{I}(t)\zeta_{k_3}^{I}(t) T\exp(-i\int_{t_0}^{t} dt H_{int}^{I}(t))\rangle$$

Can finally compute the bispectrum

$$\langle \zeta_{k_1}(0)\zeta_{k_2}(0)\zeta_{k_3}(0)\rangle =$$
$$\langle \mathsf{Re}\left[-2i\zeta^I_{k_1}(0)\zeta^I_{k_2}(0)\zeta^I_{k_3}(0)\int_{-\infty(1+i\epsilon)}^{0}\mathrm{d}\tau\,\mathrm{d}^3x a^2\epsilon^2\zeta_I(x,\tau')\zeta_I'(x,\tau')\zeta_I'(x,\tau')\right]\rangle$$

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Fourier Space $\int \mathrm{d}^3x \to \int \frac{\mathrm{d}^3q_1}{(2\pi)^3}\frac{\mathrm{d}^3q_2}{(2\pi)^3}\frac{\mathrm{d}^3q_3}{(2\pi)^3} e^{-i(q_1+q_2+q_3)x}$

Wick Contractions

$$\langle \zeta^I_{k_1}(0)\zeta^I_{k_2}(0)\zeta^I_{k_3}(0)\zeta^I_{q_1}(\tau)\zeta^I_{q_2}(\tau)'\zeta^I_{q_3}(\tau)' \rangle$$

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becomes products of

$$\langle \zeta_k(t_1)\zeta_q(t_2)\rangle = (2\pi)^3 u(t_1)u^*(t_2)\delta(k-q)$$

After doing the contractions and computing the integrals...

One term in the bispectrum of single-field inflation

$$\langle\zeta_{k_1}(0)\zeta_{k_2}(0)\zeta_{k_3}(0)\rangle =$$
$$\frac{H^4}{16\epsilon}\frac{1}{(k_1k_2k_3)^3}(2\pi)^3\delta(k_1+k_2+k_3)(k_2k_3)^2(\frac{1}{K}+\frac{k_1}{K^2}+1\to 2+1\to 3)$$

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