# Astro 321

# Set 7: Spherical Collapse & Halo Model

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# Closed Universe

- Friedmann equation in a closed universe

$$\frac{1}{a}\frac{da}{dt} = H_0 \left(\Omega_m a^{-3} + (1-\Omega_m)a^{-2}\right)^{1/2}$$

- Parametric solution in terms of a development angle $\theta = H_0\eta(\Omega_m - 1)^{1/2}$, scaled conformal time $\eta$

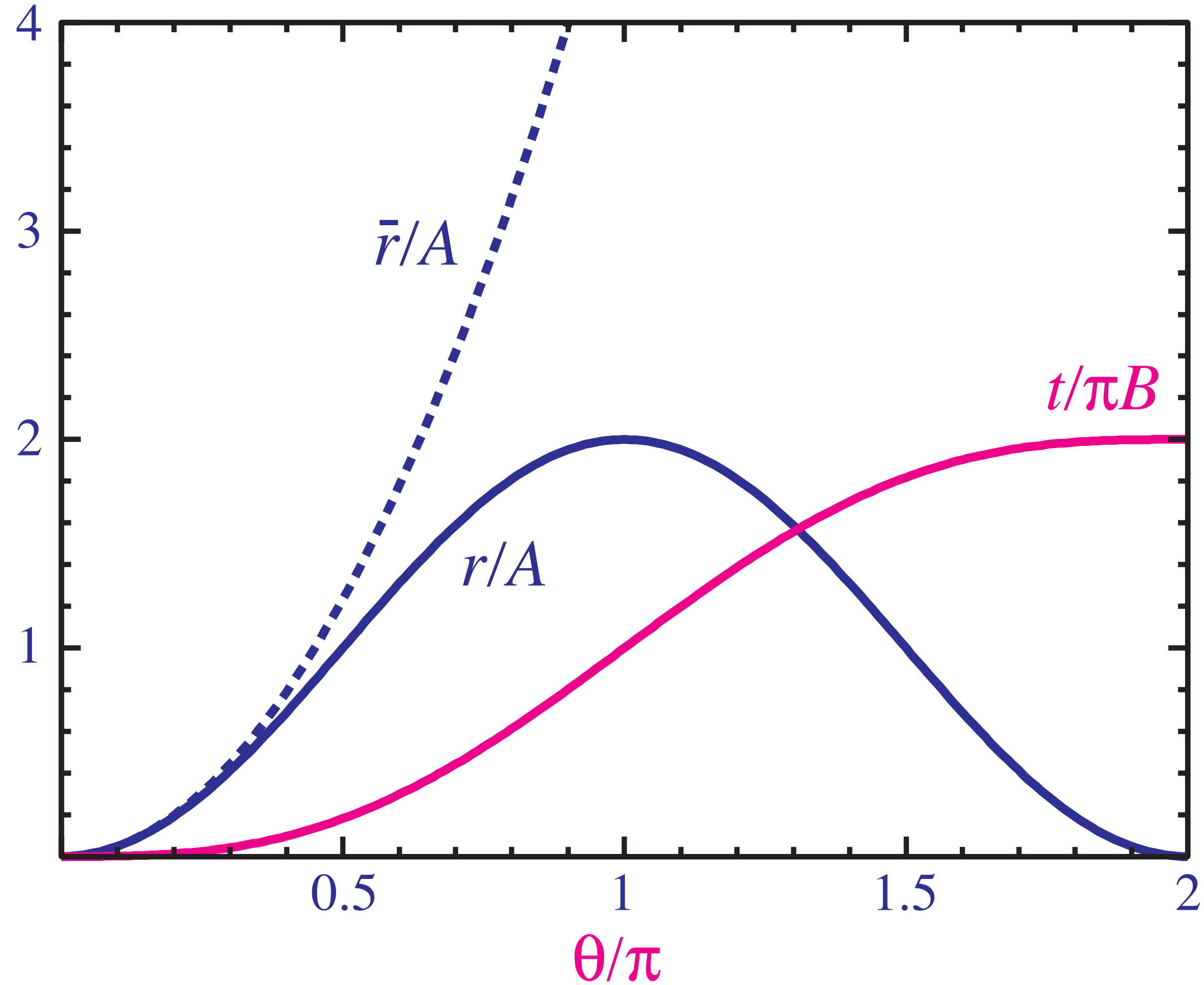
$$\begin{aligned} r(\theta) &= A(1-\cos\theta) \\ t(\theta) &= B(\theta - \sin\theta) \end{aligned}$$

where $A = r_0\Omega_m/2(\Omega_m - 1)$, $B = H_0^{-1}\Omega_m/2(\Omega_m - 1)^{3/2}$.

- Turn around at $\theta = \pi$, $r = 2A$, $t = B\pi$.
- Collapse at $\theta = 2\pi$, $r \to 0$, $t = 2\pi B$

# Spherical Collapse

- Parametric Solution:



# Correspondence

- Eliminate cosmological correspondence in $A$ and $B$ in terms of enclosed mass $M$

$$M = \frac{4\pi}{3} r_0^3 \Omega_m \rho_c = \frac{4\pi}{3} r_0^3 \Omega_m \frac{3H_0^2}{8\pi G}$$

- Related as $A^3 = GMB^2$, and to initial perturbation

$$\lim_{\theta\to 0} r(\theta) = A\left(\frac{1}{2}\theta^2 - \frac{1}{24}\theta^4\right)$$

$$\lim_{\theta\to 0} t(\theta) = B\left(\frac{1}{6}\theta^3 - \frac{1}{120}\theta^5\right)$$

- Leading Order: $r = A\theta^2/2$, $t = B\theta^3/6$

$$r = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}$$

# Next Order

- Leading order is unperturbed matter dominated expansion
  $r \propto a \propto t^{2/3}$
- Iterate $r$ and $t$ solutions

$$\lim_{\theta \to 0} t(\theta) = \frac{\theta^3}{6} B \left[ 1 - \frac{1}{20} \left( \frac{6t}{B} \right)^{2/3} \right]$$

$$\theta \approx \left( \frac{6t}{B} \right)^{1/3} \left[ 1 + \frac{1}{60} \left( \frac{6t}{B} \right)^{2/3} \right]$$

# Next Order

- Substitute back into $r(\theta)$

$$
\begin{aligned}
r(\theta) &= A\frac{\theta^2}{2}\left(1-\frac{\theta^2}{12}\right) \\
&= \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left[1-\frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right] \\
&= \frac{1}{2}(6t)^{2/3}(GM)^{1/3}\left[1-\frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right]
\end{aligned}
$$

# Density Correspondence

- Density

$$\rho_m = \frac{M}{\frac{4}{3}\pi r^3}$$

$$= \frac{1}{6\pi t^2 G}\left[1 + \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}\right]$$

- Density perturbation

$$\delta \equiv \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m} \approx \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}$$

# Density Correspondence

- Time $\rightarrow$ scale factor

$$
\begin{aligned}
t &= \frac{2}{3H_0\Omega_m^{1/2}} a^{3/2} \\
\delta &= \frac{3}{20} a \left(4/BH_0\Omega_m^{1/2}\right)^{2/3}
\end{aligned}
$$

- $A$ and $B$ constants $\rightarrow$ initial cond.

$$
\begin{aligned}
B &= \frac{1}{2H_0\Omega_m^{1/2}} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right)^{3/2} \\
A &= \frac{3}{10}\frac{r_i}{\delta_i}
\end{aligned}
$$

# Spherical Collapse Relations

- Scale factor $a \propto t^{2/3}$

$$a = \left(\frac{3}{4}\right)^{2/3} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right) (\theta - \sin\theta)^{2/3}$$

- At collapse $\theta = 2\pi$

$$a_{\rm col} = \left(\frac{3}{4}\right)^{2/3} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right) (2\pi)^{2/3} \approx 1.686 \frac{a_i}{\delta_i}$$

- Perturbation collapses when linear theory predicts $\delta_c \equiv 1.686$

# Virialization

- A real density perturbation is neither spherical nor homogeneous
- Shell crossing if $\delta_i$ doesn't monotonically decrease
- Collapse does not proceed to a point but reaches virial equilibrium

$$U = -2K, \qquad E = U + K = U(r_{\rm max}) = \frac{1}{2}U(r_{\rm vir}) \tag{1}$$

$r_{\rm vir} = \frac{1}{2}r_{\rm max}$ since $U \propto r^{-1}$. Thus $\theta_{\rm vir} = \frac{3}{2}\pi$

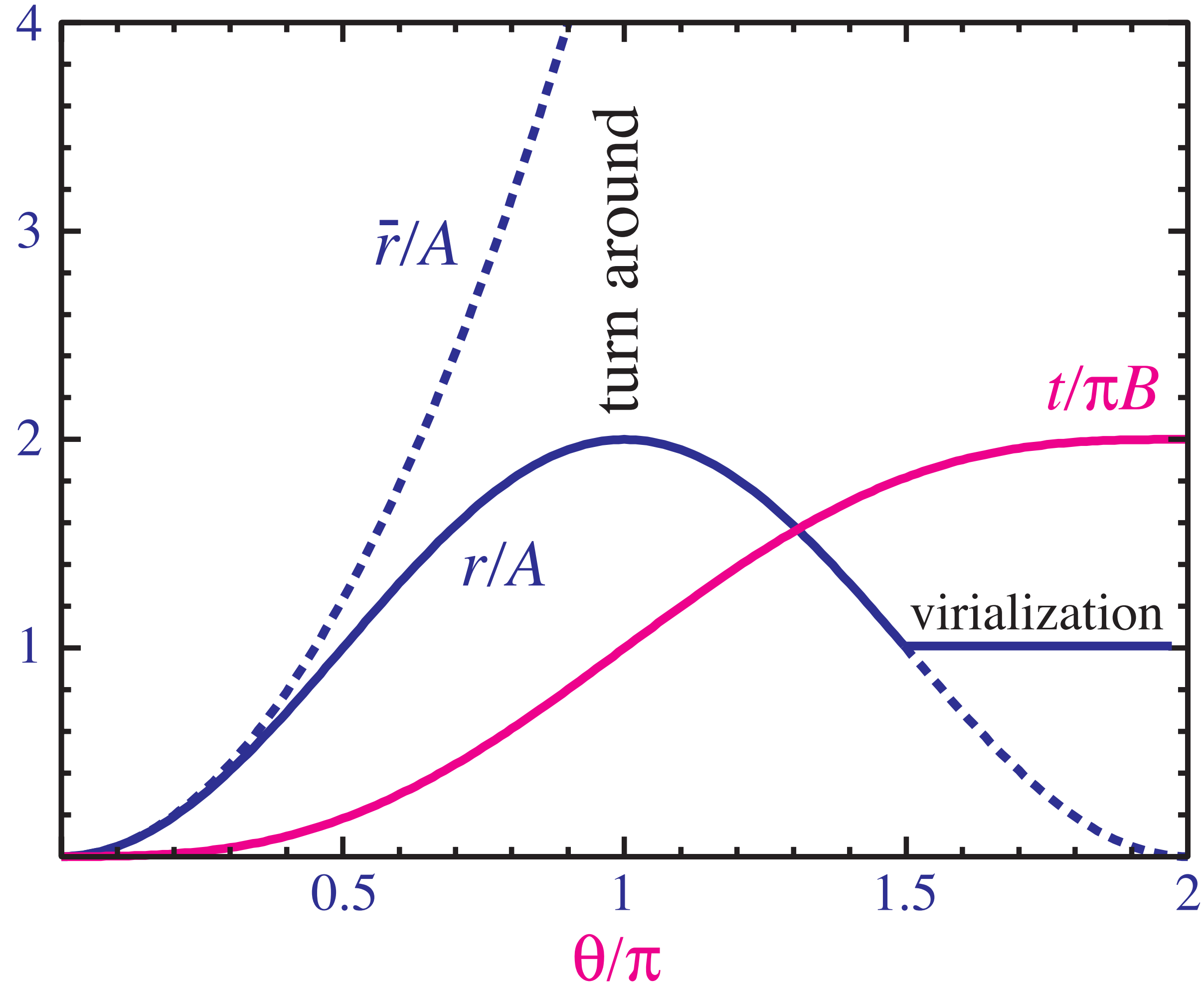
- Overdensity at virialization

$$\frac{\rho_m(\theta = 3\pi/2)}{\bar{\rho}_m(\theta = 2\pi)} = 18\pi^2 \approx 178$$

- Threshold $\Delta_v = 178$ often used to define a collapsed object
- Equivalently relation between virial mass, radius, overdensity: $M_v = \frac{4\pi}{3}r_v^3\rho_m\Delta_v$

# Virialization

- Schematic Picture:



# Generalization Beyond Matter

- In a universe with smooth components like dark energy driving the expansion but not participating in collapse we cannot consider spherical collapse to be a separate universe
- Go back to the continuity and Euler equation to derive the general equation

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla \cdot (1+\delta)\mathbf{v} = 0$$
$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a}(\mathbf{v}\cdot\nabla)\mathbf{v} + H\mathbf{v} = -\frac{1}{a}\nabla\Psi$$

which is true for any type of dark energy or even metric modified gravity

# Generalization Beyond Matter

- For a tophat density perturbation $\mathbf{v} = A(t)\mathbf{r}$ interior given the continuity equation and so

$$\frac{d^2\delta}{dt^2} - \frac{4}{3}\frac{1}{1+\delta}\left(\frac{d\delta}{dt}\right)^2 + 2H\frac{d\delta}{dt} = \frac{(1+\delta)}{a^2}\nabla^2\Psi$$

- Under ordinary gravity $\nabla^2\Psi = 4\pi G a^2 \bar{\rho}_m \delta$ and so a tophat remains a tophat
- Thus use conservation of the dark matter mass

$$M = (4\pi/3) r^3 \bar{\rho}_m (1+\delta)$$

to trade the density for the tophat radius $\delta \to R$

# Generalization Beyond Matter

- Using the Friedmann equations for the evolution of the background

$$H^2 = \frac{8\pi G}{3}(\bar\rho_m + \bar\rho_{\rm eff})$$

we obtain using the Poisson equation

$$\begin{aligned}\frac{1}{r}\frac{d^2 r}{dt^2} &= H^2 + \dot H - \frac{1}{3}\nabla^2\Psi \\ &= -\frac{4\pi G}{3}\left[\rho_m + (1+3w_{\rm eff})\bar\rho_{\rm eff}\right]\end{aligned}$$

where $\rho_m = \bar\rho_m(1+\delta)$ includes the tophat fluctuation whereas $\bar\rho_{\rm eff}$ is a smooth background contribution to the Friedmann equation

- In other words $H^2 + \dot H$ carries the acceleration effect of background total density but $\Psi$ carries only that of the collapsing component - alters the collapse relations

# Generalization Beyond Matter

- Similarly virial equilibrium altered to include smooth contribution to acceleration or effective potential

$$U = -2K$$

where

$$U = -\frac{3}{5}\frac{GM^2}{R} - \frac{4\pi G}{5}(1+3w_{\rm eff})\bar\rho_{\rm eff} M R^2$$

- Note that virial equilibrium is defined in terms of the trace of the potential tensor and is a statement of force balance

$$U \equiv -\int d^3 x \rho_m \mathbf{x} \cdot \nabla \Psi_{\rm tot}$$

# Generalization Beyond Matter

- Hence $U$ is well defined even in cases where energy is not conserved in the usual manner (though still convariantly conserved), e.g. if $\rho_{\rm eff}$ is not constant during collapse
- In general keep track of the kinetic energy during collapse and finding the virial radius as the point at which

$$U(r_{\rm vir}) = -2K(r_{\rm vir})$$

- Rather than using energy conservation (important if $w_{\rm eff} \neq -1$)

# The Mass Function

- Spherical collapse predicts the end state as virialized halos given an initial density perturbation
- Initial density perturbation is a Gaussian random field
- Compare the variance in the linear density field to threshold $\delta_c = 1.686$ to determine collapse fraction
- Combine to form the mass function, the number density of halos in a range $dM$ around $M$.
- Halo density defined entirely by linear theory
- Fudge the result to get the right answer compared with simulations (a la Press-Schechter)!

# Press-Schechter Formalism

- Smooth linear density density field on mass scale $M$ with tophat

$$R = \left(\frac{3M}{4\pi}\right)^{1/3}$$

- Result is a Gaussian random field with variance $\sigma^2(M)$
- Fluctuations above the threshold $\delta_c$ correspond to collapsed regions. The fraction in halos $> M$ becomes

$$\frac{1}{\sqrt{2\pi}\sigma(M)} \int_{\delta_c}^{\infty} d\delta \exp\left(-\frac{\delta^2}{2\sigma^2(M)}\right) = \frac{1}{2}\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$

where $\nu \equiv \delta_c/\sigma(M)$

- Problem: even as $\sigma(M) \to \infty$, $\nu \to 0$, collapse fraction $\to 1/2$ – only overdense regions participate in spherical collapse.
- Multiply by an ad hoc factor of 2!

# Press-Schechter Mass Function

- Differentiate in $M$ to find fraction in range $dM$ and multiply by $\rho_m/M$ the number density of halos if all of the mass were composed of such halos $\rightarrow$ differential number density of halos

$$\frac{dn}{d\ln M} = \frac{\rho_m}{M}\frac{d}{d\ln M}\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$
$$= \sqrt{\frac{2}{\pi}}\frac{\rho_m}{M}\frac{d\ln\sigma^{-1}}{d\ln M}\nu\exp(-\nu^2/2)$$

- High mass: exponential cut off above $M_*$ where $\sigma(M_*) = \delta_c$

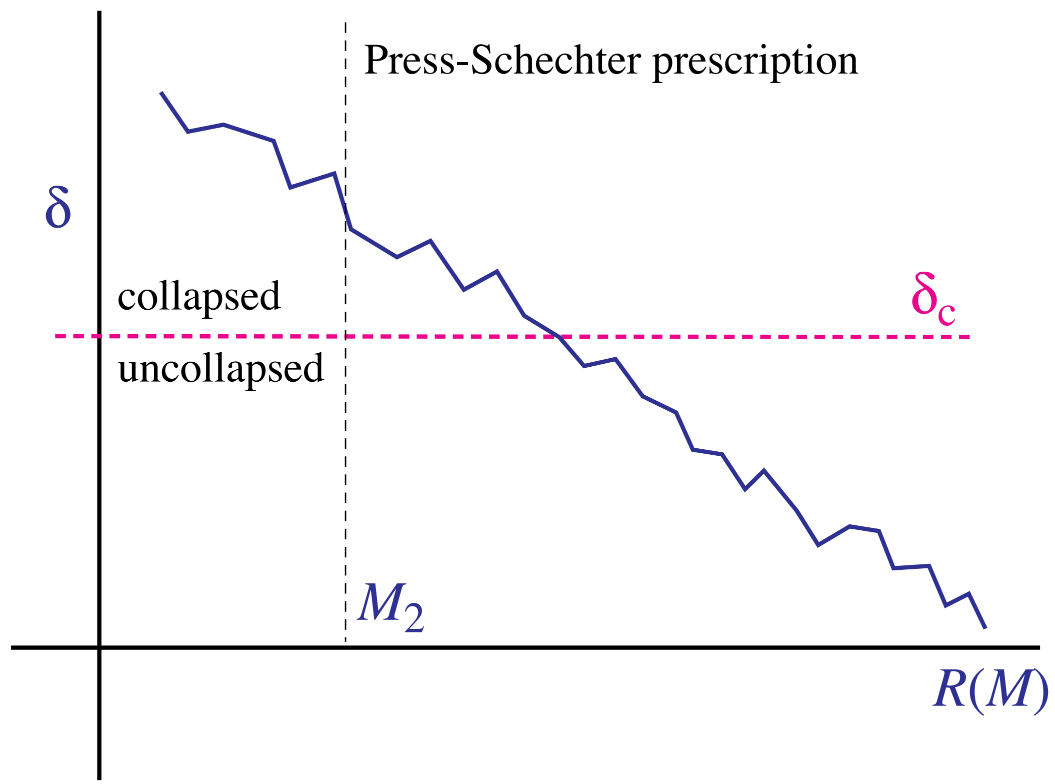
$$M_* \sim 10^{13}h^{-1}M_\odot \quad \text{today}$$

- Low mass divergence: (too many for the observations?)

$$\frac{dn}{d\ln M} \propto\sim M^{-1}$$

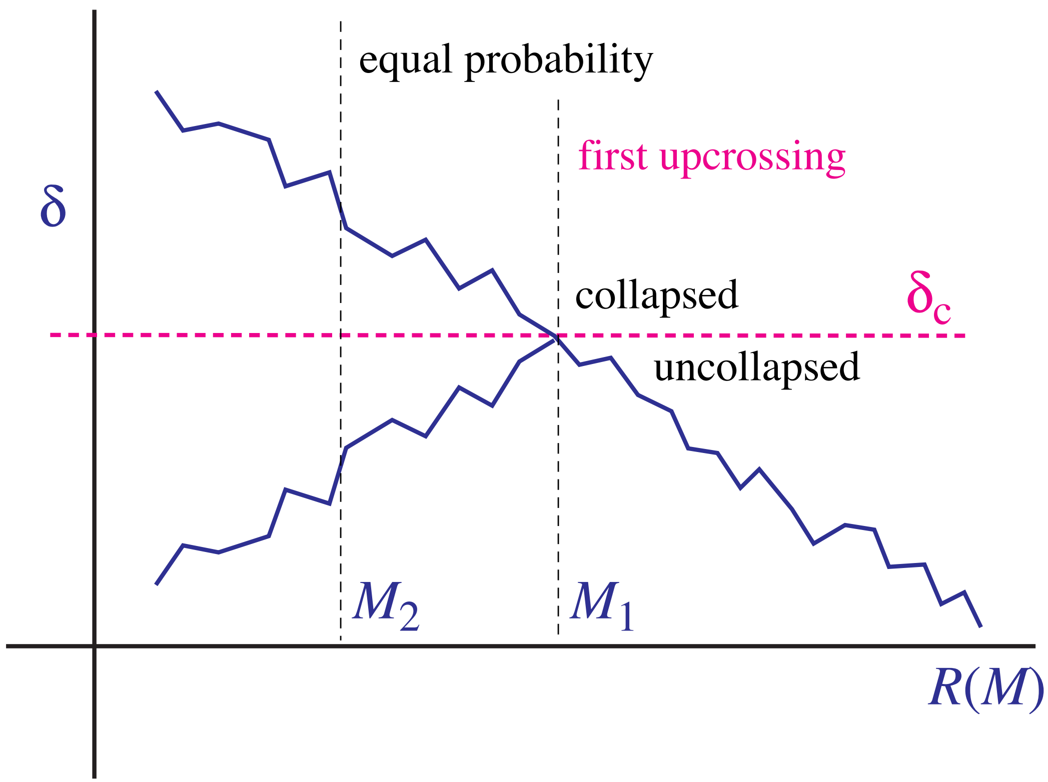
# Extended Press-Schechter Formalism

- A region that is underdense when smoothed on the scale $M$ may be overdense on a scale of a larger $M$
- If smoothing is a tophat in $k$-space, independence of $k$-modes implies fluctuation executes a random walk



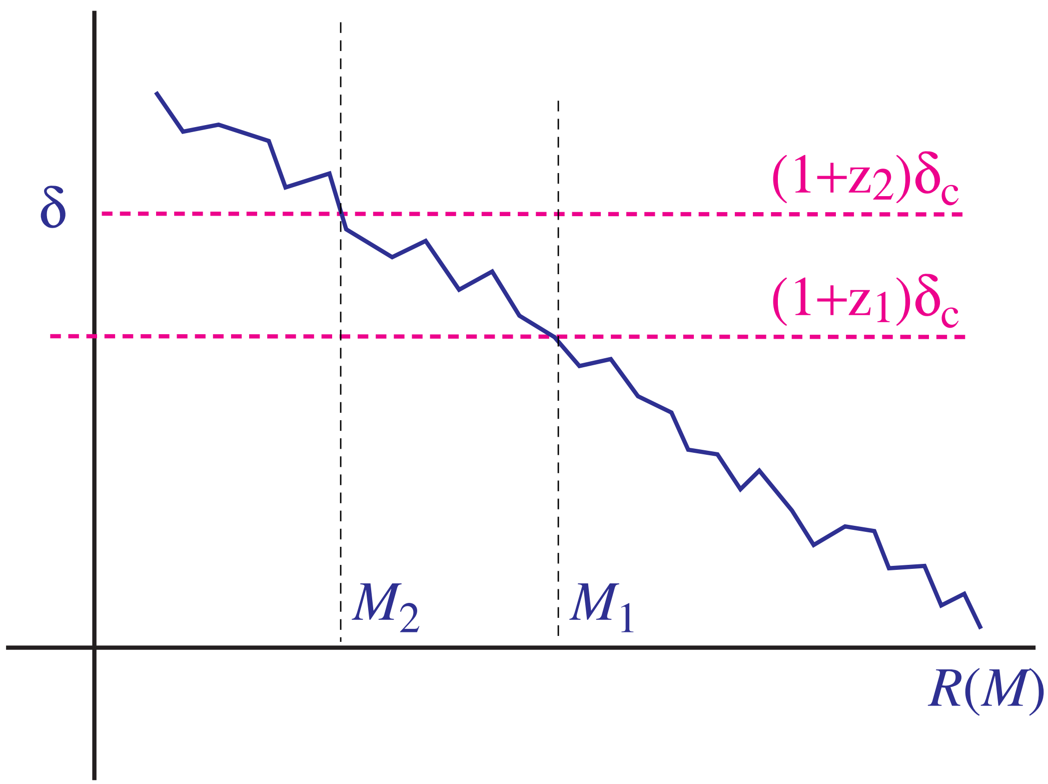
# Extended Press-Schechter Formalism

- For each trajectory that lies above threshold at $M_2$, there is an equivalent trajectory that is its mirror image reflected around $\delta_c$
- Press-Schechter ignored this branch. It supplies the missing factor of 2



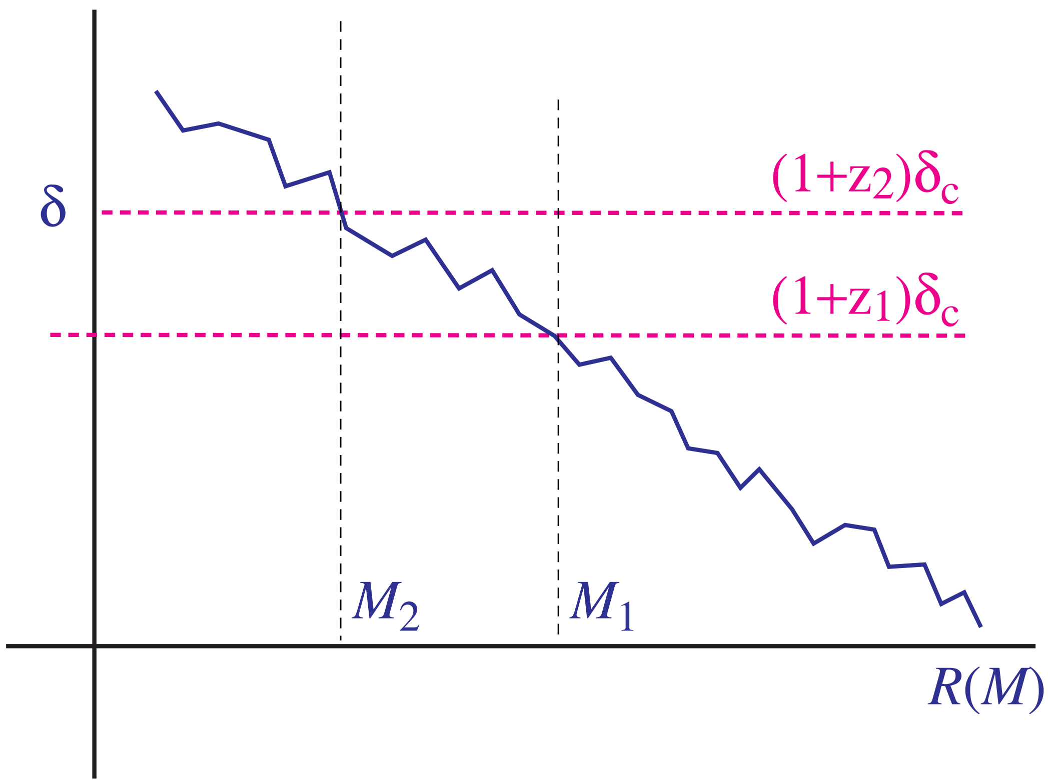
# Conditional Mass Function

- Extended Press-Schechter also gives the conditional mass function, useful for merger histories.
- Given a halo of mass $M_1$ exists at $z_1$, what is the probability that it was part of a halo of mass $M_2$ at $z_2$



# Conditional Mass Function

- Same as before but with the origin translated.
- Conditional mass function is mass function with $\delta_c$ and $\sigma^2(M)$ shifted



# Magic "2" resolved?

- Spherical collapse is defined for a real-space not $k$-space smoothing. Random walk is only a qualitative explanation.
- Modern approach: think of spherical collapse as motivating a fitting form for the mass function

$$\nu \exp(-\nu^2/2) \to A[1 + (a\nu^2)^{-p}]\sqrt{a\nu^2} \exp(-a\nu^2/2)$$

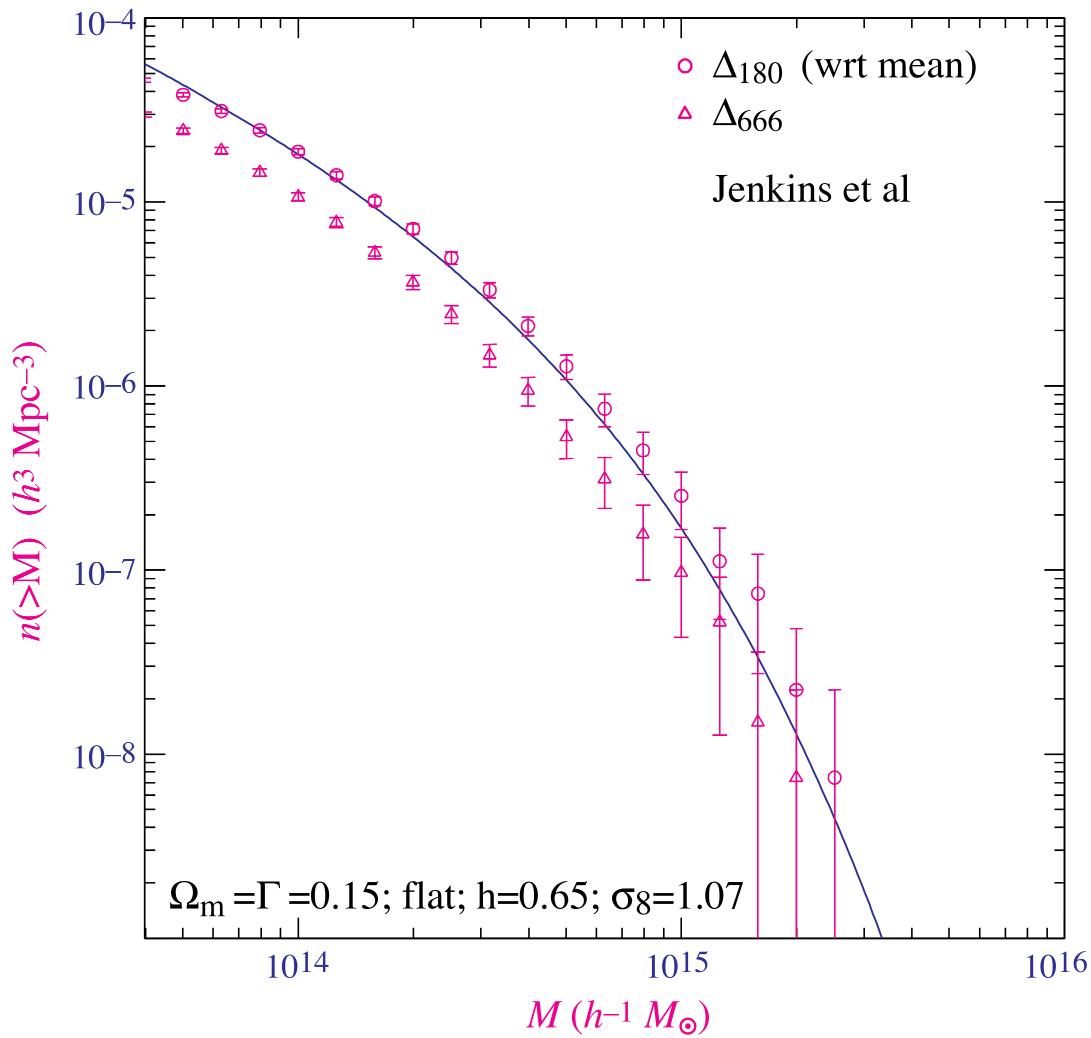
Sheth-Torman 1999, $a = 0.75$, $p = 0.3$. or a completely empirical fitting

$$\frac{dn}{d\ln M} = 0.301 \frac{\rho_m}{M} \frac{d\ln \sigma^{-1}}{d\ln M} \exp[-|\ln \sigma^{-1} + 0.64|^{3.82}]$$

Jenkins et al 2001. Choice is tied up with the question: what is the mass of a halo?

# Numerical Mass Function

- Example of difference in mass definition (from Hu & Kravstov 2002)



# Halo Bias

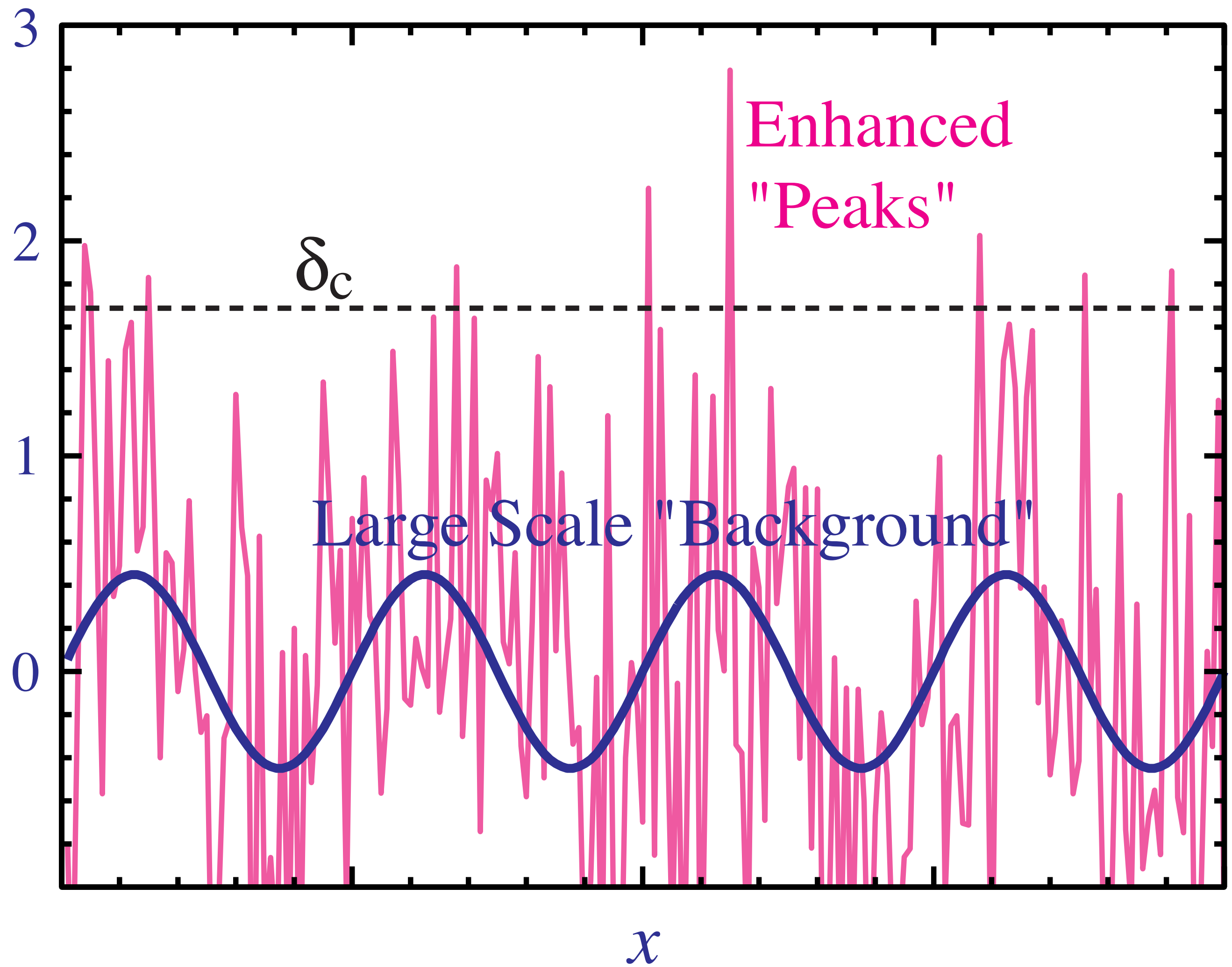
- If halos are formed without regard to the underlying density fluctuation and move under the gravitational field then their number density is an unbiased tracer of the dark matter density fluctuation

$$\left(\frac{\delta n}{n}\right)_{\rm halo} = \left(\frac{\delta \rho}{\rho}\right)$$

- However spherical collapse says the probability of forming a halo depends on the initial density field
- Large scale density field acts as "background" enhancement of probability of forming a halo or "peak"
- Peak-Background Split (Efstathiou 1998; Cole & Kaiser 1989; Mo & White 1997)

# Peak-Background Split

- Schematic Picture:



# Perturbed Mass Function

- Density fluctuation split

$$\delta = \delta_b + \delta_p$$

- Lowers the threshold for collapse

$$\delta_{cp} = \delta_c - \delta_b$$

so that $\nu = \delta_{cp}/\sigma$

- Taylor expand number density $n_M \equiv dn/d\ln M$

$$n_M + \frac{dn_M}{d\nu}\frac{d\nu}{d\delta_b}\delta_b \ldots = n_M \left[1 + \frac{(\nu^2 - 1)}{\sigma\nu}\right]$$

if mass function is given by Press-Schechter

$$n_M \propto \nu \exp(-\nu^2/2)$$

# Halo Bias

- Halos are biased tracers of the "background" dark matter field with a bias $b(M)$ that is given by spherical collapse and the form of the mass function
- Combine the enhancement with the original unbiased expectation

$$\frac{\delta n_M}{n_M} = b(M)\delta$$

- For Press-Schechter

$$b(M) = 1 + \frac{\nu^2 - 1}{\delta_c}$$

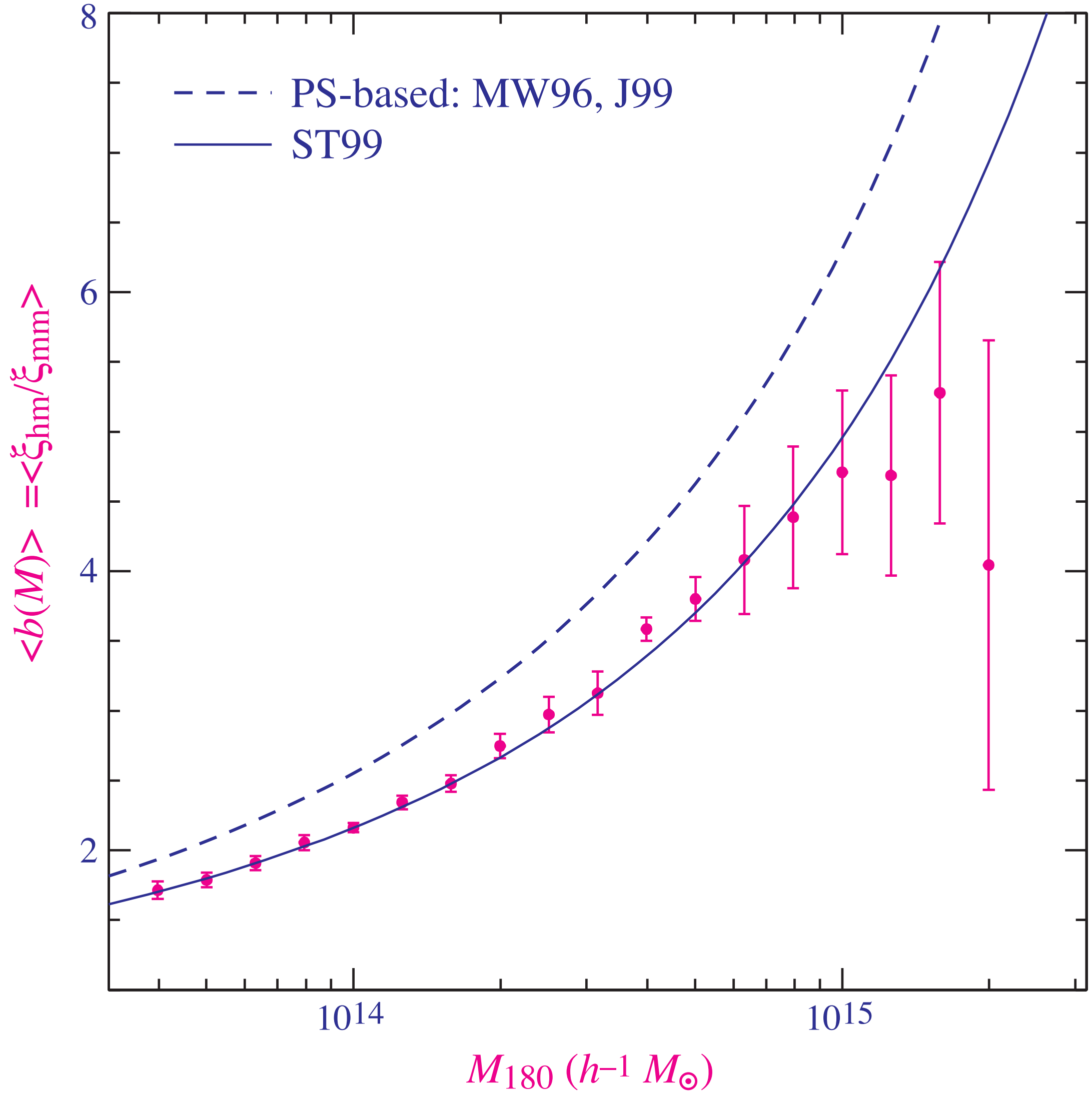
- Improved by the Sheth-Torman mass function

$$b(M) = 1 + \frac{a\nu^2 - 1}{\delta_c} + \frac{2p}{\delta_c[1 + (a\nu^2)^p]}$$

with $a = 0.75$ and $p = 0.3$ to match simulations.

# Numerical Bias

- Example of halo bias from a simulation (from Hu & Kravstov 2002)



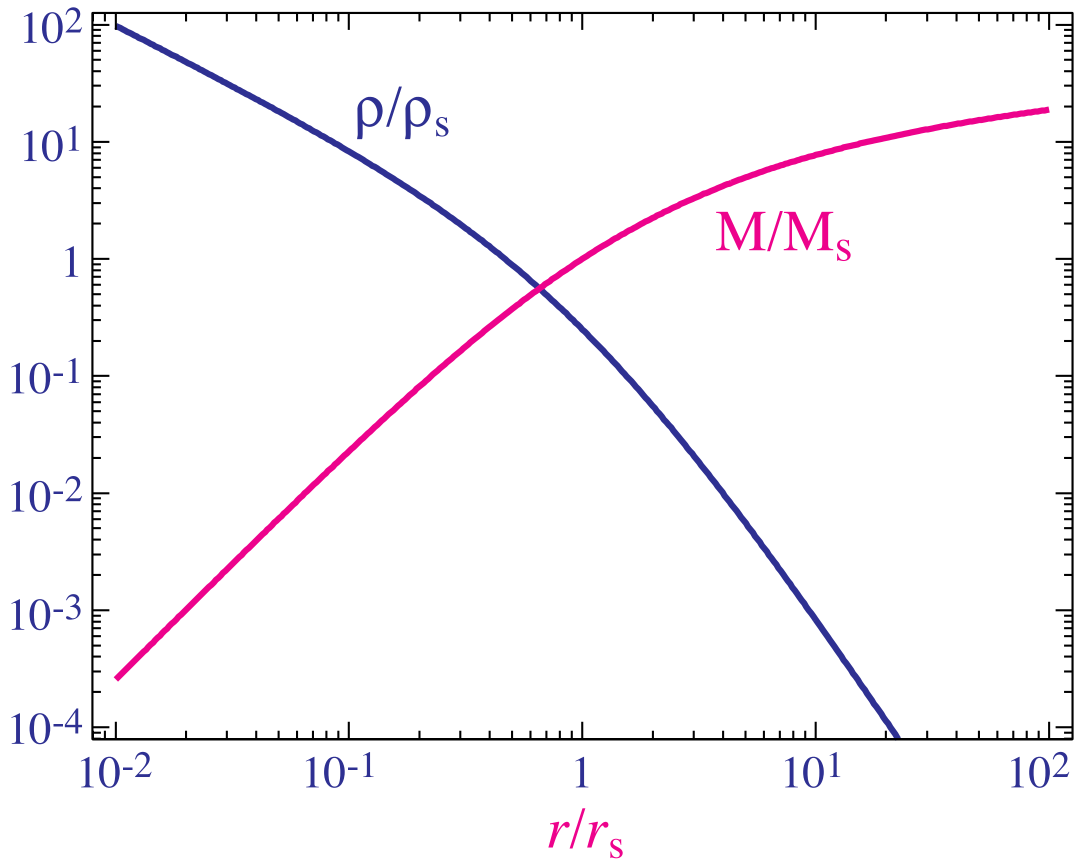
# What is a Halo?

- Mass function and halo bias depend on the definition of mass of a halo
- Agreement with simulations depend on how halos are identified
- Other observables (associated galaxies, $X$-ray, SZ) depend on the details of the density profile
- Fortunately, simulations have shown that halos take on a near universal form in their density profile at least on large scales.

# NFW Profile

- Density profile well-described by (Navarro, Frenk & White 1997)

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}$$



# Einasto Profile

- Current best simulations find that the inner slope runs rather than asymptotes to a cuspy constant
- This form is better fit by the Einasto profile (c.f. Sersic profile)

$$\ln \frac{\rho(r)}{\rho_s} = -\frac{2}{\alpha}\left[\left(\frac{r}{r_s}\right)^{\alpha} - 1\right]$$

- The local slope is given by

$$\frac{d\ln\rho}{d\ln r} = -2\left(\frac{r}{r_s}\right)^{\alpha}$$

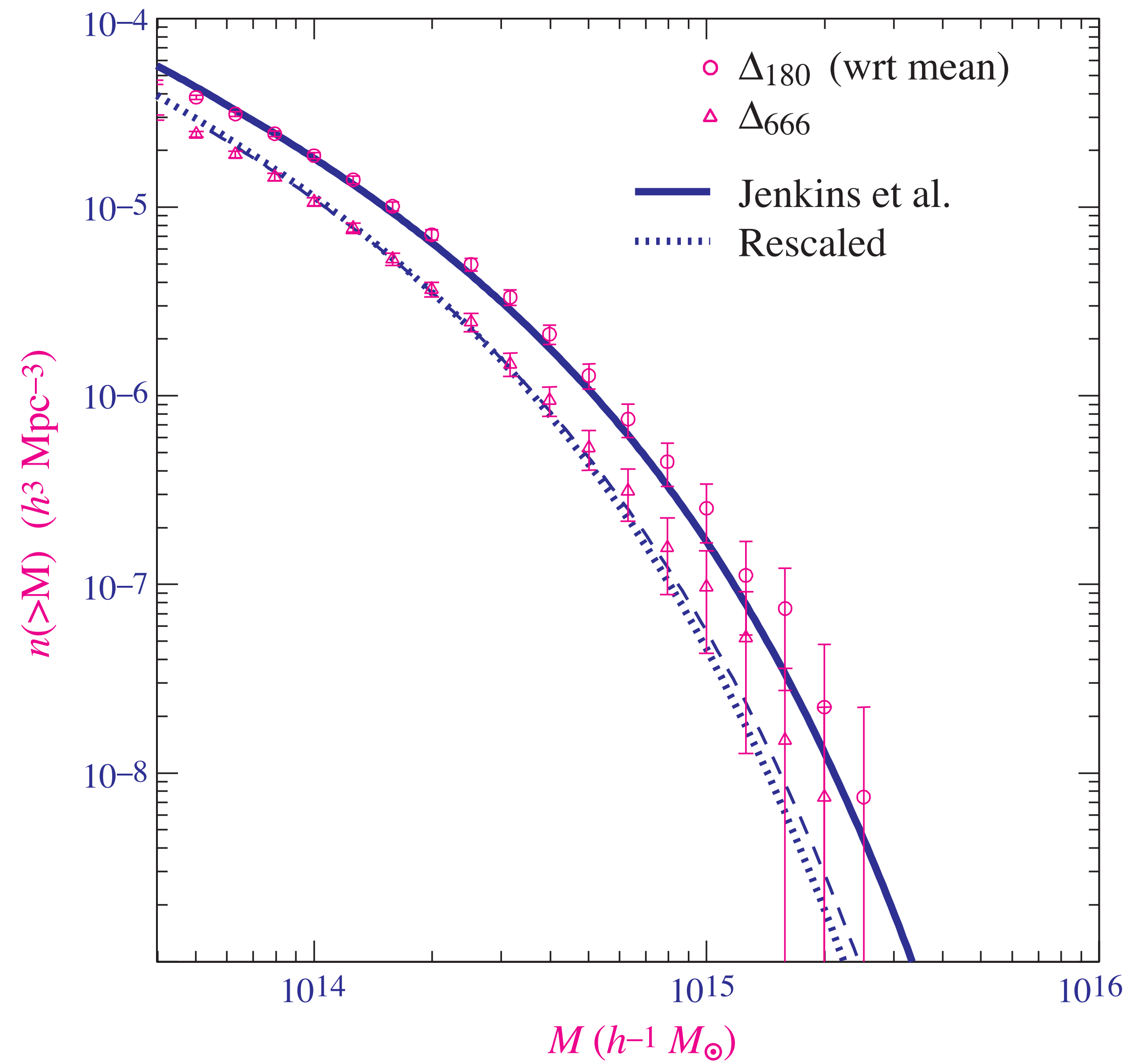
and continues to decrease as $r/r_s \to 0$

# Whence Universal Profile?

- Recent investigations by Dalal, Lithwick, Kuhlen (2010) suggests that the universal halo profile arises generically from peaks in a Gaussian random field
- Outer $r^{-3}$ profile predicted from slow accretion of material at low initial overdensity compared with peak
- Inner profile comes from adiabatic contraction (i.e. preserving adiabatic invariants during collapse) and depends on the initial density profile of peak
- Dynamical friction implies that the centroid of the initial density peak will settle to the center of the final halo

# Transforming the Masses

- NFW profile gives a way of transforming different mass definitions



# Lack of Concentration?

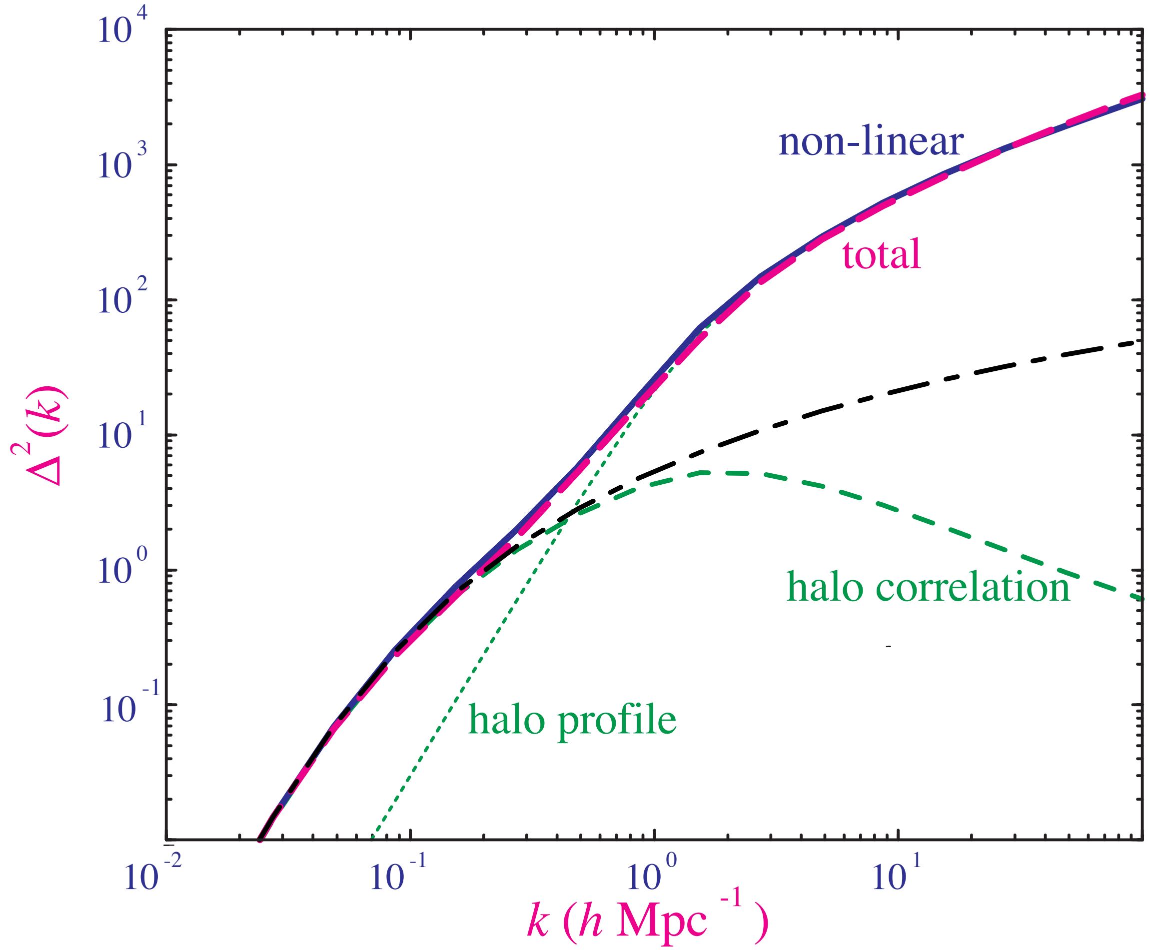
- NFW parameters may be recast into $M_v$, the mass of a halo out to the virial radius $r_v$ where the overdensity wrt mean reaches $\Delta_v = 180$.
- Concentration parameter

$$c \equiv \frac{r_v}{r_s}$$

- CDM predicts $c \sim 10$ for $M_*$ halos. Too centrally concentrated for galactic rotation curves?
- Possible discrepancy has lead to the exploration of dark matter alternatives: warm ($m$ ~keV) dark matter, self-interacting dark-matter, annihillating dark matter, ultra-light "fuzzy" dark matter, ...

# The Halo Model

- NFW halos, of abundance $n_M$ given by mass function, clustered according to the halo bias $b(M)$ and the linear theory $P(k)$
- Power spectrum example:



# Non-Linear Power Spectrum

- Non-linear power spectrum is composed of dark matter halos that are clustered according to the halo bias + the clustering due to the halo density profile

$$P_{\rm nl}(k,z) = I_2^2(k,z)P(k,z) + I_1(k,z)$$

where

$$I_2(k,z) = \int d\ln M \left(\frac{M}{\rho_m(z=0)}\right) \frac{dn}{d\ln M} b(M) y(k,M)$$

$$I_1(k,z) = \int d\ln M \left(\frac{M}{\rho_m(z=0)}\right)^2 \frac{dn}{d\ln M} y^2(k,M)$$

and $y$ is the Fourier transform of the halo profile with $y(0,M)=1$

$$y(k,M) = \frac{1}{M}\int_0^{r_h} dr 4\pi r^2 \rho(r,M) \frac{\sin(kr)}{kr}$$

# Galaxy Power Spectrum

- For galaxies, one defines a halo occupation distribution which determines the number of galaxies (satisfying a certain observational criteria) that can occupy a halo of mass $M$
- Take a simple example of a mass selection on the galaxies, then $N(M) = 0$ for $M < M_{\rm th}$ and above threshold $N(M) = C + S(M)$ where $C = 1$ accounts for the central galaxy and satellite galaxies follow a poisson distribution with mean $S(M) \approx M/30M_{\rm th}$

# Galaxy Power Spectrum

- Then assuming that satellites are distributed according to the mass profile

$$P_{\rm gal}(k,z) = I_2^2(k,z)P(k,z) + I_1(k,z)$$

where

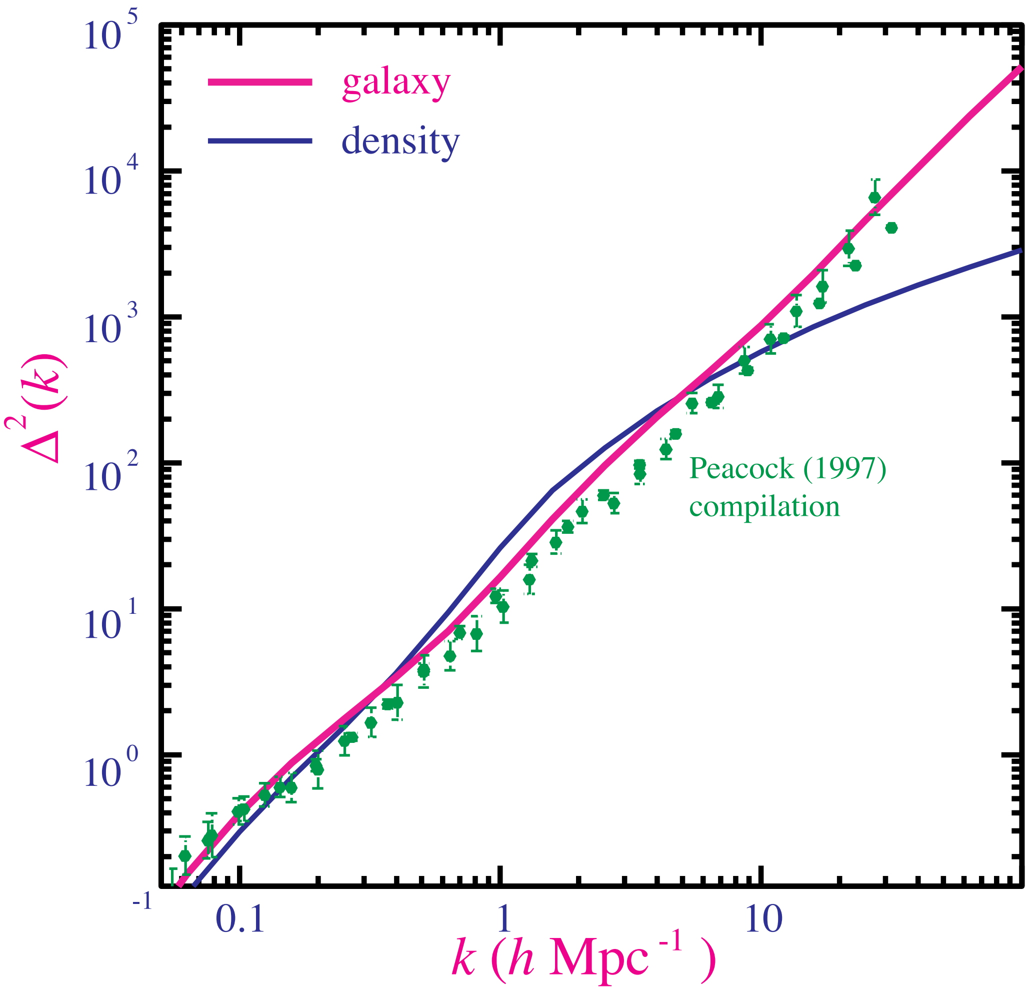
$$I_2(k,z) = \frac{1}{n_{\rm gal}}\int d\ln M \frac{dn}{d\ln M} b(M)[C + y(k,M)S(M)]$$

$$I_1(k,z) = \frac{1}{n_{\rm gal}^2}\int d\ln M \frac{dn}{d\ln M} [S^2(M)y^2(k,M) + 2CS(M)y(k,M)]$$

- Break between the one and two halo regime first seen by SDSS

# Galaxy Power Spectrum

- Example (Seljak 2001)




- An explanation of the nearly power law galaxy spectrum

# Incredible, Extensible Halo Model

- An industry developed to build semi-analytic models for wide variety of cosmological observables based on the halo model
- Idea: associate an observable (galaxies, gas, ...) with dark matter halos
- Let the halo model describe the statistics of the observable
- The overextended halo model?

# Halo Temperature

- Motivate with isothermal distribution, correct from simulations

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$

- Express in terms of virial mass $M_v$ enclosed at virial radius $r_v$

$$M_v = \frac{4\pi}{3} r_v^3 \rho_m \Delta_v = \frac{2}{G} r_v \sigma^2$$

- Eliminate $r_v$, temperature $T \propto \sigma^2$ velocity dispersion$^2$
- Then $T \propto M_v^{2/3} (\rho_m \Delta_v)^{1/3}$ or

$$\left( \frac{M_v}{10^{15} h^{-1} M_\odot} \right) = \left[ \frac{f}{(1+z)(\Omega_m \Delta_v)^{1/3}} \frac{T}{1\mathrm{keV}} \right]^{3/2}$$

- Theory ($X$-ray weighted): $f \sim 0.75$; observations $f \sim 0.54$. Difference is crucial in determining cosmology from cluster counts!

# Summary

- Dark matter simulations well-understood and can be modelled with dark matter halos
- Halo formation modelled by spherical collapse, two magic numbers $\delta_c = 1.686$ and $\Delta_v = 178$
- Halo abundance described by a mass function with exponential high mass cutoff – rare clusters extremely sensitive to power spectrum amplitude and growth rate $\rightarrow$ dark energy

  Possibly too many small halos or sub-structure?
- Halo clustering modelled with peak-background split leading to halo bias
- Halo profile described by NFW halos

  Possibly too high central concentration
- Associate an observable with a halo $\rightarrow$ a halo model