Ast 448: CMB
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Syllabus

This course will have be composed of 2 parts

- Lectures establishing a common denominator on CMB physics: temperature anisotropy, polarization...
- Current topics of your own interests, culminating in a presentation by you to the class

Prerequisites

- Cosmology - at least the advanced undergrad (Ryden) or graduate level:
  - FRW cosmology
  - Thermal history
  - Inflation

Helpful:

- Radiative processes, GR, stat mech
CMB Temperature Anisotropy

- Planck map of the temperature anisotropy (first discovered by COBE) from recombination:
CMB Temperature Anisotropy

- Power spectrum shows characteristic scales where the intensity of variations peak - reveals geometry and contents of the universe:
CMB Parameter Inferences

- Spectrum constrains the matter-energy contents of the universe
- Planck 2018 results [arXiv:1807.06209]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>TT+lowE 68% limits</th>
<th>TE+lowE 68% limits</th>
<th>EE+lowE 68% limits</th>
<th>TT,TE,EE+lowE 68% limits</th>
<th>TT,TE,EE+lowE+lensing 68% limits</th>
<th>TT,TE,EE+lowE+lensing+BAO 68% limits</th>
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<tbody>
<tr>
<td>$\Omega_b h^2$</td>
<td>0.02212 ± 0.00022</td>
<td>0.02249 ± 0.00025</td>
<td>0.0240 ± 0.0012</td>
<td>0.02236 ± 0.00015</td>
<td>0.02237 ± 0.00015</td>
<td>0.02242 ± 0.00014</td>
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<td>$\Omega_c h^2$</td>
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<td>0.1177 ± 0.0020</td>
<td>0.1158 ± 0.0046</td>
<td>0.1202 ± 0.0014</td>
<td>0.1200 ± 0.0012</td>
<td>0.11933 ± 0.00091</td>
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<td>$100 \theta_{MC}$</td>
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<td>1.04139 ± 0.00049</td>
<td>1.03999 ± 0.00089</td>
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<td>1.04092 ± 0.00031</td>
<td>1.04101 ± 0.00029</td>
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<td>$\tau$</td>
<td>0.0522 ± 0.0080</td>
<td>0.0496 ± 0.0085</td>
<td>0.0527 ± 0.0090</td>
<td>0.0544*0.0070 ±0.0081</td>
<td>0.0544 ± 0.0073</td>
<td>0.0561 ± 0.0071</td>
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<td>$\ln(10^{10} A_s)$</td>
<td>3.040 ± 0.016</td>
<td>3.018*0.020 ±0.018</td>
<td>3.052 ± 0.022</td>
<td>3.045 ± 0.016</td>
<td>3.044 ± 0.014</td>
<td>3.047 ± 0.014</td>
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<td>$n_s$</td>
<td>0.9626 ± 0.0057</td>
<td>0.967 ± 0.011</td>
<td>0.980 ± 0.015</td>
<td>0.9649 ± 0.0044</td>
<td>0.9649 ± 0.0042</td>
<td>0.9665 ± 0.0038</td>
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<td>$H_0$ [km s^{-1} Mpc^{-1}]</td>
<td>66.88 ± 0.92</td>
<td>68.44 ± 0.91</td>
<td>69.9 ± 2.7</td>
<td>67.27 ± 0.60</td>
<td>67.36 ± 0.54</td>
<td>67.66 ± 0.42</td>
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CMB Power Spectra

- Power spectra of CMB
  - temperature
  - polarization
  - lensing
Tensor Power Spectrum

- Gravitational waves from inflation (yet to be detected)
Ast 448

Set 1: Temperature Anisotropy
Schematic Outline

- Take apart features in the power spectrum
- Take apart features in the power spectrum

\[ \Delta (\mu \text{K})^{\frac{1}{10}} \]

\[ l \leq l_a \]

\[ \Theta \Theta \]

\[ EE \]

\[ ISW \]

\[ \text{damping} \]

\[ \text{driving} \]

\[ \text{damping} \]

\[ \text{tight coupling} \]

\[ l_e q, l_A, l_d \]
Last Scattering

- Angular distribution of radiation is the 3D temperature field projected onto a shell - surface of last scattering

- Shell radius is distance from the observer to recombination: called the last scattering surface

- Take the radiation distribution at last scattering to also be described by an isotropic temperature fluctuation field $\Theta(x)$
Angular Power Spectrum

- Take recombination to be instantaneous

\[ \Theta(\hat{n}) = \int dD \Theta(x) \delta(D - D_*) \]

where \( D \) is the comoving distance and \( D_* \) denotes recombination.

- Describe the temperature field by its Fourier moments

\[ \Theta(x) = \int \frac{d^3 k}{(2\pi)^3} \Theta(k) e^{i k \cdot x} \]

- Orthogonality and Completeness (forward and inverse transform):

\[ \int d^3 x e^{i(k-k') \cdot x} = (2\pi)^3 \delta(k - k') \]

\[ \int \frac{d^3 k}{(2\pi)^3} e^{i k \cdot (x-x')} = \delta(x - x') \]
Angular Power Spectrum

- Statistical homogeneity and isotropy

\[ \langle \Theta(x) \Theta(x') \rangle = C(|x - x'|) \]

function of separation only

\[ \langle \Theta(x + d) \Theta(x' + d) \rangle = \langle \Theta(x) \Theta(x') \rangle \]

\[ \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} e^{-ik \cdot x + ik' \cdot x'} e^{-i(k - k') \cdot d} \langle \Theta^*(k) \Theta(k') \rangle \]

\[ = \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} e^{-ik \cdot x + ik' \cdot x'} \langle \Theta^*(k) \Theta(k') \rangle \]

requires the 2pt Fourier correlation to be described by a power spectrum

\[ \langle \Theta^*(k) \Theta(k') \rangle = (2\pi)^3 \delta(k - k') P_T(k) \]
Angular Power Spectrum

- Correlation function and power spectrum are Fourier conjugates

\[ C(|x - x'|) = \langle \Theta(x) \Theta(x') \rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot (x' - x)} P_T(k) \]

- Log weighted power spectrum determines variance

\[ \langle \Theta(x) \Theta(x) \rangle = \int \frac{d^3k}{(2\pi)^3} P_T(k) = \int \frac{dk}{k} \frac{k^3}{2\pi^2} P_T = \int \frac{dk}{k} \Delta_T^2(k) \]

\[ \Delta_T^2 = \frac{k^3}{2\pi^2} P_T[= P_T(k)] \]

and is the contribution to the total variance per log interval in \( k \)

- \( \Delta_T^2 \) dimensionless, whereas \( P_T \) has dimensions of \([L^3]\), e.g. \((h^{-1}\text{Mpc})^3\) for the power spectrum of a redshift survey
Angular Power Spectrum

- Temperature field

\[ \Theta(\hat{n}) = \int \frac{d^3k}{(2\pi)^3} \Theta(k) e^{i\mathbf{k} \cdot D^* \hat{n}} \]

Multipole moments \( \Theta(\hat{n}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m} \)

- Orthogonality:

\[ \int d\hat{n} Y_{\ell m}^* (\hat{n}) Y_{\ell' m'} (\hat{n}) = \delta_{\ell \ell'} \delta_{m m'} \]

Completeness:

\[ \sum_{\ell m} Y_{\ell m}^* (\hat{n}) Y_{\ell m} (\hat{n}') = \delta(\phi - \phi') \delta(\cos \theta - \cos \theta') \]

- Statistical isotropy:

\[ \langle \Theta_{\ell m}^* \Theta_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell \]
Angular Power Spectrum

- Expand out plane wave in spherical coordinates

\[ e^{i k D_\bullet \hat{n}} = 4\pi \sum_{\ell m} i^\ell j_\ell (k D_\bullet) Y_{\ell m}^*(\hat{k}) Y_{\ell m}(\hat{n}) \]

- Aside: as in the figure, it will often be convenient when considering a single k mode to orient the north pole to \( \hat{k} \). This simplifies the decomposition since

\[ Y_{\ell m}^*(\hat{k}) \rightarrow Y_{\ell m}^*(0) = \delta_{m0} \sqrt{\frac{2\ell + 1}{4\pi}} \]
Angular Power Spectrum

- Power spectrum

\[ \Theta_{\ell m} = \int \frac{d^3 k}{(2\pi)^3} \Theta(k) 4\pi i^\ell j_\ell(kD_*) Y_{\ell m}^*(k) \]

\[ \langle \Theta^*_{\ell m} \Theta_{\ell' m'} \rangle = \int \frac{d^3 k}{(2\pi)^3} \left( 4\pi \right)^2 i^{\ell - \ell'} j_\ell(kD_*) j_{\ell'}(kD_*) Y_{\ell m}(k) Y_{\ell' m'}^*(k) P_T(k) \]

\[ = \delta_{\ell \ell'} \delta_{m m'} 4\pi \int d \ln k j_\ell^2(kD_*) \Delta^2_T(k) \]

with \( \int_0^\infty j_\ell^2(x) d \ln x = 1/(2\ell(\ell + 1)) \), slowly varying \( \Delta^2_T \)

- Angular power spectrum:

\[ C_\ell = \frac{4\pi \Delta^2_T(\ell/D_*)}{2\ell(\ell + 1)} = \frac{2\pi}{\ell(\ell + 1)} \Delta^2_T(\ell/D_*) \]
Angular Power Spectrum

- The log power spectrum (sometimes called $D_\ell$)

$$\frac{\ell(\ell + 1)}{2\pi}C_\ell \approx \Delta^2_T$$

so that a scale invariant spectrum $\Delta^2_T = \text{const}$ is scale invariant in the log power spectrum

- Related to the contribution to the variance per log interval in $\ell$

$$\langle \Theta(\hat{n})\Theta(\hat{n}) \rangle = \langle \Theta(0)\Theta(0) \rangle = \sum_\ell \frac{2\ell + 1}{4\pi}C_\ell = \sum_\ell \frac{1}{\ell} \frac{\ell(2\ell + 1)}{4\pi}C_\ell$$

with the two being equivalent if $\ell \gg 1$
Thomson Scattering

- Thomson scattering of photons off of free electrons is the most important CMB process with a cross section (averaged over polarization states) of

\[ \sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25}\text{cm}^2 \]

- Density of free electrons in a fully ionized \( x_e = 1 \) universe

\[ n_e = (1 - Y_p/2)x_e n_b \approx 10^{-5}\Omega_b h^2 (1 + z)^3 \text{cm}^{-3}, \]

where \( Y_p \approx 0.24 \) is the Helium mass fraction, creates a high (comoving) Thomson opacity

\[ \dot{\tau} \equiv n_e \sigma_T a \]

where dots are conformal time \( \eta \equiv \int dt/a \) derivatives and \( \tau \) is the optical depth.
Tight Coupling Approximation

- Near recombination $z \approx 10^3$ and $\Omega_b h^2 \approx 0.02$, the (comoving) mean free path of a photon

$$\lambda_C \equiv \frac{1}{\tau} \sim 2.5\text{Mpc}$$

small by cosmological standards!

- On scales $\lambda \gg \lambda_C$ photons are tightly coupled to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions

- Specifically, their bulk velocities are defined by a single fluid velocity $v_\gamma = v_b$ and the photons carry no anisotropy in the rest frame of the baryons

- $\rightarrow$ No heat conduction or viscosity (anisotropic stress) in fluid
Full Equations of Motion

- Continuity

\[ \dot{\Theta} = -\frac{k}{3} v_\gamma - \dot{\Phi}, \quad \dot{\delta}_b = -kv_b - 3\dot{\Phi} \]

which expresses number conservation in the presence of velocity divergence and local expansion, with \( \rho_b = m_b n_b \)

- Navier-Stokes (Euler + heat conduction, viscosity)

\[ \dot{v}_\gamma = k(\Theta + \Psi) - \frac{k}{6} \pi_\gamma - \dot{\tau}(v_\gamma - v_b) \]

\[ \dot{v}_b = -\frac{\dot{a}}{a} v_b + k\Psi + \dot{\tau}(v_\gamma - v_b)/R \]

where the photon momentum changes due to pressure, gravity and anisotropic stress \( \pi_\gamma \) gradients (from radiation viscosity) and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation
Zeroth Order Approximation

- **Momentum density** of a fluid is \((\rho + p)v\), where \(p\) is the pressure

- **Neglect** the momentum density of the baryons

\[
R \equiv \frac{(\rho_b + p_b)v_b}{(\rho_\gamma + p_\gamma)v_\gamma} = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}
\]

\[
\approx 0.6 \left( \frac{\Omega_b h^2}{0.02} \right) \left( \frac{a}{10^{-3}} \right)
\]

since \(\rho_\gamma \propto T^4\) is fixed by the CMB temperature \(T = 2.73(1 + z)\)K – OK substantially before recombination

- **Neglect radiation** in the expansion (not a good approx, just for pedagogical start)

\[
\frac{\rho_m}{\rho_r} = 3.6 \left( \frac{\Omega_m h^2}{0.15} \right) \left( \frac{a}{10^{-3}} \right)
\]

- **Neglect gravity** (obviously just for pedagogy)
Fluid Equations

- **Density** \(\rho_\gamma \propto T^4\) so define **temperature fluctuation** \(\Theta\)

\[
\delta_\gamma = 4 \frac{\delta T}{T} \equiv 4\Theta
\]

- **Real space continuity equation**

\[
\dot{\delta}_\gamma = -(1 + w_\gamma) k v_\gamma
\]

\[
\dot{\Theta} = -\frac{1}{3} k v_\gamma
\]

- **Euler equation (neglecting gravity)**

\[
\dot{v}_\gamma = -(1 - 3w_\gamma) \frac{\dot{a}}{a} v_\gamma + \frac{kc_s^2}{1 + w_\gamma} \delta_\gamma
\]

\[
\dot{v}_\gamma = kc_s^2 \frac{3}{4} \delta_\gamma = 3c_s^2 k\Theta
\]
Oscillator: Take One

- Combine these to form the simple harmonic oscillator equation

\[ \ddot{\Theta} + c_s^2 k^2 \Theta = 0 \]

where the sound speed is adiabatic

\[ c_s^2 = \frac{\delta p_\gamma}{\delta \rho_\gamma} = \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma} \]

here \( c_s^2 = 1/3 \) since we are photon-dominated

- General solution:

\[ \Theta(\eta) = \Theta(0) \cos(k s) + \frac{\dot{\Theta}(0)}{k c_s} \sin(k s) \]

where the sound horizon is defined as \( s \equiv \int c_s d\eta \)
Harmonic Extrema

- All modes are frozen in at recombination (denoted with a subscript $*$).
- Temperature perturbations of different amplitude for different modes.
- For the adiabatic (curvature mode) initial conditions
  $$\dot{\Theta}(0) = 0$$
- So solution
  $$\Theta(\eta_*) = \Theta(0) \cos(k s_*)$$
Harmonic Extrema

- Modes caught in the **extrema** of their oscillation will have enhanced fluctuations

\[ k_n s_* = n\pi \]

yielding a **fundamental scale** or frequency, related to the inverse sound horizon

\[ k_A = \pi / s_* \]

and a **harmonic relationship** to the other extrema as \(1 : 2 : 3\ldots\)
Peak Location

- The fundamental physical scale is translated into a fundamental angular scale by simple projection according to the angular diameter distance $D_A$

\[
\theta_A = \frac{\lambda_A}{D_A}
\]

\[
\ell_A = k_A D_A
\]

- In a flat universe, the distance is simply $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$, the horizon distance, and $k_A = \pi/s_* = \sqrt{3\pi}/\eta_*$ so

\[
\theta_A \approx \frac{\eta_*}{\eta_0}
\]

- In a matter-dominated universe $\eta \propto a^{1/2}$ so $\theta_A \approx 1/30 \approx 2^\circ$ or

\[
\ell_A \approx 200
\]
Curvature

- In a curved universe, the apparent or angular diameter distance is no longer the conformal distance $D_A = R \sin(D/R) \neq D$

- Objects in a closed universe are further than they appear! gravitational lensing of the background...

- Curvature scale of the universe must be substantially larger than current horizon
Curvature

- **Flat universe** indicates critical density and implies missing energy given local measures of the matter density “dark energy”
- $D$ also depends on dark energy density $\Omega_{DE}$ and equation of state $w = p_{DE}/\rho_{DE}$.
- Expansion rate at recombination or matter-radiation ratio enters into calculation of $k_A$. 
Fixed Deceleration Epoch

- CMB determination of matter density controls all determinations in the deceleration (matter dominated) epoch

- **Planck:** \( \Omega_m h^2 = 0.1426 \pm 0.0025 \rightarrow 1.7\% \)

- Distance to recombination \( D_\ast \) determined to \( \frac{1}{4} 1.7\% \approx 0.43\% \)
  (\( \Lambda CDM \) result 0.46%; \( \Delta h/h \approx -\Delta \Omega_m h^2 / \Omega_m h^2 \))
  [more general: \(-0.11\Delta w - 0.48\Delta \ln h - 0.15\Delta \ln \Omega_m - 1.4\Delta \ln \Omega_{tot} = 0 \)]

- Expansion rate during any redshift in the deceleration epoch determined to \( \frac{1}{2} 1.7\% \)

- Distance to any redshift in the deceleration epoch determined as

  \[
  D(z) = D_\ast - \int_z^{z_\ast} \frac{dz}{H(z)}
  \]

- Volumes determined by a combination \( dV = D_A^2 d\Omega d\tau / H(z) \)

- Structure also determined by growth of fluctuations from \( z_\ast \)
Doppler Effect

- **Bulk motion** of fluid changes the observed temperature via Doppler shifts

\[
\left( \frac{\Delta T}{T} \right)_{\text{dop}} = \hat{n} \cdot \mathbf{v}_\gamma
\]

- Averaged over directions

\[
\left( \frac{\Delta T}{T} \right)_{\text{rms}} = \frac{v_\gamma}{\sqrt{3}}
\]

- Acoustic solution

\[
\frac{v_\gamma}{\sqrt{3}} = -\frac{\sqrt{3}}{k} \dot{\Theta} = \frac{\sqrt{3}}{k} k c_s \Theta(0) \sin(k s) = \Theta(0) \sin(k s)
\]
**Doppler Peaks?**

- **Doppler effect** for the photon dominated system is of equal amplitude and $\pi/2$ out of phase: extrema of temperature are turning points of velocity.

- Effects add in **quadrature**:

  \[
  \left( \frac{\Delta T}{T} \right)^2 = \Theta^2(0)[\cos^2(ks) + \sin^2(ks)] = \Theta^2(0)
  \]

- **No peaks** in $k$ spectrum! However the Doppler effect carries an angular dependence that changes its projection on the sky:

  \[\hat{n} \cdot \mathbf{v}_\gamma \propto \hat{n} \cdot \hat{k}\]
Doppler Peaks?

- Coordinates where $\hat{z} \parallel \hat{k}$

$$Y_{10} Y_{\ell 0} \rightarrow Y_{\ell \pm 1 0}$$

Recoupling $j'_\ell Y_{\ell 0}$: no peaks in Doppler effect
Restoring Gravity

- Take a simple photon dominated system with gravity
- Continuity altered since a gravitational potential represents a stretching of the spatial fabric that dilutes number densities – formally a spatial curvature perturbation
- Think of this as a perturbation to the scale factor $a \rightarrow a(1 + \Phi)$ so that the cosmogical redshift is generalized to

$$\frac{\dot{a}}{a} \rightarrow \frac{\dot{a}}{a} + \dot{\Phi}$$

so that the continuity equation becomes

$$\dot{\Theta} = -\frac{1}{3} k v_\gamma - \dot{\Phi}$$
Restoring Gravity

- **Gravitational force** in momentum conservation \( F = -m \nabla \Psi \) generalized to momentum density modifies the Euler equation to

\[
\dot{v}_\gamma = k (\Theta + \Psi)
\]

- General relativity says that \( \Phi \) and \( \Psi \) are the relativistic analogues of the Newtonian potential and that \( \Phi \approx -\Psi \).

- In our matter-dominated approximation, \( \Phi \) represents matter density fluctuations through the cosmological Poisson equation

\[
k^2 \Phi = 4\pi G a^2 \rho_m \Delta_m
\]

where the difference comes from the use of comoving coordinates for \( k \) (\( a^2 \) factor), the removal of the background density into the background expansion (\( \rho \Delta_m \)) and finally a coordinate subtlety that enters into the definition of \( \Delta_m \).
In the matter dominated epoch potentials are constant because infall generates velocities as $v_m \sim k\eta \Psi$

Velocity divergence generates density perturbations as $\Delta_m \sim -k\eta v_m \sim -(k\eta)^2 \Psi$

And density perturbations generate potential fluctuations

$$\Phi = \frac{4\pi G a^2 \rho \Delta}{k^2} \approx \frac{3}{2} \frac{H^2 a^2}{k^2} \Delta \sim \frac{\Delta}{(k\eta)^2} \sim -\Psi$$

keeping them constant. Note that because of the expansion, density perturbations must grow to keep potentials constant.
Constant Potentials

- More generally, if stress perturbations are negligible compared with density perturbations ($\delta p \ll \delta \rho$) then potential will remain roughly constant.

- More specifically a variant called the Bardeen or comoving curvature is strictly constant:

$$\mathcal{R} = \text{const} \approx \frac{5 + 3w}{3 + 3w} \Phi$$

where the approximation holds when $w \approx \text{const}$. 
Oscillator: Take Two

- Combine these to form the simple harmonic oscillator equation

\[ \ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - \dot{\Phi} \]

- In a CDM dominated expansion \( \dot{\Phi} = \dot{\Psi} = 0 \). Also for photon domination \( c_s^2 = 1/3 \) so the oscillator equation becomes

\[ \ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0 \]

- Solution is just an offset version of the original

\[ [\Theta + \Psi](\eta) = [\Theta + \Psi](0) \cos(ks) \]

- \( \Theta + \Psi \) is also the observed temperature fluctuation since photons lose energy climbing out of gravitational potentials at recombination
Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or effective temperature

\[ \Theta + \Psi \]

- Effective temperature oscillates around zero with amplitude given by the initial conditions
- Note: initial conditions are set when the perturbation is outside of horizon, need inflation or other modification to matter-radiation FRW universe.

- GR says that initial temperature is given by initial potential
Sachs-Wolfe Effect and the Magic 1/3

- A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]
  \[
  \frac{\delta t}{t} = \Psi
  \]

- Convert this to a perturbation in the scale factor,
  \[
  t = \int \frac{da}{aH} \propto \int \frac{da}{a\rho^{1/2}} \propto a^{3(1+w)/2}
  \]
  where \( w \equiv p/\rho \) so that during matter domination
  \[
  \frac{\delta a}{a} = \frac{2}{3} \frac{\delta t}{t}
  \]

- CMB temperature is cooling as \( T \propto a^{-1} \) so
  \[
  \Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3} \Psi
  \]
Sachs-Wolfe Normalization

- Use measurements of $\Delta T / T \approx 10^{-5}$ in the Sachs-Wolfe effect to infer $\Delta_R^2$

- Recall in matter domination $\Psi = -3R/5$

$$\frac{\ell(\ell + 1)C_\ell}{2\pi} \approx \Delta_T^2 \approx \frac{1}{25} \Delta_R^2$$

- Thus, amplitude of initial curvature fluctuations is $\Delta_R \approx 5 \times 10^{-5}$

- Modern usage: acoustic peak measurements plus known radiation transfer function is used to convert $\Delta T / T$ to $\Delta_R$. Best measured at $k = 0.08\,\text{Mpc}^{-1}$ by Planck

- Current convention set in the WMAP era

$$\Delta_R^2(k) \equiv A_s \left( \frac{k}{0.05\,\text{Mpc}^{-1}} \right)^{n_s-1}$$

so $A_s \sim 2.5 \times 10^{-9}$ (slightly smaller since red tilt $n_s - 1 \approx -0.04$)
Baryon Loading

- Baryons add extra mass to the photon-baryon fluid
- Controlling parameter is the momentum density ratio:

\[ R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left( \frac{a}{10^{-3}} \right) \]

of order unity at recombination

- Momentum density of the joint system is conserved

\[
(p_\gamma + p_\gamma)v_\gamma + (\rho_b + p_b)v_b \approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma)v_\gamma
= (1 + R)(\rho_\gamma + p_\gamma)v_\gamma b
\]
New Euler Equation

- Momentum density ratio enters as

\[(1 + R)v_{\gamma b} = k\Theta + (1 + R)k\Psi\]

- Photon continuity remains the same

\[\dot{\Theta} = -\frac{k}{3}v_{\gamma b} - \dot{\Phi}\]

- Modification of oscillator equation

\[\left[(1 + R)\Theta\right] + \frac{1}{3}k^2\Theta = -\frac{1}{3}k^2(1 + R)\Psi - \left[(1 + R)\dot{\Phi}\right]\]
Oscillator: Take Three

- Combine these to form the not-quite-so simple harmonic oscillator equation

\[
c_s^2 \frac{d}{d\eta} \left( c_s^{-2} \dot{\Theta} \right) + c_s^2 k^2 \Theta = - \frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} \left( c_s^{-2} \dot{\Phi} \right)
\]

where \( c_s^2 \equiv \frac{\dot{p}_{\gamma b}}{\dot{\rho}_{\gamma b}} \)

\[
c_s^2 = \frac{1}{3} \frac{1}{1 + R}
\]

- In a CDM dominated expansion \( \dot{\Phi} = \dot{\Psi} = 0 \) and the adiabatic approximation \( \dot{R}/R \ll \omega = kc_s \)

\[
[\Theta + (1 + R)\Psi](\eta) = [\Theta + (1 + R)\Psi](0) \cos(ks)
\]
Baryon Peak Phenomenology

- Photon-baryon ratio enters in three ways
- Overall larger amplitude:

\[ [\Theta + (1 + R)\Psi](0) = \frac{1}{3}(1 + 3R)\Psi(0) \]

- Even-odd peak modulation of effective temperature

\[ [\Theta + \Psi]_{\text{peaks}} = [\pm(1 + 3R) - 3R] \frac{1}{3} \Psi(0) \]

\[ [\Theta + \Psi]_1 - [\Theta + \Psi]_2 = [-6R] \frac{1}{3} \Psi(0) \]

- Shifting of the sound horizon down or \( \ell_A \) up

\[ \ell_A \propto \sqrt{1 + R} \]
Photon Baryon Ratio Evolution

- Actual effects smaller since $R$ evolves

- Oscillator equation has time evolving mass

\[ c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = 0 \]

- Effective mass is $m_{\text{eff}} = 3c_s^{-2} = (1 + R)$

- Adiabatic invariant

\[ \frac{E}{\omega} = \frac{1}{2} m_{\text{eff}} \omega A^2 = \frac{1}{2} 3c_s^{-2} k c_s A^2 \propto A^2 (1 + R)^{1/2} = \text{const.} \]

- Amplitude of oscillation $A \propto (1 + R)^{-1/4}$ decays adiabatically as the photon-baryon ratio changes
Baryons in the Power Spectrum

- Relative heights of peaks
The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator:

\[ c_s^2 \frac{d}{d\eta} \left( c_s^{-2} \dot{\Theta} \right) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} \left( c_s^{-2} \Phi \right) \]

changes in the gravitational potentials alter the form of the acoustic oscillations.

If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator.

Term involving \( \Psi \) is the ordinary gravitational force.

Term involving \( \Phi \) involves the \( \dot{\Phi} \) term in the continuity equation as a (curvature) perturbation to the scale factor.
Potential Decay

• Matter-to-radiation ratio

\[ \frac{\rho_m}{\rho_r} \approx 24 \Omega_m h^2 \left( \frac{a}{10^{-3}} \right) \]

of order unity at recombination in a low \( \Omega_m \) universe

• Radiation is not stress free and so impedes the growth of structure

\[ k^2 \Phi = 4\pi G a^2 \rho_r \Delta_r \]

\( \Delta_r \sim 4\Theta \) oscillates around a constant value, \( \rho_r \propto a^{-4} \) so the Newtonian curvature decays.

• General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale.
Radiation Driving

- Decay is timed precisely to drive the oscillator - close to fully coherent

\[ |[\Theta + \Psi](\eta)| = |[\Theta + \Psi](0) + \Delta \Psi - \Delta \Phi| \]
\[ = \left| \frac{1}{3} \Psi(0) - 2\Psi(0) \right| = \left| \frac{5}{3} \Psi(0) \right| \]

- $5 \times$ the amplitude of the Sachs-Wolfe effect!
External Potential Approach

• Solution to homogeneous equation

\[(1 + R)^{-1/4} \cos(ks), \quad (1 + R)^{-1/4} \sin(ks)\]

• Give the general solution for an external potential by propagating impulsive forces

\[(1 + R)^{1/4} \Theta(\eta) = \Theta(0) \cos(ks) + \frac{\sqrt{3}}{k} \left[ \dot{\Theta}(0) + \frac{1}{4} \dot{R}(0) \Theta(0) \right] \sin ks \]

\[+ \frac{\sqrt{3}}{k} \int_0^{\eta} d\eta' (1 + R')^{3/4} \sin[ks - ks'] F(\eta')\]

where

\[F = -\ddot{\Phi} - \frac{\dot{R}}{1 + R} \dot{\Phi} - \frac{k^2}{3} \Psi\]

• Useful if general form of potential evolution is known
Matter-Radiation in the Power Spectrum

- Coherent approximation is **exact** for a photon-baryon fluid but reality is reduced to $\sim 4 \times$ because **neutrino contribution** is free streaming not fluid like.

- Neutrinos drive the oscillator less efficiently and also slightly change the phase of the oscillation.

- Actual **initial conditions** are $\Theta + \Psi = \Psi / 2$ for radiation domination but comparison to matter dominated SW correct.

- With 3 peaks, it is possible to solve for both the baryons and dark matter densities, providing a calibration for the sound horizon.

- Higher peaks check consistency with assumptions: e.g. extra relativistic d.o.f.s.
Damping

- Tight coupling equations assume a **perfect fluid**: no viscosity, no heat conduction.
- Fluid imperfections are related to the **mean free path of the photons in the baryons**

\[ \lambda_C = \frac{1}{\dot{\tau}} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a \]

is the conformal opacity to Thomson scattering.
- Dissipation related to **diffusion length**: random walk approx

\[ \lambda_D = \sqrt{N} \lambda_C = \sqrt{\eta/\lambda_C} \lambda_C = \sqrt{\eta \lambda_C} \]

the geometric mean between the horizon and mean free path.
- \( \lambda_C / \eta_* \sim \% \), so expect **peaks > 3** to be affected by dissipation.
- \( \sqrt{\eta} \) enters here and \( \eta \) in the acoustic scale → expansion rate and extra relativistic species.
Equations of Motion

- Continuity

\[ \dot{\Theta} = -\frac{k}{3} v_\gamma - \dot{\Phi}, \quad \dot{\delta}_b = -kv_b - 3\dot{\Phi} \]

where the photon equation remains unchanged and the baryons follow number conservation with \( \rho_b = m_b n_b \)

- Navier-Stokes (Euler + heat conduction, viscosity)

\[ \dot{v}_\gamma = k(\Theta + \Psi) - \frac{k}{6} \pi_\gamma - \dot{\tau}(v_\gamma - v_b) \]
\[ \dot{v}_b = -\frac{\dot{a}}{a} v_b + k\Psi + \dot{\tau}(v_\gamma - v_b)/R \]

where the photons gain an anisotropic stress term \( \pi_\gamma \) from radiation viscosity and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation
Viscosity

- **Viscosity** is generated from radiation **streaming** from hot to cold regions

- Expect

\[
\pi_\gamma \sim v_\gamma \frac{k}{\dot{T}}
\]

generated by streaming, suppressed by **scattering** in a wavelength of the fluctuation. **Radiative transfer** says

\[
\pi_\gamma \approx 2A_v v_\gamma \frac{k}{\dot{T}}
\]

where \(A_v = 16/15\)

\[
\dot{v}_\gamma = k(\Theta + \Psi) - \frac{k}{3} A_v \frac{k}{\dot{T}} v_\gamma
\]
Oscillator: Penultimate Take

- Adiabatic approximation ($\omega \gg \dot{a}/a$)
  \[ \dot{\Theta} \approx -\frac{k}{3} v_\gamma \]

- Oscillator equation contains a $\dot{\Theta}$ damping term
  \[
  c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} A_v \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})
  \]

- Heat conduction term similar in that it is proportional to $v_\gamma$ and is suppressed by scattering $k/\dot{\tau}$. Expansion of Euler equations to leading order in $k/\dot{\tau}$ gives
  \[
  A_h = \frac{R^2}{1 + R}
  \]
  since the effects are only significant if the baryons are dynamically important.
Oscillator: Final Take

- **Final oscillator equation**

\[ c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} [A_v + A_h] \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi}) \]

- **Solve in the adiabatic approximation**

\[ \Theta \propto \exp(i \int \omega d\eta) \]

\[ -\omega^2 + \frac{k^2 c_s^2}{\dot{\tau}} (A_v + A_h) i\omega + k^2 c_s^2 = 0 \]
Dispersion Relation

- Solve

\[ \omega^2 = k^2 c_s^2 \left[ 1 + i \frac{\omega}{\dot{\tau}} (A_v + A_h) \right] \]

\[ \omega = \pm kc_s \left[ 1 + \frac{i \omega}{2 \dot{\tau}} (A_v + A_h) \right] \]

\[ = \pm kc_s \left[ 1 \pm \frac{i k c_s}{2 \dot{\tau}} (A_v + A_h) \right] \]

- Exponentiate

\[ \exp(i \int \omega d\eta) = e^{\pm i ks} \exp[-k^2 \int d\eta \frac{1}{2} \frac{c_s^2}{\dot{\tau}} (A_v + A_h)] \]

\[ = e^{\pm i ks} \exp[-(k/k_D)^2] \]

- Damping is exponential under the scale \( k_D \)
Diffusion Scale

- Diffusion wavenumber

\[ k_D^{-2} = \int d\eta \frac{1}{\dot{\tau}} \frac{1}{6(1 + R)} \left( \frac{16}{15} + \frac{R^2}{(1 + R)} \right) \]

- Limiting forms

\[ \lim_{R \to 0} k_D^{-2} = \frac{1}{6} \frac{16}{15} \int d\eta \frac{1}{\dot{\tau}} \]

\[ \lim_{R \to \infty} k_D^{-2} = \frac{1}{6} \int d\eta \frac{1}{\dot{\tau}} \]

- Geometric mean between horizon and mean free path as expected from a random walk

\[ \lambda_D = \frac{2\pi}{k_D} \sim \frac{2\pi}{\sqrt{6}} (\eta \dot{\tau}^{-1})^{1/2} \]