Ast 448: CMB

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Syllabus

This course will have be composed of 2 parts

- Lectures establishing a common denominator on CMB physics: temperature anisotropy, polarization...
- Current topics of your own interests, culminating in a presentation by you to the class

Prerequisites

- Cosmology - at least the advanced undergrad (Ryden) or graduate level:

  FRW cosmology

  Thermal history

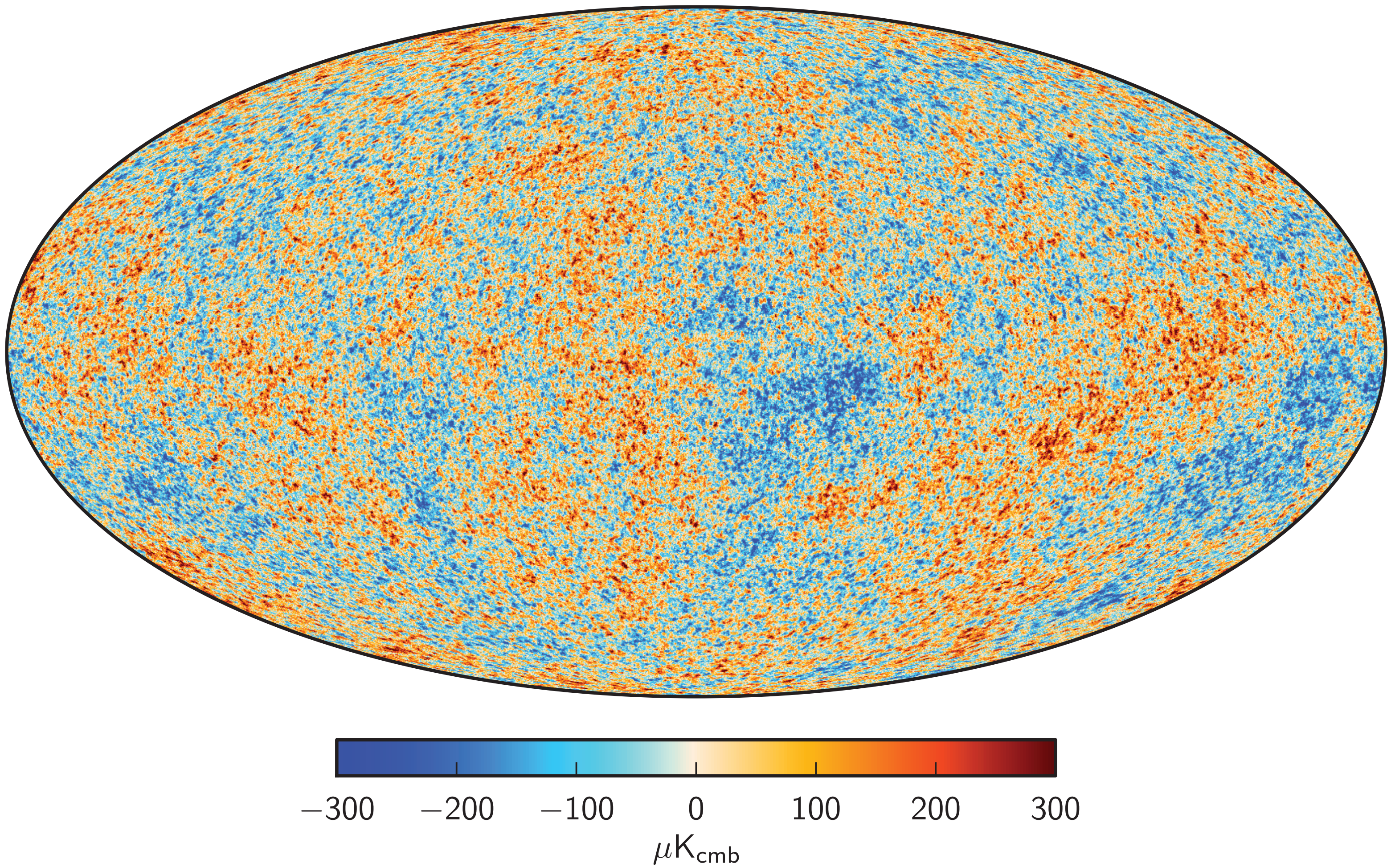
  Inflation

Helpful:

- Radiative processes, GR, stat mech

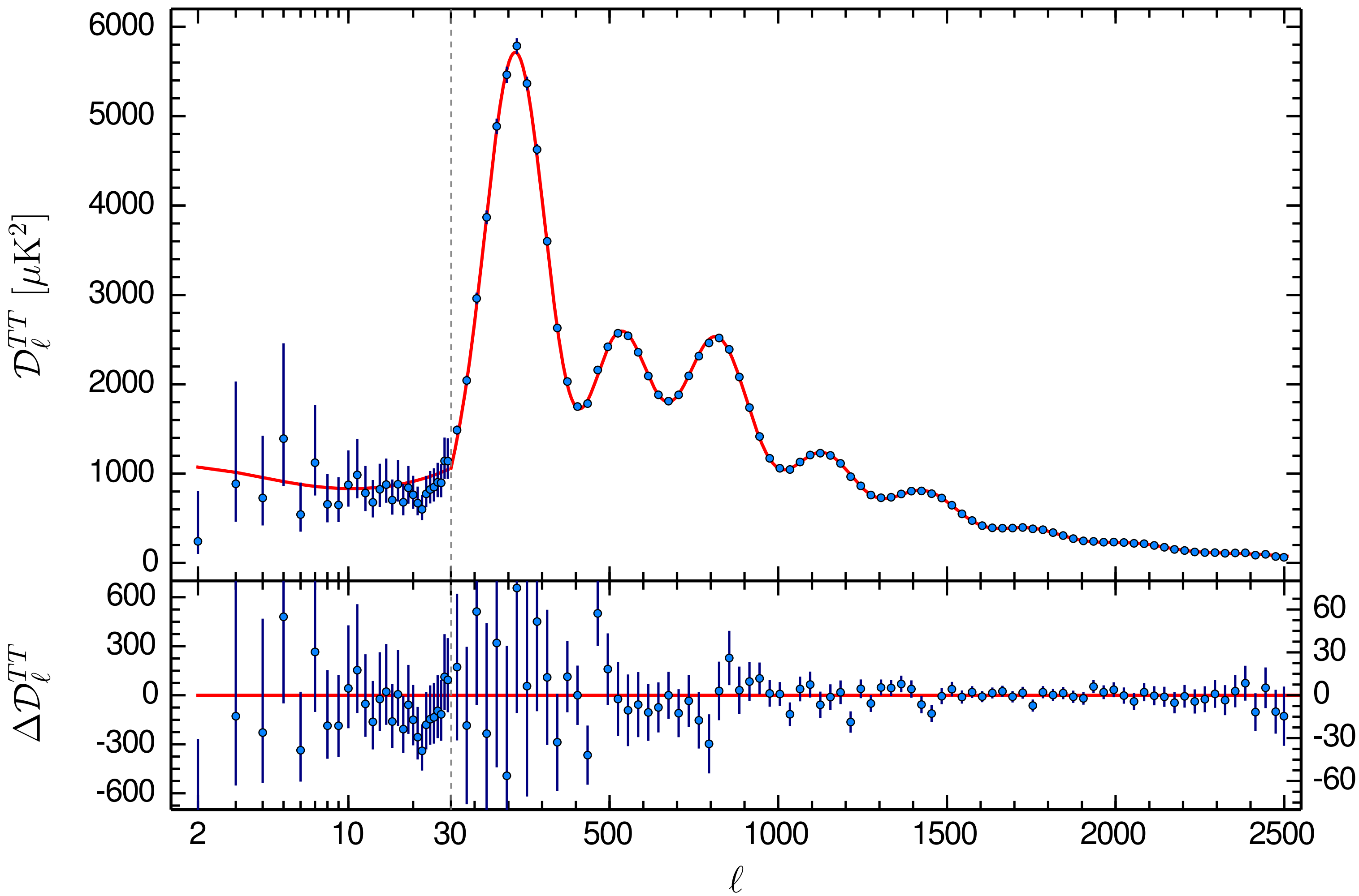
CMB Temperature Anisotropy

- Planck map of the temperature anisotropy (first discovered by COBE) from recombination:



CMB Temperature Anisotropy

- Power spectrum shows characteristic scales where the intensity of variations peak - reveals geometry and contents of the universe:



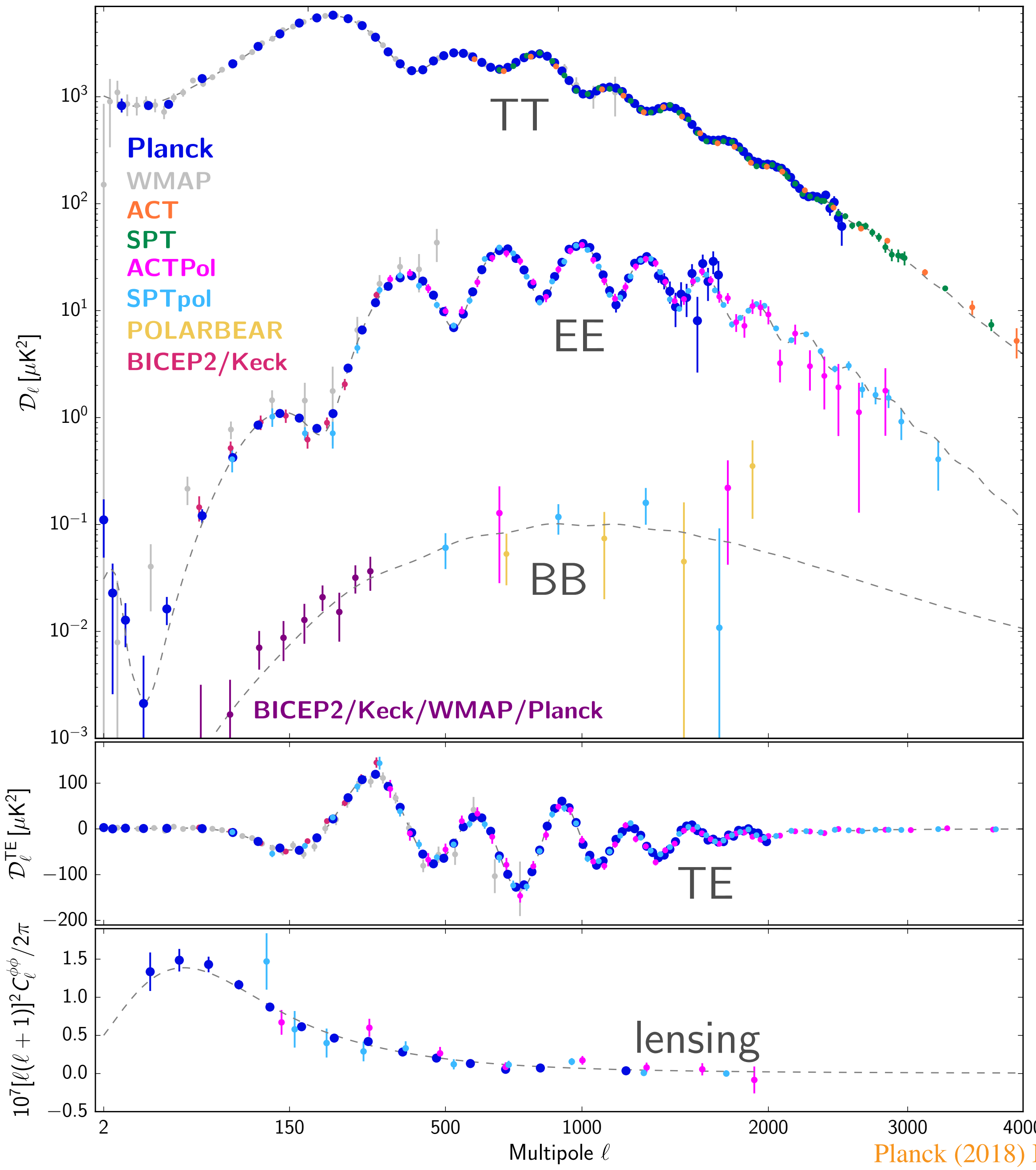
CMB Parameter Inferences

- Spectrum constrains the matter-energy contents of the universe
- Planck 2018 results [arXiv:1807.06209]

Parameter	TT+lowE 68% limits	TE+lowE 68% limits	EE+lowE 68% limits	TT,TE,EE+lowE 68% limits	TT,TE,EE+lowE+lensing 68% limits	TT,TE,EE+lowE+lensing+BAO 68% limits
$\Omega_b h^2$	0.02212 ± 0.00022	0.02249 ± 0.00025	0.0240 ± 0.0012	0.02236 ± 0.00015	0.02237 ± 0.00015	0.02242 ± 0.00014
$\Omega_c h^2$	0.1206 ± 0.0021	0.1177 ± 0.0020	0.1158 ± 0.0046	0.1202 ± 0.0014	0.1200 ± 0.0012	0.11933 ± 0.00091
$100\theta_{MC}$	1.04077 ± 0.00047	1.04139 ± 0.00049	1.03999 ± 0.00089	1.04090 ± 0.00031	1.04092 ± 0.00031	1.04101 ± 0.00029
τ	0.0522 ± 0.0080	0.0496 ± 0.0085	0.0527 ± 0.0090	$0.0544^{+0.0070}_{-0.0081}$	0.0544 ± 0.0073	0.0561 ± 0.0071
$\ln(10^{10} A_s)$	3.040 ± 0.016	$3.018^{+0.020}_{-0.018}$	3.052 ± 0.022	3.045 ± 0.016	3.044 ± 0.014	3.047 ± 0.014
n_s	0.9626 ± 0.0057	0.967 ± 0.011	0.980 ± 0.015	0.9649 ± 0.0044	0.9649 ± 0.0042	0.9665 ± 0.0038
H_0 [km s^{-1} Mpc^{-1}] . .	66.88 ± 0.92	68.44 ± 0.91	69.9 ± 2.7	67.27 ± 0.60	67.36 ± 0.54	67.66 ± 0.42

CMB Power Spectra

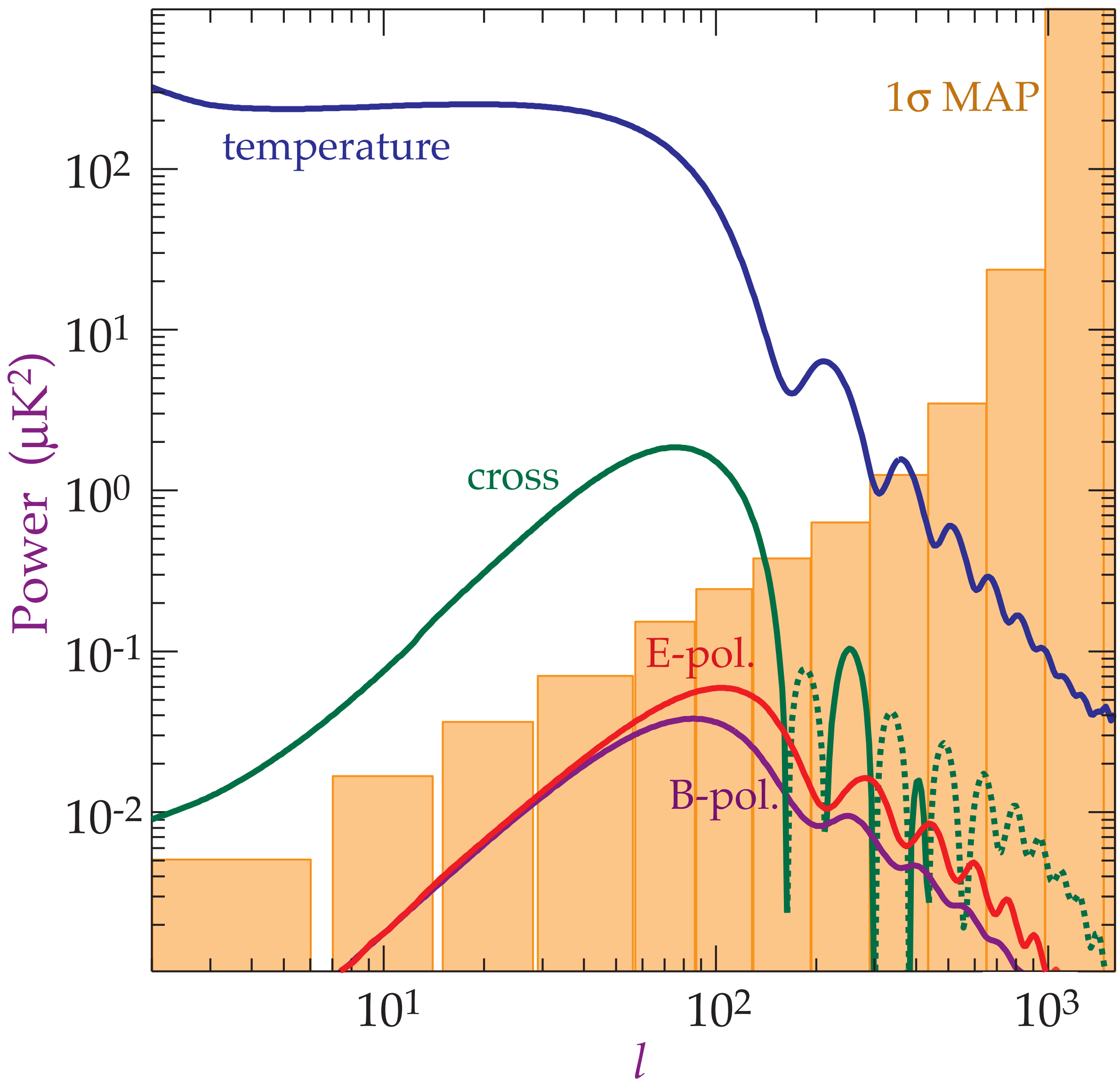
- Power spectra of CMB
  - – temperature
  - – polarization
  - – lensing



Planck (2018) I

Tensor Power Spectrum

- Gravitational waves from inflation (yet to be detected)

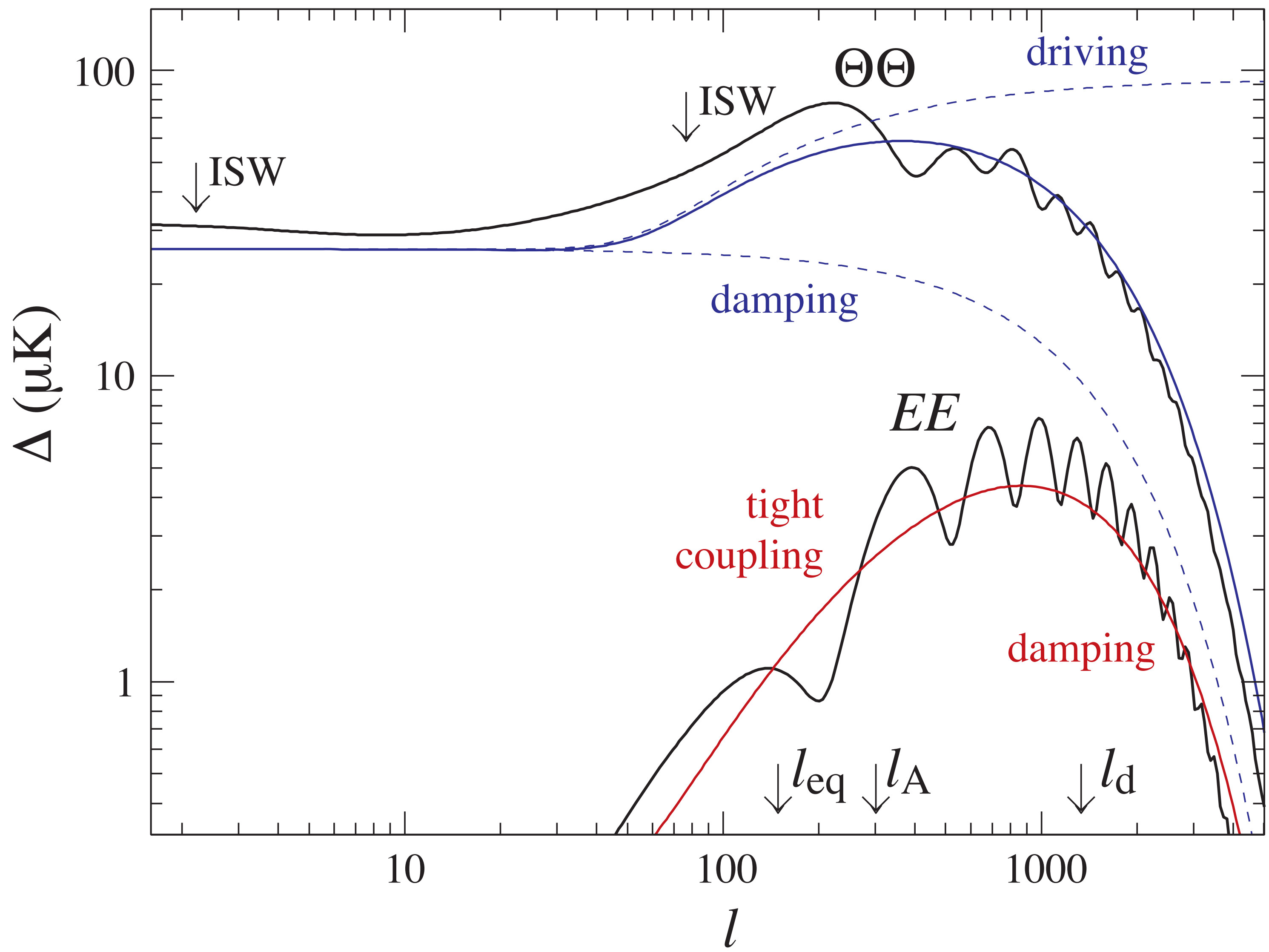


Ast 448

Set 1: Temperature Anisotropy

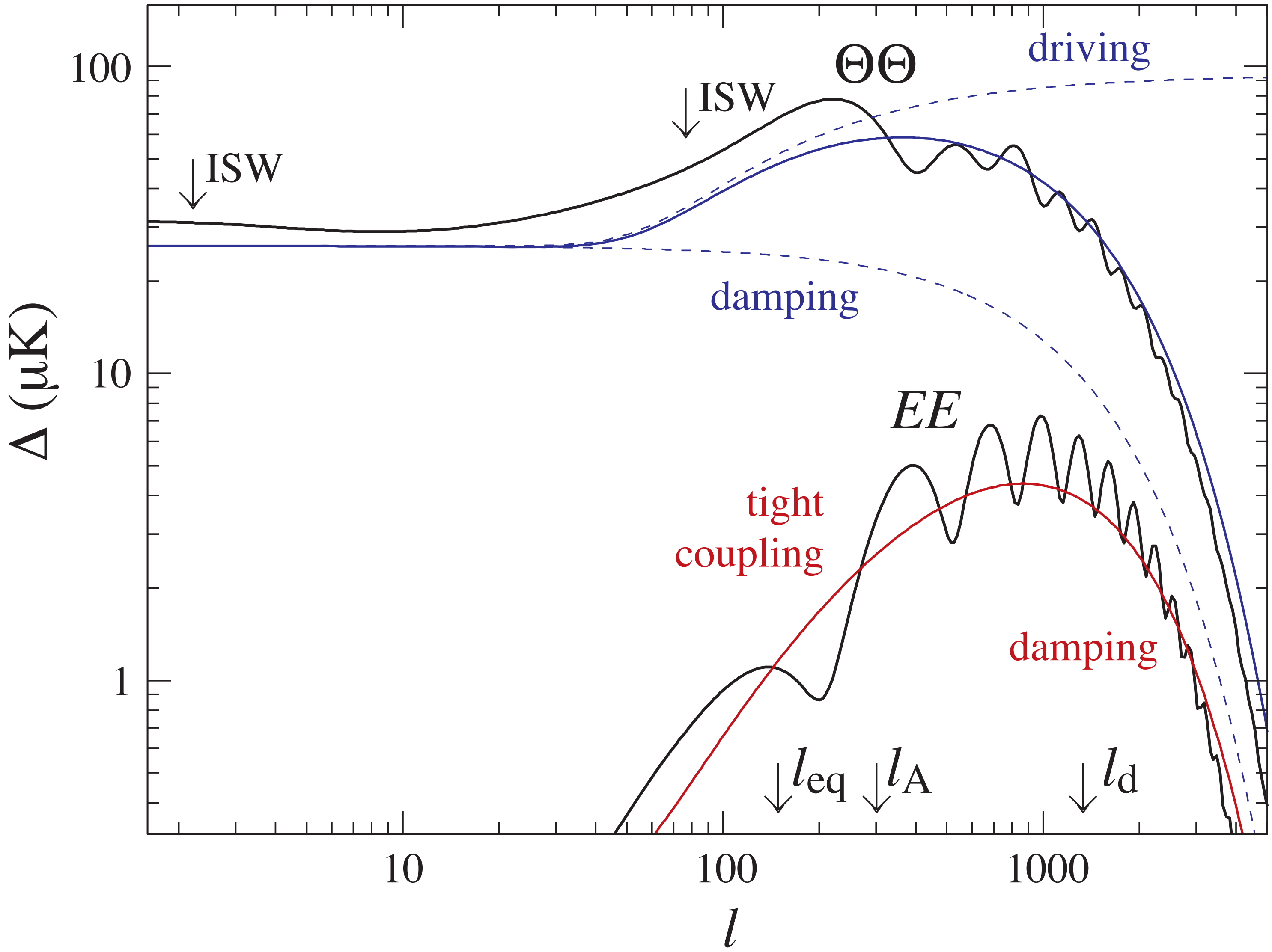
Schematic Outline

- Take apart features in the power spectrum



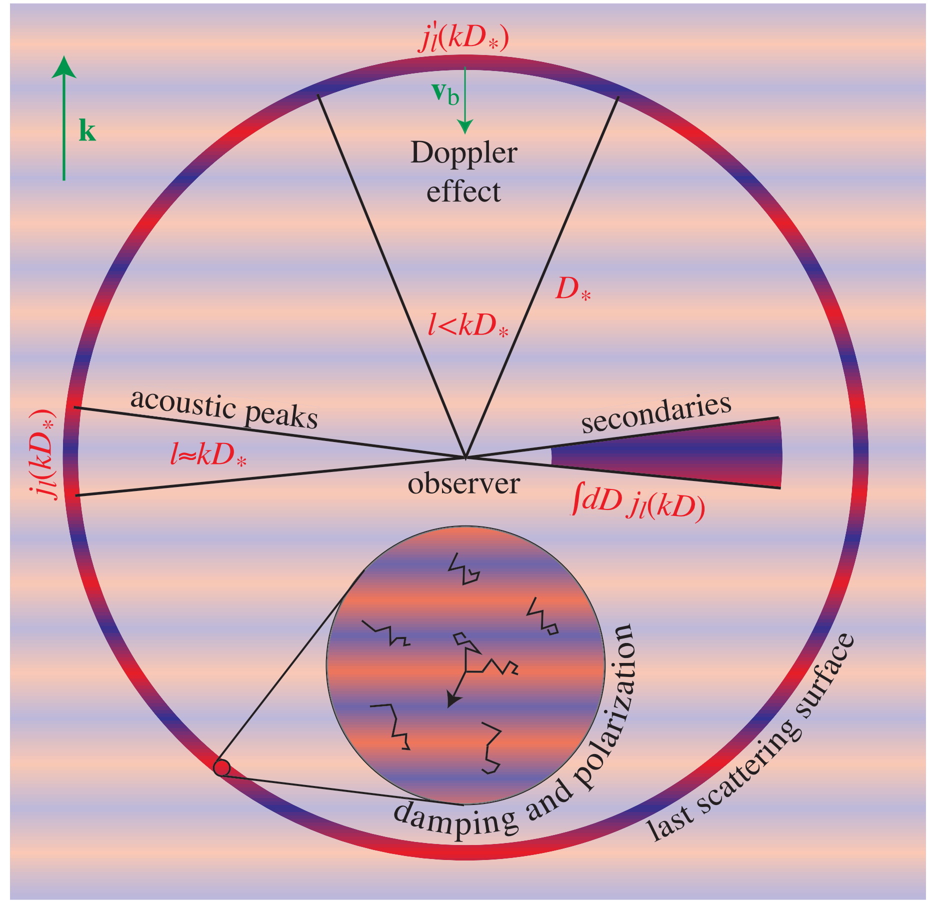
Schematic Outline

- Take apart features in the power spectrum



Last Scattering

- Angular distribution of radiation is the 3D temperature field projected onto a shell - surface of last scattering



- Shell radius is distance from the observer to recombination: called the last scattering surface
- Take the radiation distribution at last scattering to also be described by an isotropic temperature fluctuation field $\Theta(\mathbf{x})$

Angular Power Spectrum

- Take recombination to be instantaneous

$$\Theta(\hat{\mathbf{n}}) = \int dD\, \Theta(\mathbf{x})\delta(D - D_*)$$

where D is the comoving distance and D_* denotes recombination.

- Describe the temperature field by its Fourier moments

$$\Theta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

- Orthogonality and Completeness (forward and inverse transform):

$$\int d^3x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')$$

$$\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \delta(\mathbf{x} - \mathbf{x}')$$

Angular Power Spectrum

- Statistical homogeneity and isotropy

$$\langle \Theta(\mathbf{x})\Theta(\mathbf{x}')\rangle = C(|\mathbf{x}-\mathbf{x}'|)$$

function of separation only

$$\langle \Theta(\mathbf{x}+\mathbf{d})\Theta(\mathbf{x}'+\mathbf{d})\rangle = \langle \Theta(\mathbf{x})\Theta(\mathbf{x}')\rangle$$

$$\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{x}'} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{d}} \langle \Theta^*(\mathbf{k})\Theta(\mathbf{k}')\rangle$$
$$= \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{x}'} \langle \Theta^*(\mathbf{k})\Theta(\mathbf{k}')\rangle$$

requires the 2pt Fourier correlation to be described by a power spectrum

$$\langle \Theta^*(\mathbf{k})\Theta(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P_T(k)$$

Angular Power Spectrum

- Correlation function and power spectrum are Fourier conjugates

$$C(|\mathbf{x}-\mathbf{x}'|) = \langle \Theta(\mathbf{x})\Theta(\mathbf{x}')\rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})} P_T(k)$$

- Log weighted power spectrum determines variance

$$\langle \Theta(\mathbf{x})\Theta(\mathbf{x})\rangle = \int \frac{d^3k}{(2\pi)^3} P_T(k) = \int \frac{dk}{k}\frac{k^3}{2\pi^2} P_T = \int \frac{dk}{k}\Delta_T^2(k)$$

$$\Delta_T^2 = \frac{k^3}{2\pi^2} P_T [= \mathcal{P}_T(k)]$$

and is the contribution to the total variance per log interval in k

- Δ_T^2 dimensionless, whereas P_T has dimensions of $[L^3]$, e.g. $(h^{-1}\mathrm{Mpc})^3$ for the power spectrum of a redshift survey

Angular Power Spectrum

- Temperature field

$$\Theta(\hat{\mathbf{n}}) = \int \frac{d^3k}{(2\pi)^3}\Theta(\mathbf{k})e^{i\mathbf{k}\cdot D_*\hat{\mathbf{n}}}$$

Multipole moments $\Theta(\hat{\mathbf{n}}) = \sum_{\ell m}\Theta_{\ell m}Y_{\ell m}$

- Orthogonality:

$$\int d\hat{\mathbf{n}} Y^*_{\ell m}(\hat{\mathbf{n}})Y_{\ell' m'}(\hat{\mathbf{n}}) = \delta_{\ell\ell'}\delta_{mm'}$$

Completeness:

$$\sum_{\ell m} Y^*_{\ell m}(\hat{\mathbf{n}})Y_{\ell m}(\hat{\mathbf{n}}') = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta')$$

- Statistical isotropy:

$$\langle\Theta^*_{\ell m}\Theta_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'}C_\ell$$

Angular Power Spectrum

- Expand out plane wave in spherical coordinates

$$e^{i\mathbf{k}D_*\cdot\hat{\mathbf{n}}} = 4\pi\sum_{\ell m} i^\ell j_\ell(kD_*)Y^*_{\ell m}(\hat{\mathbf{k}})Y_{\ell m}(\hat{\mathbf{n}})$$

- Aside: as in the figure, it will often be convenient when considering a single $\mathbf{k}$ mode to orient the north pole to $\hat{\mathbf{k}}$. This simplifies the decomposition since

$$Y^*_{\ell m}(\hat{\mathbf{k}}) \to Y^*_{\ell m}(0) = \delta_{m0}\sqrt{\frac{2\ell+1}{4\pi}}$$

Angular Power Spectrum

- Power spectrum

$$\Theta_{\ell m} = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) 4\pi i^\ell j_\ell(kD_*) Y^*_{\ell m}(\mathbf{k})$$

$$\begin{aligned}\langle \Theta^*_{\ell m} \Theta_{\ell' m'} \rangle &= \int \frac{d^3k}{(2\pi)^3} (4\pi)^2 i^{\ell-\ell'} j_\ell(kD_*) j_{\ell'}(kD_*) Y_{\ell m}(\mathbf{k}) Y^*_{\ell' m'}(\mathbf{k}) P_T(k) \\ &= \delta_{\ell\ell'} \delta_{mm'} 4\pi \int d\ln k\, j_\ell^2(kD_*) \Delta_T^2(k)\end{aligned}$$

with $\int_0^\infty j_\ell^2(x) d\ln x = 1/(2\ell(\ell+1))$, slowly varying Δ_T^2

- Angular power spectrum:

$$C_\ell = \frac{4\pi \Delta_T^2(\ell/D_*)}{2\ell(\ell+1)} = \frac{2\pi}{\ell(\ell+1)} \Delta_T^2(\ell/D_*)$$

Angular Power Spectrum

- The log power spectrum (sometimes called $\mathcal{D}_\ell$)

$$\frac{\ell(\ell+1)}{2\pi} C_\ell \approx \Delta_T^2$$

so that a scale invariant spectrum Δ_T^2 =const is scale invariant in the log power spectrum

- Related to the contribution to the variance per log interval in ℓ

$$\langle \Theta(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}}) \rangle = \langle \Theta(0)\Theta(0) \rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell = \sum_\ell \frac{1}{\ell} \frac{\ell(2\ell+1)}{4\pi} C_\ell$$

with the two being equivalent if $\ell \gg 1$

Thomson Scattering

- Thomson scattering of photons off of free electrons is the most important CMB process with a cross section (averaged over polarization states) of

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25}\text{cm}^2$$

- Density of free electrons in a fully ionized $x_e = 1$ universe

$$n_e = (1 - Y_p/2)x_e n_b \approx 10^{-5}\Omega_b h^2(1+z)^3\text{cm}^{-3}\,,$$

where $Y_p \approx 0.24$ is the Helium mass fraction, creates a high (comoving) Thomson opacity

$$\dot{\tau} \equiv n_e \sigma_T a$$

where dots are conformal time $\eta \equiv \int dt/a$ derivatives and τ is the optical depth.

Tight Coupling Approximation

- Near recombination $z \approx 10^3$ and $\Omega_b h^2 \approx 0.02$, the (comoving) mean free path of a photon

$$\lambda_C \equiv \frac{1}{\dot{\tau}} \sim 2.5\mathrm{Mpc}$$

  small by cosmological standards!

- On scales $\lambda \gg \lambda_C$ photons are tightly coupled to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions
- Specifically, their bulk velocities are defined by a single fluid velocity $v_\gamma = v_b$ and the photons carry no anisotropy in the rest frame of the baryons
- $\to$ No heat conduction or viscosity (anisotropic stress) in fluid

Full Equations of Motion

- Continuity

$$\dot{\Theta} = -\frac{k}{3} v_\gamma - \dot{\Phi}\,, \quad \dot{\delta}_b = -k v_b - 3\dot{\Phi}$$

which expresses number conservation in the presence of velocity divergence and local expansion, with $\rho_b = m_b n_b$

- Navier-Stokes (Euler + heat conduction, viscosity)

$$\begin{aligned} \dot{v}_\gamma &= k(\Theta + \Psi) - \frac{k}{6}\pi_\gamma - \dot{\tau}(v_\gamma - v_b) \\ \dot{v}_b &= -\frac{\dot{a}}{a} v_b + k\Psi + \dot{\tau}(v_\gamma - v_b)/R \end{aligned}$$

where the photon momentum changes due to pressure, gravity and anisotropic stress π_γ gradients (from radiation viscosity) and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation

Zeroth Order Approximation

- Momentum density of a fluid is $(\rho + p)v$, where p is the pressure
- Neglect the momentum density of the baryons

$$R \equiv \frac{(\rho_b + p_b)v_b}{(\rho_\gamma + p_\gamma)v_\gamma} = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}$$
$$\approx 0.6 \left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{a}{10^{-3}}\right)$$

since $\rho_\gamma \propto T^4$ is fixed by the CMB temperature $T = 2.73(1+z)$K – OK substantially before recombination

- Neglect radiation in the expansion (not a good approx, just for pedagogical start)

$$\frac{\rho_m}{\rho_r} = 3.6 \left(\frac{\Omega_m h^2}{0.15}\right)\left(\frac{a}{10^{-3}}\right)$$

- Neglect gravity (obviously just for pedagogy)

Fluid Equations

- Density $\rho_\gamma \propto T^4$ so define temperature fluctuation Θ

$$\delta_\gamma = 4\frac{\delta T}{T} \equiv 4\Theta$$

- Real space continuity equation

$$\dot{\delta}_\gamma = -(1 + w_\gamma)kv_\gamma$$

$$\dot{\Theta} = -\frac{1}{3}kv_\gamma$$

- Euler equation (neglecting gravity)

$$\dot{v}_\gamma = -(1 - 3w_\gamma)\frac{\dot{a}}{a}v_\gamma + \frac{kc_s^2}{1 + w_\gamma}\delta_\gamma$$

$$\dot{v}_\gamma = kc_s^2\frac{3}{4}\delta_\gamma = 3c_s^2 k\Theta$$

Oscillator: Take One

- Combine these to form the simple harmonic oscillator equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = 0$$

where the sound speed is adiabatic

$$c_s^2 = \frac{\delta p_\gamma}{\delta \rho_\gamma} = \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma}$$

here $c_s^2 = 1/3$ since we are photon-dominated

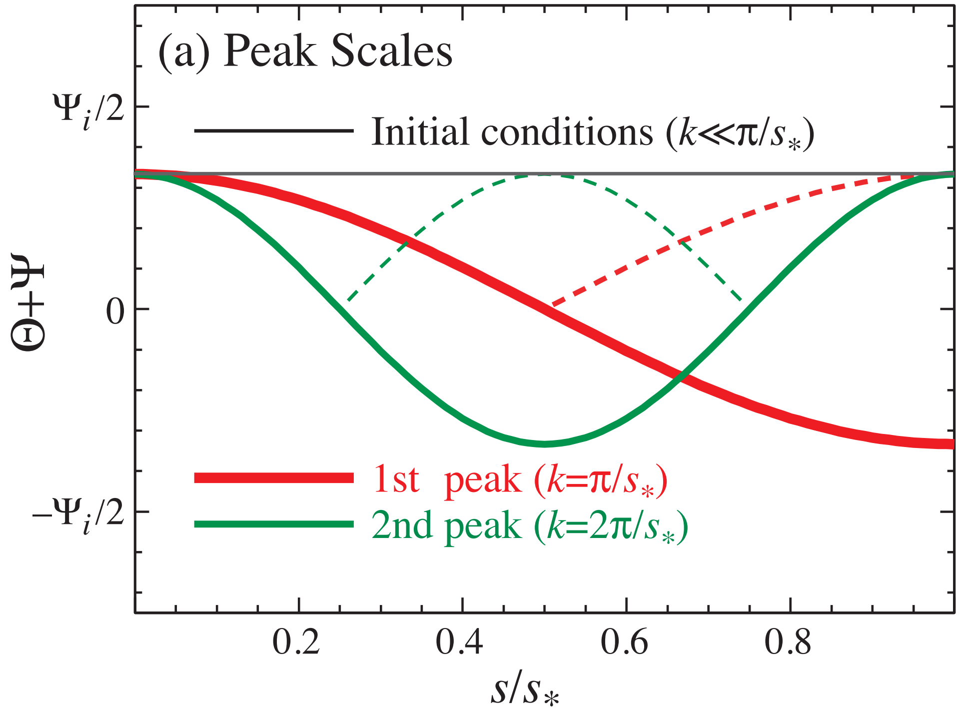
- General solution:

$$\Theta(\eta) = \Theta(0)\cos(ks) + \frac{\dot{\Theta}(0)}{kc_s}\sin(ks)$$

where the sound horizon is defined as $s \equiv \int c_s d\eta$

Harmonic Extrema

- All modes are frozen in at recombination (denoted with a subscript $*$)
- Temperature perturbations of different amplitude for different modes.
- For the adiabatic (curvature mode) initial conditions



$$\dot\Theta(0) = 0$$

- So solution

$$\Theta(\eta_*) = \Theta(0)\cos(ks_*)$$

Harmonic Extrema

- Modes caught in the extrema of their oscillation will have enhanced fluctuations

$$k_n s_* = n\pi$$

yielding a fundamental scale or frequency, related to the inverse sound horizon

$$k_A = \pi / s_*$$

and a harmonic relationship to the other extrema as $1 : 2 : 3 \ldots$

Peak Location

- The fundmental physical scale is translated into a fundamental angular scale by simple projection according to the angular diameter distance D_A

$$\begin{aligned}\theta_A &= \lambda_A/D_A \\ \ell_A &= k_A D_A\end{aligned}$$

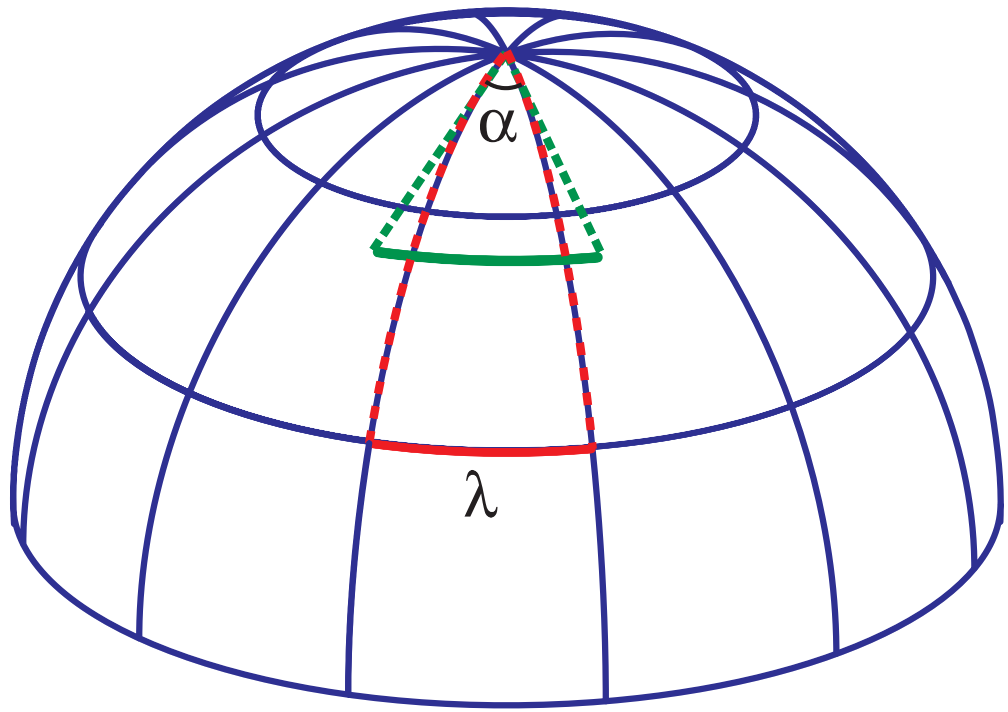
- In a flat universe, the distance is simply $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$, the horizon distance, and $k_A = \pi/s_* = \sqrt{3}\pi/\eta_*$ so

$$\theta_A \approx \frac{\eta_*}{\eta_0}$$

- In a matter-dominated universe $\eta \propto a^{1/2}$ so $\theta_A \approx 1/30 \approx 2^\circ$ or

$$\ell_A \approx 200$$

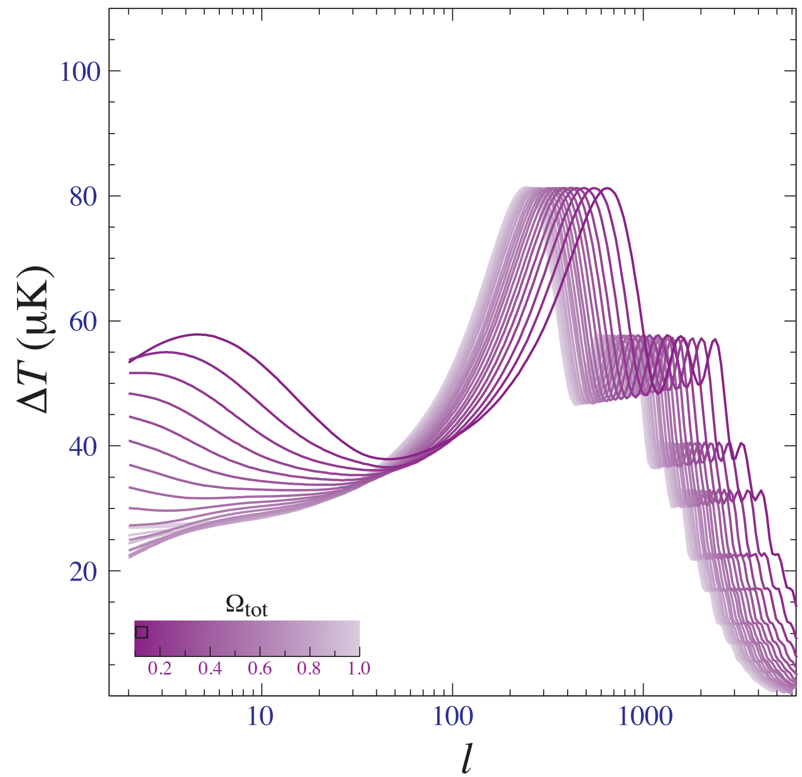
Curvature



- In a curved universe, the apparent or angular diameter distance is no longer the conformal distance $D_A = R\sin(D/R) \neq D$
- Objects in a closed universe are further than they appear! gravitational lensing of the background...
- Curvature scale of the universe must be substantially larger than current horizon

Curvature

- Flat universe indicates critical density and implies missing energy given local measures of the matter density "dark energy"
- D also depends on dark energy density $\Omega_{\rm DE}$ and equation of state $w = p_{\rm DE}/\rho_{\rm DE}$.



- Expansion rate at recombination or matter-radiation ratio enters into calculation of k_A.

Fixed Deceleration Epoch

- CMB determination of matter density controls all determinations in the deceleration (matter dominated) epoch
- Planck: $\Omega_m h^2 = 0.1426 \pm 0.0025 \rightarrow 1.7\%$
- Distance to recombination D_* determined to $\frac{1}{4} 1.7\% \approx 0.43\%$ (ΛCDM result 0.46%; $\Delta h/h \approx -\Delta\Omega_m h^2/\Omega_m h^2$)

  [more general: $-0.11\Delta w - 0.48\Delta \ln h - 0.15\Delta \ln \Omega_m - 1.4\Delta \ln \Omega_{\rm tot} = 0$]
- Expansion rate during any redshift in the deceleration epoch determined to $\frac{1}{2} 1.7\%$
- Distance to any redshift in the deceleration epoch determined as

$$D(z) = D_* - \int_z^{z_*} \frac{dz}{H(z)}$$

- Volumes determined by a combination $dV = D_A^2 d\Omega dz/H(z)$
- Structure also determined by growth of fluctuations from z_*

Doppler Effect

- Bulk motion of fluid changes the observed temperature via Doppler shifts

$$\left(\frac{\Delta T}{T}\right)_{\rm dop} = \hat{\mathbf{n}} \cdot \mathbf{v}_\gamma$$

- Averaged over directions

$$\left(\frac{\Delta T}{T}\right)_{\rm rms} = \frac{v_\gamma}{\sqrt{3}}$$

- Acoustic solution

$$\frac{v_\gamma}{\sqrt{3}} = -\frac{\sqrt{3}}{k}\dot{\Theta} = \frac{\sqrt{3}}{k} k c_s \, \Theta(0) \sin(ks)$$
$$= \Theta(0) \sin(ks)$$

Doppler Peaks?

- Doppler effect for the photon dominated system is of equal amplitude and $\pi/2$ out of phase: extrema of temperature are turning points of velocity
- Effects add in quadrature:

$$\left(\frac{\Delta T}{T}\right)^2 = \Theta^2(0)[\cos^2(ks) + \sin^2(ks)] = \Theta^2(0)$$

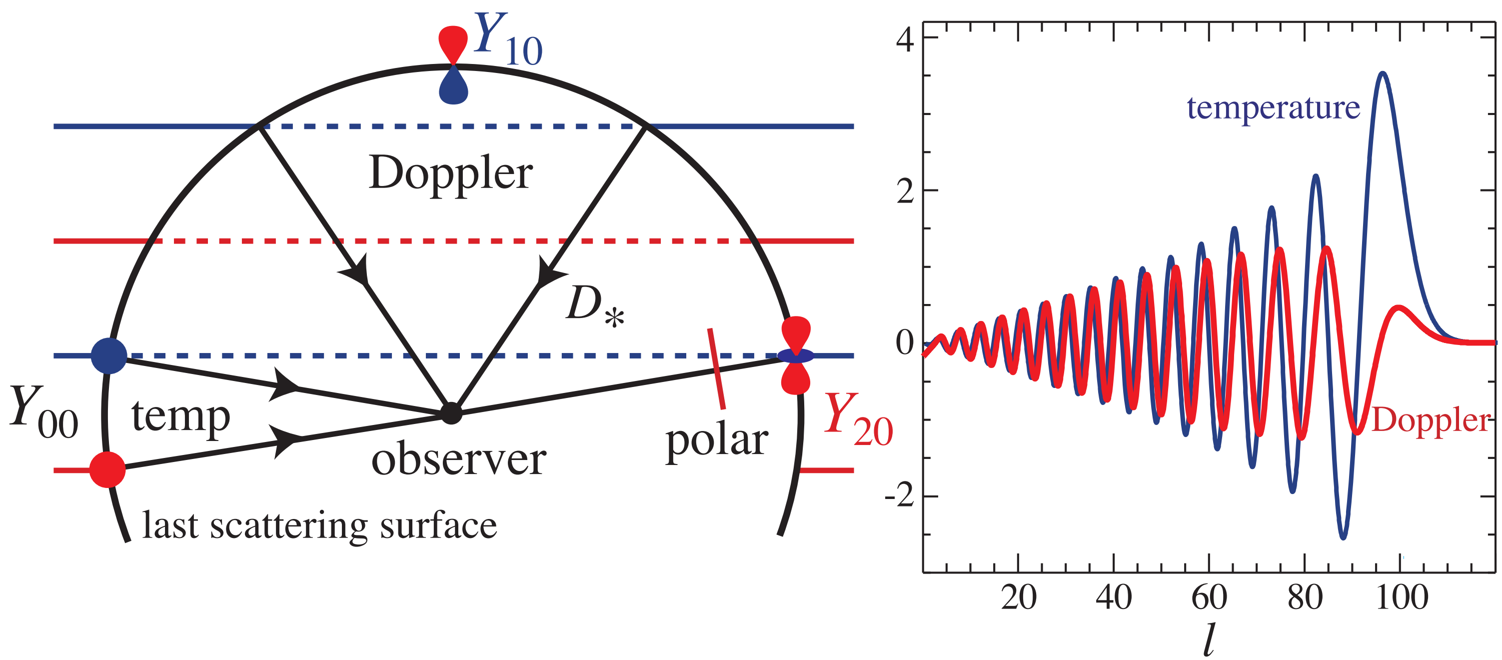
- No peaks in k spectrum! However the Doppler effect carries an angular dependence that changes its projection on the sky $\hat{\mathbf{n}} \cdot \mathbf{v}_\gamma \propto \hat{\mathbf{n}} \cdot \hat{\mathbf{k}}$

Doppler Peaks?

- Coordinates where $\hat{\mathbf{z}} \parallel \hat{\mathbf{k}}$

$$Y_{10} Y_{\ell 0} \to Y_{\ell \pm 1\, 0}$$

recoupling $j_\ell' Y_{\ell 0}$: no peaks in Doppler effect



Restoring Gravity

- Take a simple photon dominated system with gravity
- Continuity altered since a gravitational potential represents a stretching of the spatial fabric that dilutes number densities – formally a spatial curvature perturbation
- Think of this as a perturbation to the scale factor $a \to a(1+\Phi)$ so that the cosmogical redshift is generalized to

$$\frac{\dot{a}}{a} \to \frac{\dot{a}}{a} + \dot{\Phi}$$

so that the continuity equation becomes

$$\dot{\Theta} = -\frac{1}{3}kv_\gamma - \dot{\Phi}$$

Restoring Gravity

- Gravitational force in momentum conservation $\mathbf{F} = -m\nabla\Psi$ generalized to momentum density modifies the Euler equation to

$$\dot{v}_\gamma = k(\Theta + \Psi)$$

- General relativity says that Φ and Ψ are the relativistic analogues of the Newtonian potential and that $\Phi \approx -\Psi$.
- In our matter-dominated approximation, Φ represents matter density fluctuations through the cosmological Poisson equation

$$k^2\Phi = 4\pi G a^2 \rho_m \Delta_m$$

where the difference comes from the use of comoving coordinates for k (a^2 factor), the removal of the background density into the background expansion $(\rho\Delta_m)$ and finally a coordinate subtlety that enters into the definition of Δ_m

Constant Potentials

- In the matter dominated epoch potentials are constant because infall generates velocities as $v_m \sim k\eta\Psi$
- Velocity divergence generates density perturbations as $\Delta_m \sim -k\eta v_m \sim -(k\eta)^2\Psi$
- And density perturbations generate potential fluctuations

$$\Phi = \frac{4\pi G a^2 \rho \Delta}{k^2} \approx \frac{3}{2}\frac{H^2 a^2}{k^2}\Delta \sim \frac{\Delta}{(k\eta)^2} \sim -\Psi$$

keeping them constant. Note that because of the expansion, density perturbations must grow to keep potentials constant.

Constant Potentials

- More generally, if stress perturbations are negligible compared with density perturbations ($\delta p \ll \delta\rho$) then potential will remain roughly constant
- More specifically a variant called the Bardeen or comoving curvature is strictly constant

$$\mathcal{R} = \mathrm{const} \approx \frac{5+3w}{3+3w}\Phi$$

where the approximation holds when $w \approx$const.

Oscillator: Take Two

- Combine these to form the simple harmonic oscillator equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - \ddot{\Phi}$$

- In a CDM dominated expansion $\dot{\Phi} = \dot{\Psi} = 0$. Also for photon domination $c_s^2 = 1/3$ so the oscillator equation becomes

$$\ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0$$

- Solution is just an offset version of the original

$$[\Theta + \Psi](\eta) = [\Theta + \Psi](0)\, \cos(ks)$$

- $\Theta + \Psi$ is also the observed temperature fluctuation since photons lose energy climbing out of gravitational potentials at recombination

Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or effective temperature

$$\Theta + \Psi$$

- Effective temperature oscillates around zero with amplitude given by the initial conditions
- Note: initial conditions are set when the perturbation is outside of horizon, need inflation or other modification to matter-radiation FRW universe.
- GR says that initial temperature is given by initial potential

Sachs-Wolfe Effect and the Magic 1/3

- A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]

$$\frac{\delta t}{t} = \Psi$$

- Convert this to a perturbation in the scale factor,

$$t = \int \frac{da}{aH} \propto \int \frac{da}{a\rho^{1/2}} \propto a^{3(1+w)/2}$$

where $w \equiv p/\rho$ so that during matter domination

$$\frac{\delta a}{a} = \frac{2}{3}\frac{\delta t}{t}$$

- CMB temperature is cooling as $T \propto a^{-1}$ so

$$\Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3}\Psi$$

Sachs-Wolfe Normalization

- Use measurements of $\Delta T/T \approx 10^{-5}$ in the Sachs-Wolfe effect to infer $\Delta^2_{\mathcal{R}}$
- Recall in matter domination $\Psi = -3\mathcal{R}/5$

$$\frac{\ell(\ell+1)C_\ell}{2\pi} \approx \Delta_T^2 \approx \frac{1}{25}\Delta_R^2$$

- Thus, amplitude of initial curvature fluctuations is $\Delta_R \approx 5 \times 10^{-5}$
- Modern usage: acoustic peak measurements plus known radiation transfer function is used to convert $\Delta T/T$ to Δ_R. Best measured at $k = 0.08$ Mpc^{-1} by Planck
- Current convention set in the WMAP era

$$\Delta_R^2(k) \equiv A_s \left(\frac{k}{0.05\mathrm{Mpc}^{-1}}\right)^{n_s-1}$$

so $A_s \sim 2.5 \times 10^{-9}$ (slightly smaller since red tilt $n_s - 1 \approx -0.04$)

Baryon Loading

- Baryons add extra mass to the photon-baryon fluid
- Controlling parameter is the momentum density ratio:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}} \right)$$

of order unity at recombination

- Momentum density of the joint system is conserved

$$\begin{aligned}
(\rho_\gamma + p_\gamma) v_\gamma + (\rho_b + p_b) v_b &\approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma) v_\gamma \\
&= (1 + R)(\rho_\gamma + p_\gamma) v_{\gamma b}
\end{aligned}$$

New Euler Equation

- Momentum density ratio enters as

$$[(1+R)v_{\gamma b}]^{\cdot} = k\Theta + (1+R)k\Psi$$

- Photon continuity remains the same

$$\dot{\Theta} = -\frac{k}{3}v_{\gamma b} - \dot{\Phi}$$

- Modification of oscillator equation

$$[(1+R)\dot{\Theta}]^{\cdot} + \frac{1}{3}k^2\Theta = -\frac{1}{3}k^2(1+R)\Psi - [(1+R)\dot{\Phi}]^{\cdot}$$

Oscillator: Take Three

- Combine these to form the not-quite-so simple harmonic oscillator equation

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

where $c_s^2 \equiv \dot{p}_{\gamma b}/\dot{\rho}_{\gamma b}$

$$c_s^2 = \frac{1}{3}\frac{1}{1+R}$$

- In a CDM dominated expansion $\dot{\Phi} = \dot{\Psi} = 0$ and the adiabatic approximation $\dot{R}/R \ll \omega = kc_s$

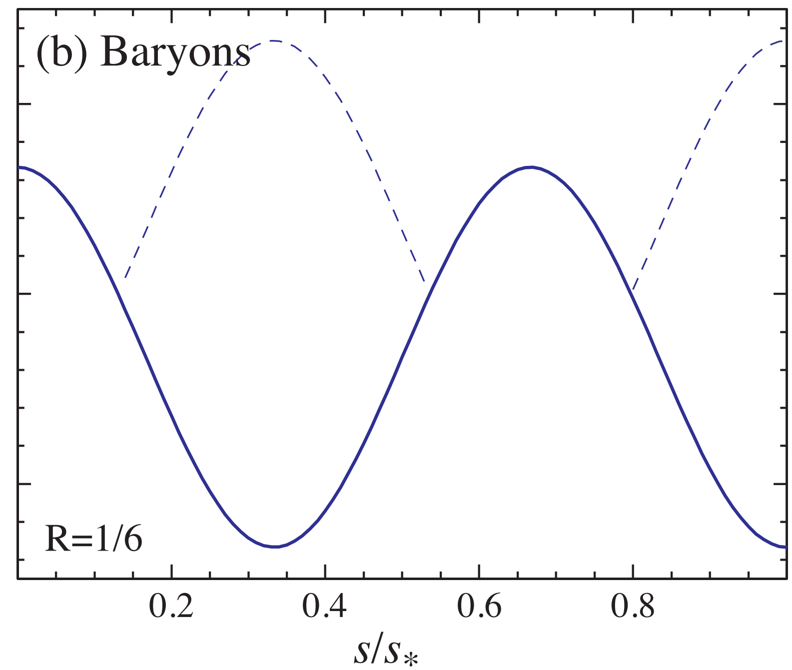
$$[\Theta + (1+R)\Psi](\eta) = [\Theta + (1+R)\Psi](0)\cos(ks)$$

Baryon Peak Phenomenology

- Photon-baryon ratio enters in three ways
- Overall larger amplitude:

$$[\Theta + (1+R)\Psi](0) = \frac{1}{3}(1+3R)\Psi(0)$$

- Even-odd peak modulation of effective temperature



$$[\Theta + \Psi]_{\rm peaks} = [\pm(1+3R) - 3R]\,\frac{1}{3}\Psi(0)$$

$$[\Theta + \Psi]_1 - [\Theta + \Psi]_2 = [-6R]\frac{1}{3}\Psi(0)$$

- Shifting of the sound horizon down or ℓ_A up

$$\ell_A \propto \sqrt{1+R}$$

Photon Baryon Ratio Evolution

- Actual effects smaller since R evolves
- Oscillator equation has time evolving mass

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = 0$$

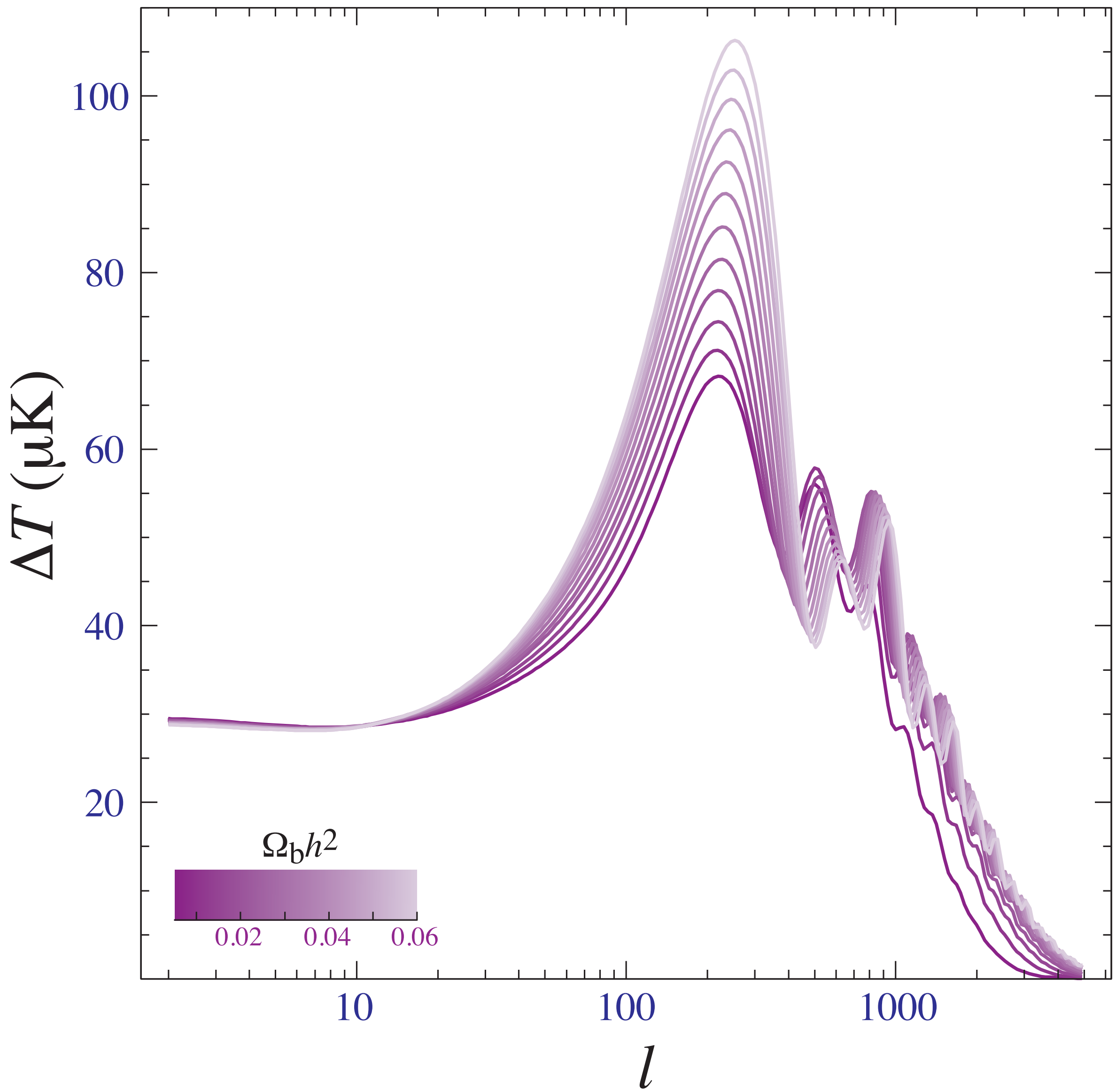
- Effective mass is is $m_{\rm eff} = 3c_s^{-2} = (1 + R)$
- Adiabatic invariant

$$\frac{E}{\omega} = \frac{1}{2} m_{\rm eff} \omega A^2 = \frac{1}{2} 3 c_s^{-2} k c_s A^2 \propto A^2 (1+R)^{1/2} = const.$$

- Amplitude of oscillation $A \propto (1+R)^{-1/4}$ decays adiabatically as the photon-baryon ratio changes

Baryons in the Power Spectrum

- Relative heights of peaks



Oscillator: Take Three and a Half

- The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\Phi)$$

  changes in the gravitational potentials alter the form of the acoustic oscillations

- If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator
- Term involving Ψ is the ordinary gravitational force
- Term involving Φ involves the $\dot{\Phi}$ term in the continuity equation as a (curvature) perturbation to the scale factor

Potential Decay

- Matter-to-radiation ratio

$$\frac{\rho_m}{\rho_r} \approx 24\Omega_m h^2 \left(\frac{a}{10^{-3}}\right)$$

  of order unity at recombination in a low Ω_m universe

- Radiation is not stress free and so impedes the growth of structure

$$k^2\Phi = 4\pi G a^2 \rho_r \Delta_r$$

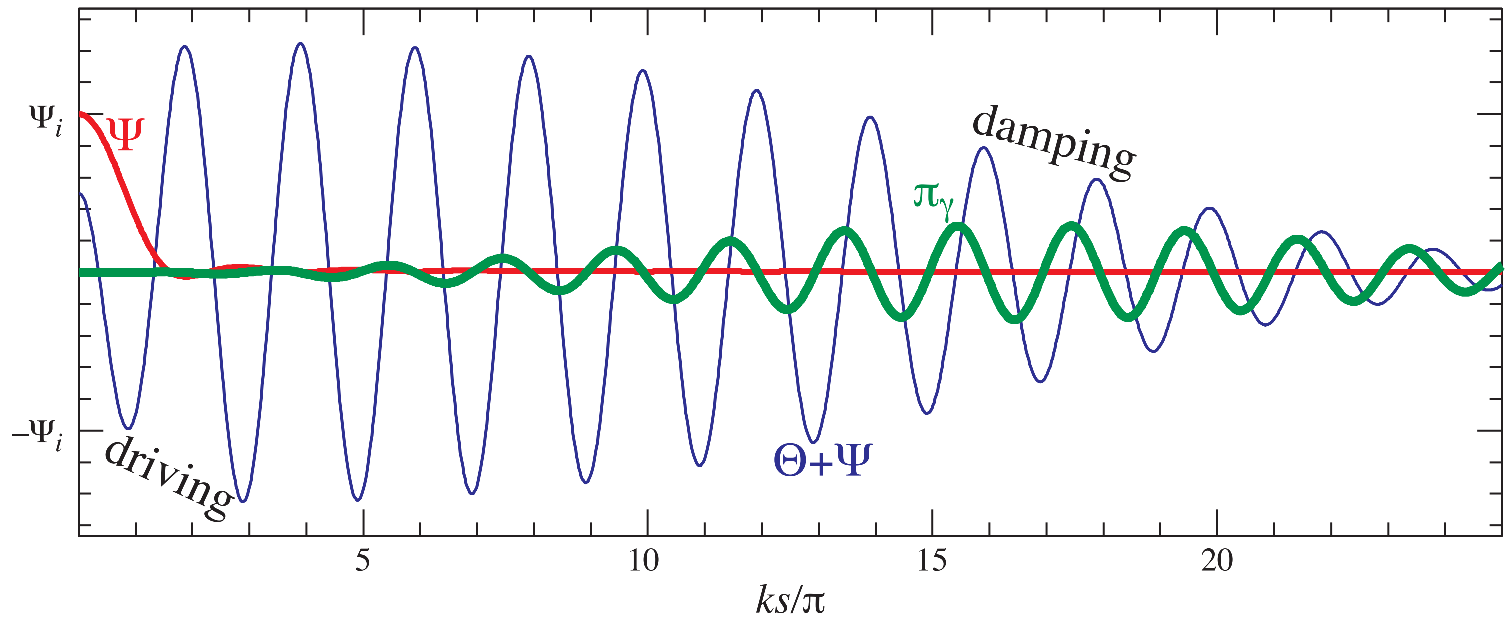
  $\Delta_r \sim 4\Theta$ oscillates around a constant value, $\rho_r \propto a^{-4}$ so the Netwonian curvature decays.

- General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale

Radiation Driving

- Decay is timed precisely to drive the oscillator - close to fully coherent

$$
\begin{aligned}
|[\Theta + \Psi](\eta)| &= |[\Theta + \Psi](0) + \Delta\Psi - \Delta\Phi| \\
&= |\frac{1}{3}\Psi(0) - 2\Psi(0)| = |\frac{5}{3}\Psi(0)|
\end{aligned}
$$



- $5\times$ the amplitude of the Sachs-Wolfe effect!

External Potential Approach

- Solution to homogeneous equation

$$(1+R)^{-1/4}\cos(ks)\,, \qquad (1+R)^{-1/4}\sin(ks)$$

- Give the general solution for an external potential by propagating impulsive forces

$$(1+R)^{1/4}\Theta(\eta) = \Theta(0)\cos(ks) + \frac{\sqrt{3}}{k}\left[\dot\Theta(0) + \frac{1}{4}\dot R(0)\Theta(0)\right]\sin ks$$
$$+ \frac{\sqrt{3}}{k}\int_0^\eta d\eta'(1+R')^{3/4}\sin[ks - ks']F(\eta')$$

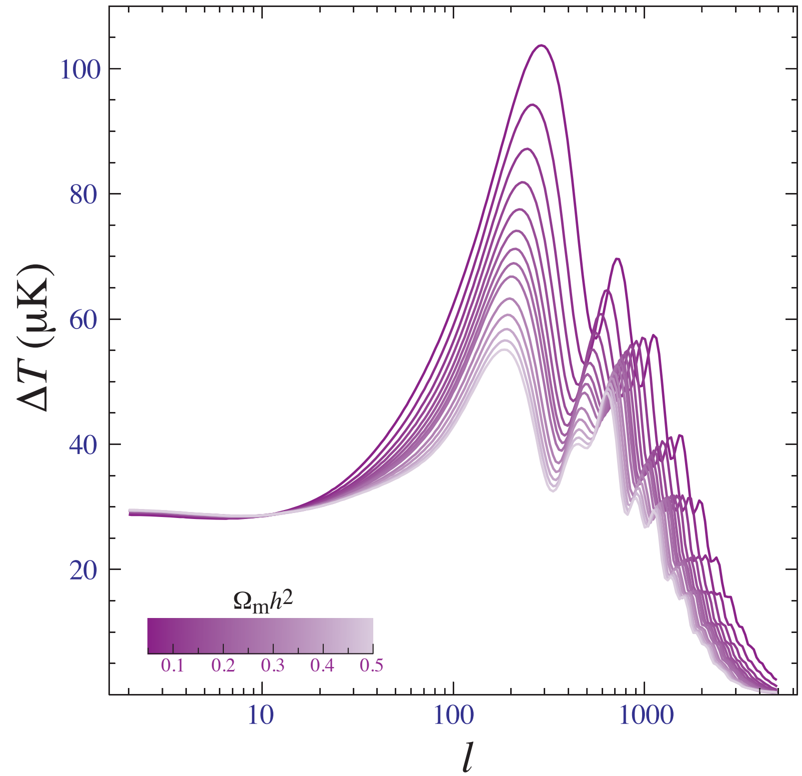
where

$$F = -\ddot\Phi - \frac{\dot R}{1+R}\dot\Phi - \frac{k^2}{3}\Psi$$

- Useful if general form of potential evolution is known

Matter-Radiation in the Power Spectrum

- Coherent approximation is exact for a photon-baryon fluid but reality is reduced to $\sim 4\times$ because neutrino contribution is free streaming not fluid like
- Neutrinos drive the oscillator less efficiently and also slightly change the phase of the oscillation



- Actual initial conditions are $\Theta + \Psi = \Psi/2$ for radiation domination but comparison to matter dominated SW correct
- With 3 peaks, it is possible to solve for both the baryons and dark matter densities, providing a calibration for the sound horizon
- Higher peaks check consistency with assumptions: e.g. extra relativistic d.o.f.s

Damping

- Tight coupling equations assume a perfect fluid: no viscosity, no heat conduction
- Fluid imperfections are related to the mean free path of the photons in the baryons

$$\lambda_C = \dot{\tau}^{-1} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a$$

  is the conformal opacity to Thomson scattering
- Dissipation related to diffusion length: random walk approx

$$\lambda_D = \sqrt{N} \lambda_C = \sqrt{\eta / \lambda_C}\, \lambda_C = \sqrt{\eta \lambda_C}$$

  the geometric mean between the horizon and mean free path
- $\lambda_C / \eta_* \sim \%$, so expect peaks > 3 to be affected by dissipation
- $\sqrt{\eta}$ enters here and η in the acoustic scale $\rightarrow$ expansion rate and extra relativistic species

Equations of Motion

- Continuity

$$\dot\Theta = -\frac{k}{3}v_\gamma - \dot\Phi\,, \quad \dot\delta_b = -kv_b - 3\dot\Phi$$

where the photon equation remains unchanged and the baryons follow number conservation with $\rho_b = m_b n_b$

- Navier-Stokes (Euler + heat conduction, viscosity)

$$\begin{aligned}
\dot v_\gamma &= k(\Theta + \Psi) - \frac{k}{6}\pi_\gamma - \dot\tau(v_\gamma - v_b) \\
\dot v_b &= -\frac{\dot a}{a}v_b + k\Psi + \dot\tau(v_\gamma - v_b)/R
\end{aligned}$$

where the photons gain an anisotropic stress term π_γ from radiation viscosity and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation

Viscosity

- Viscosity is generated from radiation streaming from hot to cold regions
- Expect

$$\pi_\gamma \sim v_\gamma \frac{k}{\dot{\tau}}$$

generated by streaming, suppressed by scattering in a wavelength of the fluctuation. Radiative transfer says

$$\pi_\gamma \approx 2 A_v v_\gamma \frac{k}{\dot{\tau}}$$

where $A_v = 16/15$

$$\dot{v}_\gamma = k(\Theta + \Psi) - \frac{k}{3} A_v \frac{k}{\dot{\tau}} v_\gamma$$

Oscillator: Penultimate Take

- Adiabatic approximation ($\omega \gg \dot{a}/a$)

$$\dot{\Theta} \approx -\frac{k}{3} v_\gamma$$

- Oscillator equation contains a $\dot{\Theta}$ damping term

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} A_v \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

- Heat conduction term similar in that it is proportional to v_γ and is suppressed by scattering $k/\dot{\tau}$. Expansion of Euler equations to leading order in $k\dot{\tau}$ gives

$$A_h = \frac{R^2}{1+R}$$

since the effects are only significant if the baryons are dynamically important

Oscillator: Final Take

- Final oscillator equation

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}}[A_v + A_h]\dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

- Solve in the adiabatic approximation

$$\Theta \propto \exp(i \int \omega d\eta)$$

$$-\omega^2 + \frac{k^2 c_s^2}{\dot{\tau}}(A_v + A_h) i\omega + k^2 c_s^2 = 0$$

Dispersion Relation

- Solve

$$\omega^2 = k^2 c_s^2 \left[1 + i\frac{\omega}{\dot\tau}(A_v + A_h)\right]$$
$$\omega = \pm k c_s \left[1 + \frac{i}{2}\frac{\omega}{\dot\tau}(A_v + A_h)\right]$$
$$= \pm k c_s \left[1 \pm \frac{i}{2}\frac{k c_s}{\dot\tau}(A_v + A_h)\right]$$

- Exponentiate

$$\exp(i \int \omega d\eta) = e^{\pm iks} \exp[-k^2 \int d\eta \frac{1}{2}\frac{c_s^2}{\dot\tau}(A_v + A_h)]$$
$$= e^{\pm iks} \exp[-(k/k_D)^2]$$

- Damping is exponential under the scale k_D

Diffusion Scale

- Diffusion wavenumber

$$k_D^{-2} = \int d\eta \frac{1}{\dot\tau} \frac{1}{6(1+R)} \left(\frac{16}{15} + \frac{R^2}{(1+R)} \right)$$

- Limiting forms

$$\lim_{R\to 0} k_D^{-2} = \frac{1}{6}\frac{16}{15} \int d\eta \frac{1}{\dot\tau}$$

$$\lim_{R\to \infty} k_D^{-2} = \frac{1}{6} \int d\eta \frac{1}{\dot\tau}$$

- Geometric mean between horizon and mean free path as expected from a random walk

$$\lambda_D = \frac{2\pi}{k_D} \sim \frac{2\pi}{\sqrt{6}} (\eta \dot\tau^{-1})^{1/2}$$