KIPMU

Set 1: CMB Statistics

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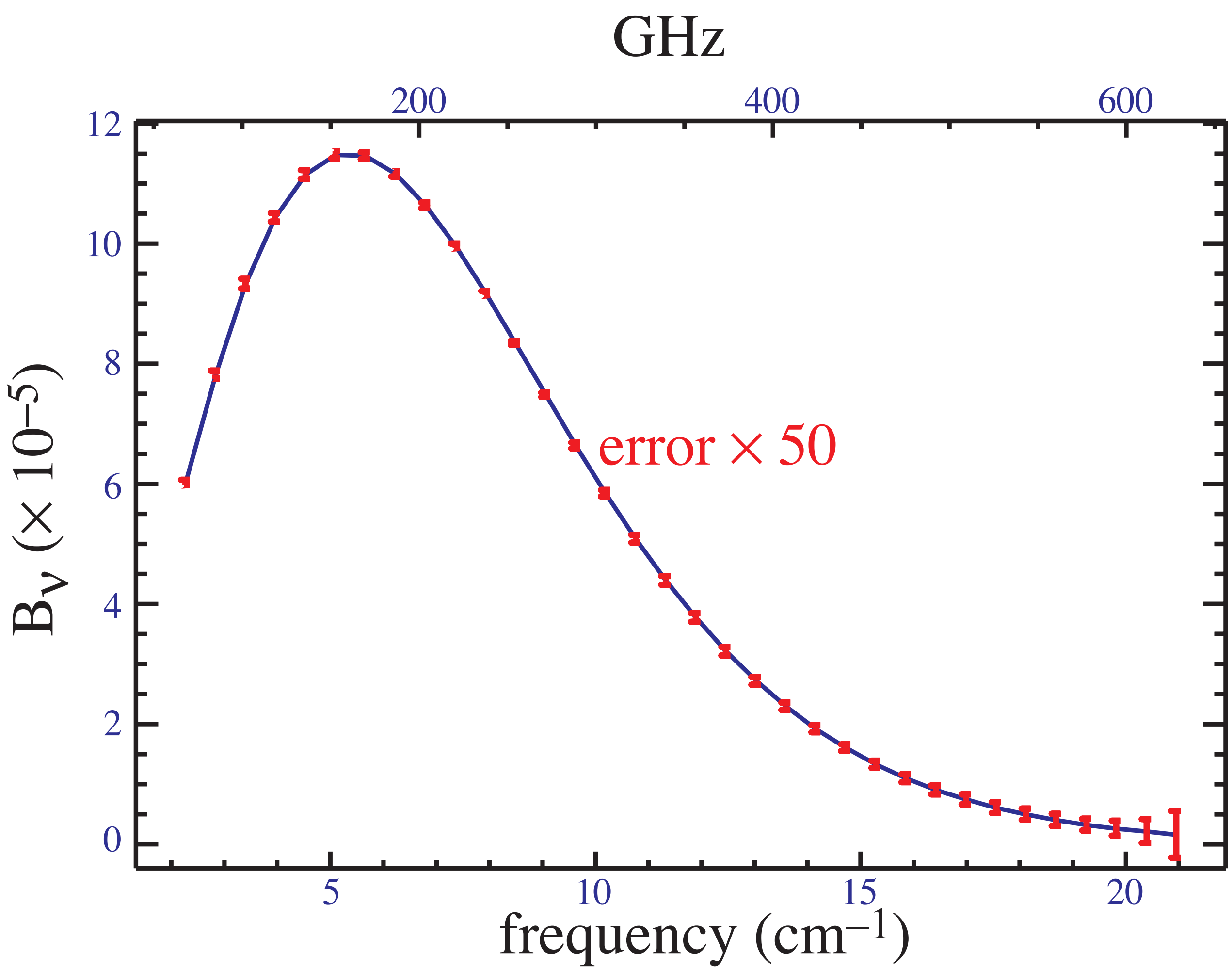
CMB Blackbody

- COBE FIRAS spectral measurement. yellBlackbody spectrum. $T = 2.725\mathrm{K}$ giving $\Omega_\gamma h^2 = 2.471 \times 10^{-5}$



CMB Blackbody

- CMB is a (nearly) perfect blackbody characterized by a phase space distribution function

$$f = \frac{1}{e^{E/T} - 1}$$

  where the temperature $T(\mathbf{x}, \hat{\mathbf{n}}, t)$ is observed at our position $\mathbf{x} = 0$ and time t_0 to be nearly isotropic with a mean temperature of $\bar{T} = 2.725\mathrm{K}$

- Our observable then is the temperature anisotropy

$$\Theta(\hat{\mathbf{n}}) \equiv \frac{T(0, \hat{\mathbf{n}}, t_0) - \bar{T}}{\bar{T}}$$

- Given that physical processes essentially put a band limit on this function it is useful to decompose it into a complete set of harmonic coefficients

Spherical Harmonics

- Laplace Eigenfunctions

$$\nabla^2 Y_\ell^m = -[l(l+1)] Y_\ell^m$$

- Orthogonal and complete

$$\int d\hat{\mathbf{n}} Y_\ell^{m*}(\hat{\mathbf{n}}) Y_\ell^m(\hat{\mathbf{n}}) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\sum_{\ell m} Y_\ell^{m*}(\hat{\mathbf{n}}) Y_\ell^m(\hat{\mathbf{n}}') = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta')$$

Generalizable to tensors on the sphere (polarization), modes on a curved FRW metric

- Conjugation

$$Y_\ell^{m*} = (-1)^m Y_\ell^{-m}$$

Multipole Moments

- Decompose into multipole moments

$$\Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_\ell^m(\hat{\mathbf{n}})$$

- So $\Theta_{\ell m}$ is complex but $\Theta(\hat{\mathbf{n}})$ real:

$$\begin{aligned}
\Theta^*(\hat{\mathbf{n}}) &= \sum_{\ell m} \Theta^*_{\ell m} Y_\ell^{m*}(\hat{\mathbf{n}}) \\
&= \sum_{\ell m} \Theta^*_{\ell m} (-1)^m Y_\ell^{-m}(\hat{\mathbf{n}}) \\
&= \Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_\ell^m(\hat{\mathbf{n}}) = \sum_{\ell -m} \Theta_{\ell -m} Y_\ell^{-m}(\hat{\mathbf{n}})
\end{aligned}$$

so m and $-m$ are not independent

$$\Theta^*_{\ell m} = (-1)^m \Theta_{\ell\, -m}$$

N-pt correlation

- Since the fluctuations are random and zero mean we are interested in characterizing the N-point correlation

$$\langle \Theta(\hat{\mathbf{n}}_1) \ldots \Theta(\hat{\mathbf{n}}_n) \rangle = \sum_{\ell_1 \ldots \ell_n} \sum_{m_1 \ldots m_n} \langle \Theta_{\ell_1 m_1} \ldots \Theta_{\ell_n m_n} \rangle Y_{\ell_1}^{m_1}(\hat{\mathbf{n}}_1) \ldots Y_{\ell_n}^{m_n}(\hat{\mathbf{n}}_n)$$

- Statistical isotropy implies that we should get the same result in a rotated frame

$$R[Y_\ell^m(\hat{\mathbf{n}})] = \sum_{m'} D^\ell_{m' m}(\alpha, \beta, \gamma) Y_\ell^{m'}(\hat{\mathbf{n}})$$

where α, β and γ are the Euler angles of the rotation and D is the Wigner function (note Y_ℓ^m is a D function)

$$\langle \Theta_{\ell_1 m_1} \ldots \Theta_{\ell_n m_n} \rangle = \sum_{m_1' \ldots m_n'} \langle \Theta_{\ell_1 m_1'} \ldots \Theta_{\ell_n m_n'} \rangle D^{\ell_1}_{m_1 m_1'} \ldots D^{\ell_n}_{m_n m_n'}$$

N-pt correlation

- For any N-point function, combine rotation matrices (group multiplication; angular momentum addition) and orthogonality

$$\sum_m (-1)^{m_2-m} D^{\ell_1}_{m_1 m} D^{\ell_1}_{-m_2 -m} = \delta_{m_1 m_2}$$

- The simplest case is the 2pt function:

$$\langle \Theta_{\ell_1 m_1} \Theta_{\ell_2 m_2} \rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 -m_2} (-1)^{m_1} C_{\ell_1}$$

where C_ℓ is the power spectrum. Check

$$\begin{aligned} &= \sum_{m_1' m_2'} \delta_{\ell_1 \ell_2} \delta_{m_1' -m_2'} (-1)^{m_1'} C_{\ell_1} D^{\ell_1}_{m_1 m_1'} D^{\ell_2}_{m_2 m_2'} \\ &= \delta_{\ell_1 \ell_2} C_{\ell_1} \sum_{m_1'} (-1)^{m_1'} D^{\ell_1}_{m_1 m_1'} D^{\ell_2}_{m_2 -m_1'} = \delta_{\ell_1 \ell_2} \delta_{m_1 -m_2} (-1)^{m_1} C_{\ell_1} \end{aligned}$$

N-pt correlation

- Using the reality of the field

$$\langle \Theta^*_{\ell_1 m_1} \Theta_{\ell_2 m_2} \rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} C_{\ell_1} \,.$$

- If the statistics were Gaussian then all the N-point functions would be defined in terms of the products of two-point contractions, e.g.

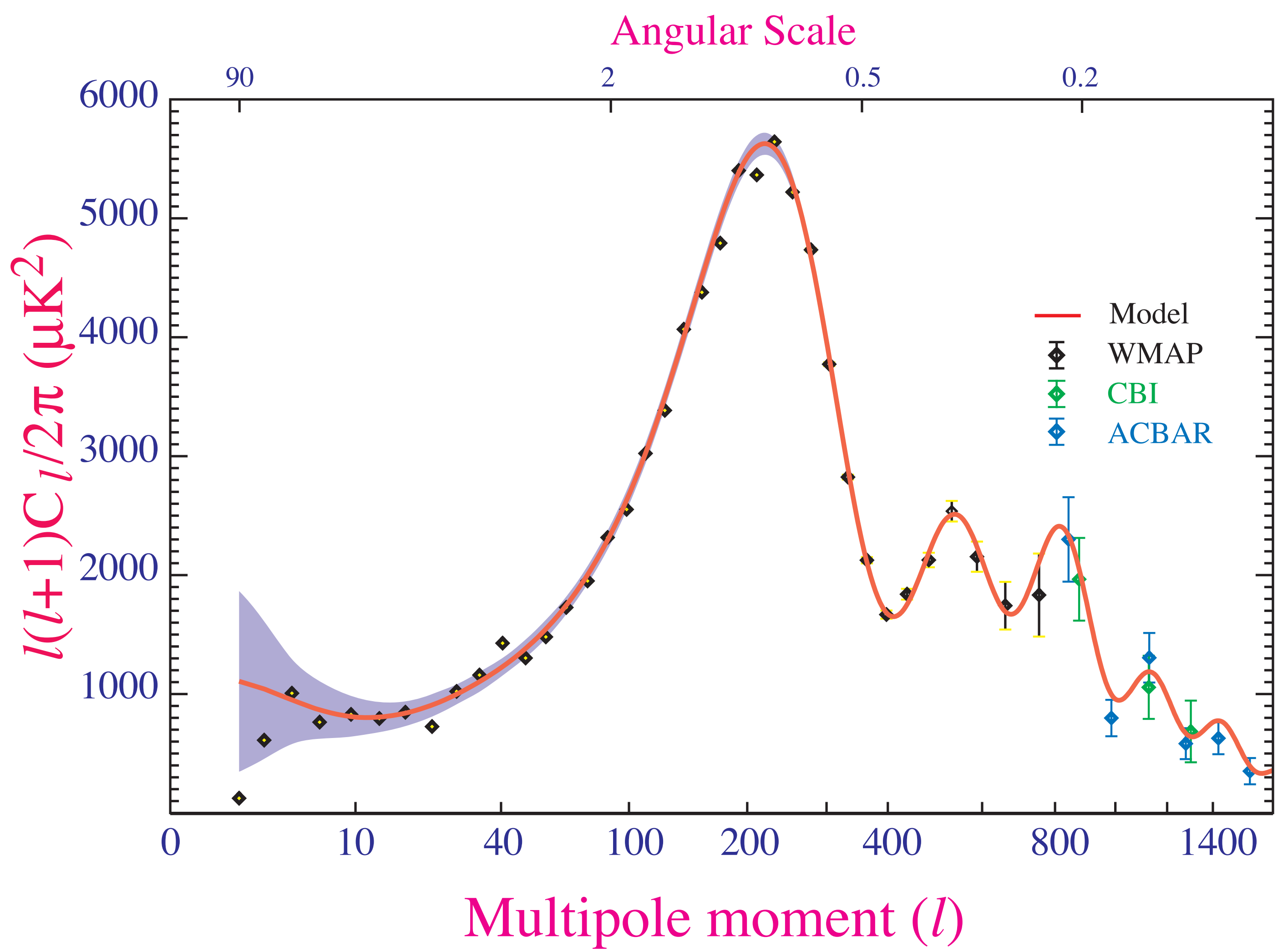
$$\langle \Theta_{\ell_1 m_1} \Theta_{\ell_2 m_2} \Theta_{\ell_3 m_3} \Theta_{\ell_4 m_4} \rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} \delta_{\ell_3 \ell_4} \delta_{m_3 m_4} C_{\ell_1} C_{\ell_3} + \text{perm.}$$

- More generally we can define the isotropy condition beyond Gaussianity, e.g. the bispectrum

$$\langle \Theta_{\ell_1 m_1} \ldots \Theta_{\ell_3 m_3} \rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1 \ell_2 \ell_3}$$

CMB Temperature Fluctuations

- Angular Power Spectrum



Why $\ell^2 C_\ell/2\pi$?

- Variance of the temperature fluctuation field

$$\begin{aligned}
\langle \Theta(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}})\rangle &= \sum_{\ell m}\sum_{\ell' m'} \langle \Theta_{\ell m}\Theta^*_{\ell' m'}\rangle Y_\ell^m(\hat{\mathbf{n}}) Y_{\ell'}^{m'*}(\hat{\mathbf{n}}) \\
&= \sum_\ell C_\ell \sum_m Y_\ell^m(\hat{\mathbf{n}}) Y_\ell^{m*}(\hat{\mathbf{n}}) \\
&= \sum_\ell \frac{2\ell+1}{4\pi} C_\ell
\end{aligned}$$

  via the angle addition formula for spherical harmonics

- For some range $\Delta\ell \approx \ell$ the contribution to the variance is

$$\langle \Theta(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}})\rangle_{\ell\pm\Delta\ell/2} \approx \Delta\ell \frac{2\ell+1}{4\pi} C_\ell \approx \frac{\ell^2}{2\pi} C_\ell$$

- Conventional to use $\ell(\ell+1)/2\pi$ for reasons below

Cosmic Variance

- We only have access to our sky, not the ensemble average
- There are $2\ell + 1$ m-modes of given ℓ mode, so average

$$\hat{C}_\ell = \frac{1}{2\ell+1}\sum_m \Theta^*_{\ell m}\Theta_{\ell m}$$

- $\langle \hat{C}_\ell \rangle = C_\ell$ but now there is a cosmic variance

$$\sigma^2_{C_\ell} = \frac{\langle (\hat{C}_\ell - C_\ell)(\hat{C}_\ell - C_\ell)\rangle}{C_\ell^2} = \frac{\langle \hat{C}_\ell \hat{C}_\ell \rangle - C_\ell^2}{C_\ell^2}$$

- For Gaussian statistics

$$\begin{aligned}
\sigma^2_{C_\ell} &= \frac{1}{(2\ell+1)^2 C_\ell^2}\langle \sum_{mm'} \Theta^*_{\ell m}\Theta_{\ell m}\Theta^*_{\ell m'}\Theta_{\ell m'}\rangle - 1 \\
&= \frac{1}{(2\ell+1)^2}\sum_{mm'}(\delta_{mm'} + \delta_{m-m'}) = \frac{2}{2\ell+1}
\end{aligned}$$

Cosmic Variance

- Note that the distribution of $\hat{C}_\ell$ is that of a sum of squares of Gaussian variates
- Distributed as a χ^2 of $2\ell + 1$ degrees of freedom
- Approaches a Gaussian for $2\ell + 1 \to \infty$ (central limit theorem)
- Anomalously low quadrupole is not that unlikely
- σ_{C_ℓ} is a useful quantification of errors at high ℓ
- Suppose C_ℓ depends on a set of cosmological parameters c_i then we can estimate errors of c_i measurements by error propagation

$$\begin{aligned} F_{ij} &= \mathrm{Cov}^{-1}(c_i, c_j) = \sum_{\ell\ell'} \frac{\partial C_\ell}{\partial c_i} \mathrm{Cov}^{-1}(\mathrm{C}_\ell, \mathrm{C}_{\ell'}) \frac{\partial C_{\ell'}}{\partial c_j} \\ &= \sum_\ell \frac{(2\ell+1)}{2C_\ell^2} \frac{\partial C_\ell}{\partial c_i} \frac{\partial C_\ell}{\partial c_j} \end{aligned}$$

Idealized Statistical Errors

- Take a noisy estimator of the multipoles in the map

$$\hat{\Theta}_{\ell m} = \Theta_{\ell m} + N_{\ell m}$$

and take the noise to be statistically isotropic

$$\langle N^*_{\ell m} N_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell^{NN}$$

- Construct an unbiased estimator of the power spectrum $\langle \hat{C}_\ell \rangle = C_\ell$

$$\hat{C}_\ell = \frac{1}{2\ell + 1} \sum_{m=-l}^{l} \hat{\Theta}^*_{\ell m} \hat{\Theta}_{\ell m} - C_\ell^{NN}$$

- Covariance in estimator

$$\mathrm{Cov}(C_\ell, C_{\ell'}) = \frac{2}{2\ell + 1} (C_\ell + C_\ell^{NN})^2 \delta_{\ell\ell'}$$

Incomplete Sky

- On a small section of sky, the number of independent modes of a given ℓ is no longer $2\ell + 1$
- As in Fourier analysis, there are two limitations: the lowest ℓ mode that can be measured is the wavelength that fits in angular patch θ

$$\ell_{\rm min} = \frac{2\pi}{\theta};$$

  modes separated by $\Delta\ell < \ell_{\rm min}$ cannot be measured independently
- Estimates of C_ℓ covary on a scale imposed by $\Delta\ell < \ell_{\rm min}$
- Crude approximation: account only for the loss of independent modes by rescaling the errors rather than introducing covariance

$$\mathrm{Cov}(C_\ell, C_{\ell'}) = \frac{2}{(2\ell+1) f_{\rm sky}} (C_\ell + C_\ell^{NN})^2 \delta_{\ell\ell'}$$

Stokes Parameters

- Specific intensity is related to quadratic combinations of the field.
- Define the intensity matrix (time averaged over oscillations) $\langle \mathbf{E}\,\mathbf{E}^\dagger \rangle$
- Hermitian matrix can be decomposed into Pauli matrices

$$\mathbf{P} = \langle \mathbf{E}\,\mathbf{E}^\dagger \rangle = \frac{1}{2}\left(I\boldsymbol{\sigma}_0 + Q\,\boldsymbol{\sigma}_3 + U\,\boldsymbol{\sigma}_1 - V\,\boldsymbol{\sigma}_2\right) ,$$

where

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Stokes parameters recovered as $\mathrm{Tr}(\sigma_i \mathbf{P})$

Stokes Parameters

- Consider a general plane wave solution

$$\mathbf{E}(t,z) = E_1(t,z)\hat{\mathbf{e}}_1 + E_2(t,z)\hat{\mathbf{e}}_2$$
$$E_1(t,z) = A_1 e^{i\phi_1} e^{i(kz-\omega t)}$$
$$E_2(t,z) = A_2 e^{i\phi_2} e^{i(kz-\omega t)}$$

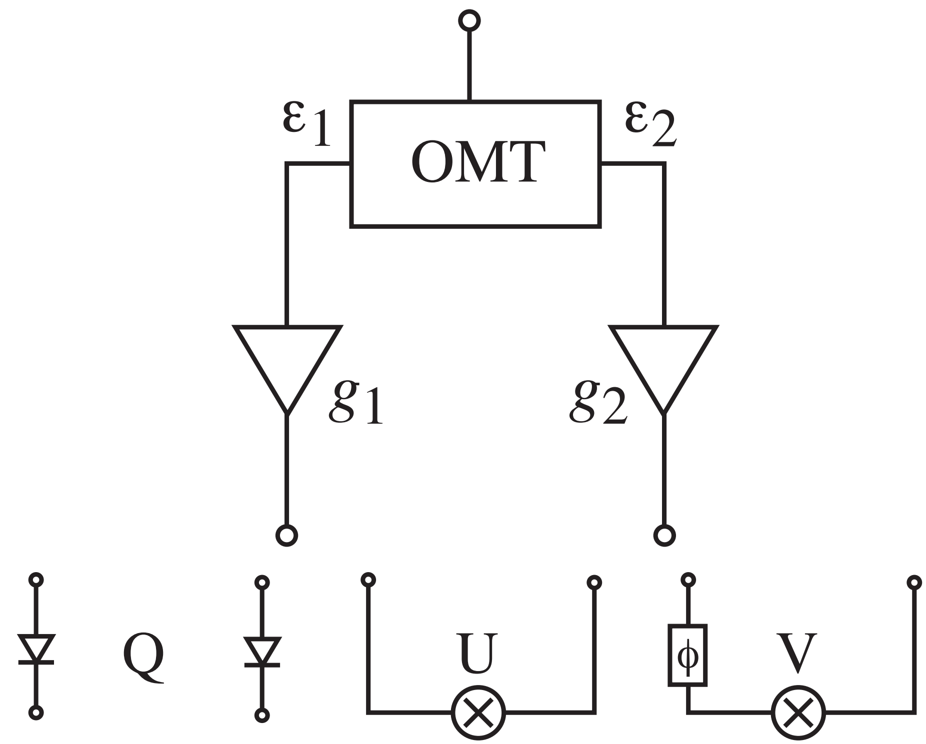
- Explicitly:

$$I = \langle E_1 E_1^* + E_2 E_2^* \rangle = A_1^2 + A_2^2$$
$$Q = \langle E_1 E_1^* - E_2 E_2^* \rangle = A_1^2 - A_2^2$$
$$U = \langle E_1 E_2^* + E_2 E_1^* \rangle = 2A_1 A_2 \cos(\phi_2 - \phi_1)$$
$$V = -i \langle E_1 E_2^* - E_2 E_1^* \rangle = 2A_1 A_2 \sin(\phi_2 - \phi_1)$$

so that the Stokes parameters define the state up to an unobservable overall phase of the wave

Detection

- This suggests that abstractly there are two different ways to detect polarization: separate and difference orthogonal modes (bolometers I, Q) or correlate the separated components (U, V).



- In the correlator example the natural output would be U but one can recover V by introducing a phase lag $\phi = \pi/2$ on one arm, and Q by having the OMT pick out directions rotated by $\pi/4$.
- Likewise, in the bolometer example, one can rotate the polarizer and also introduce a coherent front end to change V to U.

Detection

- Techniques also differ in the systematics that can convert unpolarized sky to fake polarization
- Differencing detectors are sensitive to relative gain fluctuations
- Correlation detectors are sensitive to cross coupling between the arms
- More generally, the intended block diagram and systematic problems map components of the polarization matrix onto others and are kept track of through "Jones" or instrumental response matrices $\mathbf{E}_{\rm det} = \mathbf{J}\mathbf{E}_{\rm in}$

$$\mathbf{P}_{\rm det} = \mathbf{J}\mathbf{P}_{\rm in}\mathbf{J}^\dagger$$

  where the end result is either a differencing or a correlation of the $\mathbf{P}_{\rm det}$.

Polarization

- Radiation field involves a directed quantity, the electric field vector, which defines the polarization
- Consider a general plane wave solution

$$\mathbf{E}(t,z) = E_1(t,z)\hat{\mathbf{e}}_1 + E_2(t,z)\hat{\mathbf{e}}_2$$

$$E_1(t,z) = \mathrm{Re} A_1 e^{i\phi_1} e^{i(kz-\omega t)}$$

$$E_2(t,z) = \mathrm{Re} A_2 e^{i\phi_2} e^{i(kz-\omega t)}$$

or at $z = 0$ the field vector traces out an ellipse

$$\mathbf{E}(t,0) = A_1 \cos(\omega t - \phi_1)\hat{\mathbf{e}}_1 + A_2 \cos(\omega t - \phi_2)\hat{\mathbf{e}}_2$$

with principal axes defined by

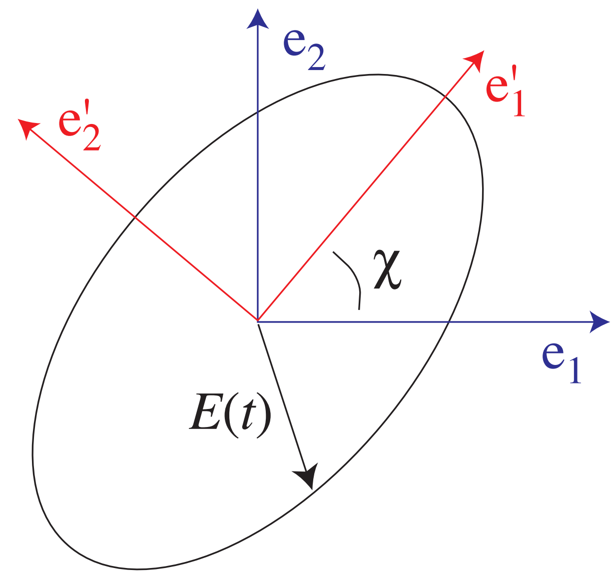
$$\mathbf{E}(t,0) = A_1' \cos(\omega t)\hat{\mathbf{e}}_1' - A_2' \sin(\omega t)\hat{\mathbf{e}}_2'$$

so as to trace out a clockwise rotation for $A_1', A_2' > 0$

Polarization

- Define polarization angle

$$\hat{\mathbf{e}}_1' = \cos\chi\hat{\mathbf{e}}_1 + \sin\chi\hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}_2' = -\sin\chi\hat{\mathbf{e}}_1 + \cos\chi\hat{\mathbf{e}}_2$$



- Match

$$\begin{aligned}\mathbf{E}(t,0) &= A_1'\cos\omega t[\cos\chi\hat{\mathbf{e}}_1 + \sin\chi\hat{\mathbf{e}}_2] \\ &\quad - A_2'\cos\omega t[-\sin\chi\hat{\mathbf{e}}_1 + \cos\chi\hat{\mathbf{e}}_2] \\ &= A_1[\cos\phi_1\cos\omega t + \sin\phi_1\sin\omega t]\hat{\mathbf{e}}_1 \\ &\quad + A_2[\cos\phi_2\cos\omega t + \sin\phi_2\sin\omega t]\hat{\mathbf{e}}_2\end{aligned}$$

Polarization

- Define relative strength of two principal states

$$A_1' = E_0 \cos\beta \qquad A_2' = E_0 \sin\beta$$

- Characterize the polarization by two angles

$$\begin{aligned} A_1 \cos\phi_1 &= E_0 \cos\beta \cos\chi, & A_1 \sin\phi_1 &= E_0 \sin\beta \sin\chi, \\ A_2 \cos\phi_2 &= E_0 \cos\beta \sin\chi, & A_2 \sin\phi_2 &= -E_0 \sin\beta \cos\chi \end{aligned}$$

Or Stokes parameters by

$$I = E_0^2\,, \qquad Q = E_0^2 \cos 2\beta \cos 2\chi$$

$$U = E_0^2 \cos 2\beta \sin 2\chi\,, \qquad V = E_0^2 \sin 2\beta$$

- So $I^2 = Q^2 + U^2 + V^2$, double angles reflect the spin 2 field or headless vector nature of polarization

Polarization

Special cases

- If $\beta = 0, \pi/2, \pi$ then only one principal axis, ellipse collapses to a line and $V = 0 \rightarrow$ linear polarization oriented at angle χ

  If $\chi = 0, \pi/2, \pi$ then $I = \pm Q$ and $U = 0$

  If $\chi = \pi/4, 3\pi/4 ...$ then $I = \pm U$ and $Q = 0$ - so U is Q in a frame rotated by 45 degrees

- If $\beta = \pi/4, 3\pi/4$, then principal components have equal strength and E field rotates on a circle: $I = \pm V$ and $Q = U = 0 \rightarrow$ circular polarization

- $U/Q = \tan 2\chi$ defines angle of linear polarization and $V/I = \sin 2\beta$ defines degree of circular polarization

Natural Light

- A monochromatic plane wave is completely polarized
  $I^2 = Q^2 + U^2 + V^2$
- Polarization matrix is like a density matrix in quantum mechanics and allows for pure (coherent) states and mixed states
- Suppose the total $\mathbf{E}_{\rm tot}$ field is composed of different (frequency) components

$$\mathbf{E}_{\rm tot} = \sum_i \mathbf{E}_i$$

- Then components decorrelate in time average

$$\left\langle \mathbf{E}_{\rm tot}\mathbf{E}_{\rm tot}^\dagger \right\rangle = \sum_{ij} \left\langle \mathbf{E}_i \mathbf{E}_j^\dagger \right\rangle = \sum_i \left\langle \mathbf{E}_i \mathbf{E}_i^\dagger \right\rangle$$

Natural Light

- So Stokes parameters of incoherent contributions add

$$I = \sum_i I_i \quad Q = \sum_i Q_i \quad U = \sum_i U_i \quad V = \sum_i V_i$$

and since individual Q, U and V can have either sign:
$I^2 \geq Q^2 + U^2 + V^2$, all 4 Stokes parameters needed

Linear Polarization

- $Q \propto \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle$, $U \propto \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle$.
- Counterclockwise rotation of axes by $\theta = 45°$

$$E_1 = (E_1' - E_2')/\sqrt{2}\,, \quad E_2 = (E_1' + E_2')/\sqrt{2}$$

- $U \propto \langle E_1' E_1'^* \rangle - \langle E_2' E_2'^* \rangle$, difference of intensities at $45°$ or Q'
- More generally, $\mathbf{P}$ transforms as a tensor under rotations and

$$Q' = \cos(2\theta)Q + \sin(2\theta)U$$
$$U' = -\sin(2\theta)Q + \cos(2\theta)U$$

or

$$Q' \pm iU' = e^{\mp 2i\theta}[Q \pm iU]$$

acquires a phase under rotation and is a spin ± 2 object

Coordinate Independent Representation

- Two directions: orientation of polarization and change in amplitude, i.e. Q and U in the basis of the Fourier wavevector (pointing with angle ϕ_l) for small sections of sky are called E and B components

$$
\begin{aligned}
E(\mathbf{l}) \pm iB(\mathbf{l}) &= -\int d\hat{\mathbf{n}}[Q'(\hat{\mathbf{n}}) \pm iU'(\hat{\mathbf{n}})]e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}} \\
&= -e^{\mp 2i\phi_l}\int d\hat{\mathbf{n}}[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})]e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}}
\end{aligned}
$$

- For the B-mode to not vanish, the polarization must point in a direction not related to the wavevector - not possible for density fluctuations in linear theory
- Generalize to all-sky: plane waves are eigenmodes of the Laplace operator on the tensor $\mathbf{P}$.

Spin Harmonics

- Laplace Eigenfunctions

$$\nabla^2{}_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1] = -[l(l+1)-4]_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1]$$

- Spin s spherical harmonics: orthogonal and complete

$$\int d\hat{\mathbf{n}}\,{}_sY^*_{\ell m}(\hat{\mathbf{n}})\,{}_sY_{\ell m}(\hat{\mathbf{n}}) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\sum_{\ell m} {}_sY^*_{\ell m}(\hat{\mathbf{n}})\,{}_sY_{\ell m}(\hat{\mathbf{n}}') = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta')$$

where the ordinary spherical harmonics are $Y_{\ell m} = {}_0Y_{\ell m}$

- Given in terms of the rotation matrix

$${}_sY_{\ell m}(\beta\alpha) = (-1)^m\sqrt{\frac{2\ell+1}{4\pi}}D^\ell_{-ms}(\alpha\beta 0)$$

Statistical Representation

- All-sky decomposition

$$[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})] = \sum_{\ell m} [E_{\ell m} \pm iB_{\ell m}]_{\pm 2}Y_{\ell m}(\hat{\mathbf{n}})$$

- Power spectra

$$\langle E_{\ell m}^* E_{\ell m} \rangle = \delta_{\ell\ell'}\delta_{mm'} C_\ell^{EE}$$

$$\langle B_{\ell m}^* B_{\ell m} \rangle = \delta_{\ell\ell'}\delta_{mm'} C_\ell^{BB}$$

- Cross correlation

$$\langle \Theta_{\ell m}^* E_{\ell m} \rangle = \delta_{\ell\ell'}\delta_{mm'} C_\ell^{\Theta E}$$

others vanish if parity is conserved

Thomson Scattering

- Polarization state of radiation in direction $\hat{\mathbf{n}}$ described by the intensity matrix $\langle E_i(\hat{\mathbf{n}})E_j^*(\hat{\mathbf{n}})\rangle$, where $\mathbf{E}$ is the electric field vector and the brackets denote time averaging.
- Differential cross section

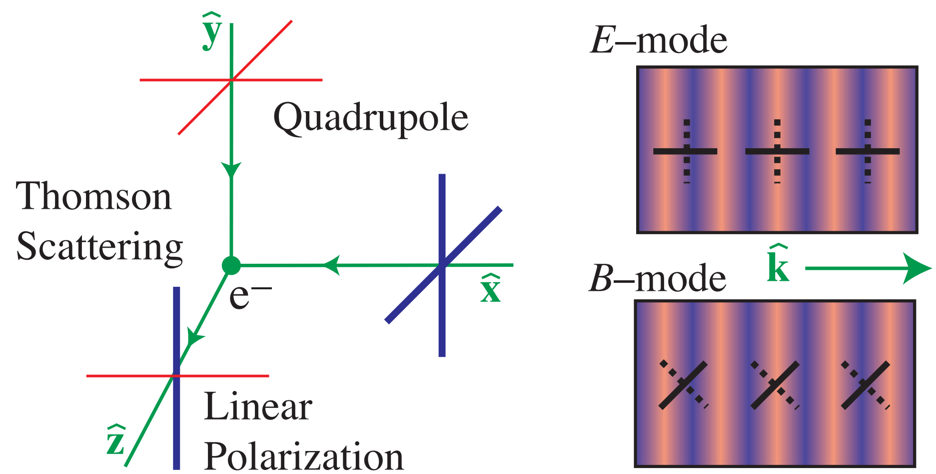
$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi}|\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \,,$$

  where $\sigma_T = 8\pi\alpha^2/3m_e$ is the Thomson cross section, $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.
- Summed over angle and incoming polarization

$$\sum_{i=1,2} \int d\hat{\mathbf{n}}' \frac{d\sigma}{d\Omega} = \sigma_T$$

Polarization Generation

- Heuristic:
  incoming radiation shakes
  an electron in direction
  of electric field vector $\hat{\mathbf{E}}'$



- Radiates photon with
  polarization also in direction $\hat{\mathbf{E}}'$
- But photon cannot be longitudinally polarized so that scattering into $90°$ can only pass one polarization
- Linearly polarized radiation like polarization by reflection
- Unlike reflection of sunlight, incoming radiation is nearly isotropic
- Missing from direction orthogonal to original incoming direction
- Only quadrupole anisotropy generates polarization by Thomson scattering