Ast 448

Set 1: CMB Statistics

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Stokes Parameters

- Specific intensity is related to quadratic combinations of the electric field.
- Define the intensity matrix (time averaged over oscillations) $\langle \mathbf{E}\,\mathbf{E}^\dagger \rangle$
- Hermitian matrix can be decomposed into Pauli matrices

$$\mathbf{P} = \langle \mathbf{E}\,\mathbf{E}^\dagger \rangle = \frac{1}{2}\left(I\boldsymbol{\sigma}_0 + Q\,\boldsymbol{\sigma}_3 + U\,\boldsymbol{\sigma}_1 - V\,\boldsymbol{\sigma}_2\right)\ ,$$

where

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Stokes parameters recovered as $\mathrm{Tr}(\sigma_i \mathbf{P})$
- Choose units of temperature for Stokes parameters $I \to \Theta$

Stokes Parameters

- Consider a general plane wave solution

$$\mathbf{E}(t,z) = E_1(t,z)\hat{\mathbf{e}}_1 + E_2(t,z)\hat{\mathbf{e}}_2$$

$$E_1(t,z) = A_1 e^{i\phi_1} e^{i(kz-\omega t)}$$

$$E_2(t,z) = A_2 e^{i\phi_2} e^{i(kz-\omega t)}$$

- Explicitly:

$$I = \langle E_1 E_1^* + E_2 E_2^* \rangle = A_1^2 + A_2^2$$

$$Q = \langle E_1 E_1^* - E_2 E_2^* \rangle = A_1^2 - A_2^2$$

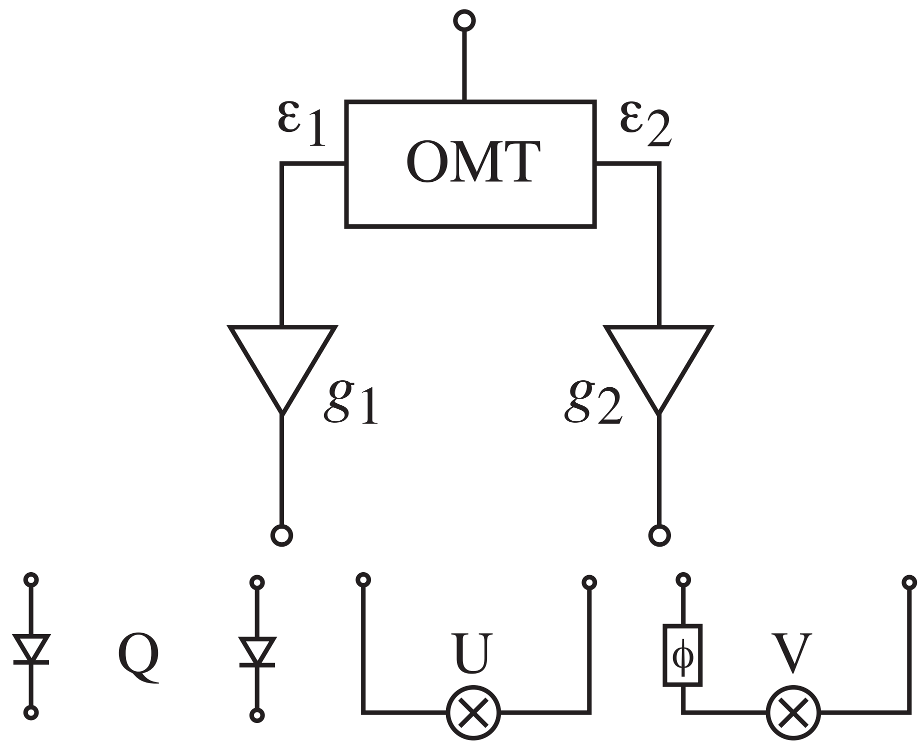
$$U = \langle E_1 E_2^* + E_2 E_1^* \rangle = 2A_1 A_2 \cos(\phi_2 - \phi_1)$$

$$V = -i \langle E_1 E_2^* - E_2 E_1^* \rangle = 2A_1 A_2 \sin(\phi_2 - \phi_1)$$

so that the Stokes parameters define the state up to an unobservable overall phase of the wave

Detection

- This suggests that abstractly there are two different ways to detect polarization: separate and difference orthogonal modes (bolometers I, Q) or correlate the separated components (U, V).



- In the correlator example the natural output would be U but one can recover V by introducing a phase lag $\phi = \pi/2$ on one arm, and Q by having the OMT pick out directions rotated by $\pi/4$.
- Likewise, in the bolometer example, one can rotate the polarizer and also introduce a coherent front end to change V to U.

Detection

- Techniques also differ in the systematics that can convert unpolarized sky to fake polarization
- Differencing detectors are sensitive to relative gain fluctuations
- Correlation detectors are sensitive to cross coupling between the arms
- More generally, the intended block diagram and systematic problems map components of the polarization matrix onto others and are kept track of through "Jones" or instrumental response matrices $\mathbf{E}_{\rm det} = \mathbf{J}\mathbf{E}_{\rm in}$

$$\mathbf{P}_{\rm det} = \mathbf{J}\mathbf{P}_{\rm in}\mathbf{J}^\dagger$$

where the end result is either a differencing or a correlation of the $\mathbf{P}_{\rm det}$.

Polarization

- Radiation field involves a directed quantity, the electric field vector, which defines the polarization
- Consider a general plane wave solution

$$\mathbf{E}(t,z) = E_1(t,z)\hat{\mathbf{e}}_1 + E_2(t,z)\hat{\mathbf{e}}_2$$

$$E_1(t,z) = \mathrm{Re} A_1 e^{i\phi_1} e^{i(kz-\omega t)}$$

$$E_2(t,z) = \mathrm{Re} A_2 e^{i\phi_2} e^{i(kz-\omega t)}$$

or at $z = 0$ the field vector traces out an ellipse

$$\mathbf{E}(t,0) = A_1 \cos(\omega t - \phi_1)\hat{\mathbf{e}}_1 + A_2 \cos(\omega t - \phi_2)\hat{\mathbf{e}}_2$$

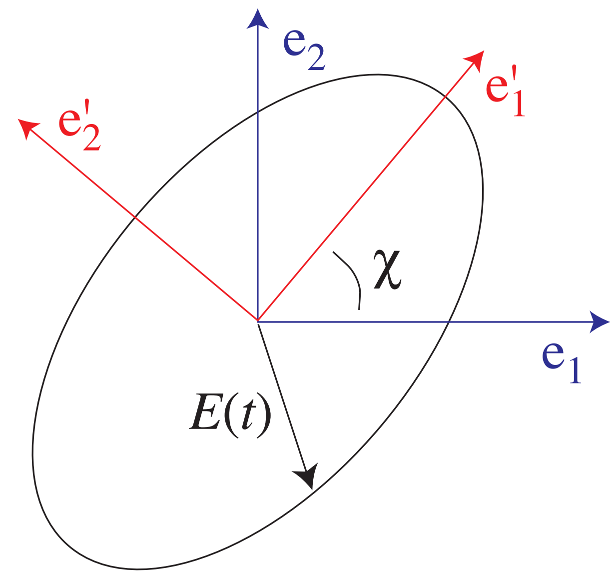
with principal axes defined by

$$\mathbf{E}(t,0) = A_1' \cos(\omega t)\hat{\mathbf{e}}_1' - A_2' \sin(\omega t)\hat{\mathbf{e}}_2'$$

so as to trace out a clockwise rotation for $A_1', A_2' > 0$

Polarization

- Define polarization angle



$$\hat{\mathbf{e}}'_1 = \cos\chi\hat{\mathbf{e}}_1 + \sin\chi\hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}'_2 = -\sin\chi\hat{\mathbf{e}}_1 + \cos\chi\hat{\mathbf{e}}_2$$

- Match

$$\begin{aligned}
\mathbf{E}(t,0) &= A'_1 \cos\omega t[\cos\chi\hat{\mathbf{e}}_1 + \sin\chi\hat{\mathbf{e}}_2] \\
&\quad - A'_2 \cos\omega t[-\sin\chi\hat{\mathbf{e}}_1 + \cos\chi\hat{\mathbf{e}}_2] \\
&= A_1[\cos\phi_1 \cos\omega t + \sin\phi_1 \sin\omega t]\hat{\mathbf{e}}_1 \\
&\quad + A_2[\cos\phi_2 \cos\omega t + \sin\phi_2 \sin\omega t]\hat{\mathbf{e}}_2
\end{aligned}$$

Polarization

- Define relative strength of two principal states

$$A'_1 = E_0 \cos\beta \quad A'_2 = E_0 \sin\beta$$

- Characterize the polarization by two angles

$$A_1 \cos\phi_1 = E_0 \cos\beta \cos\chi, \qquad A_1 \sin\phi_1 = E_0 \sin\beta \sin\chi,$$
$$A_2 \cos\phi_2 = E_0 \cos\beta \sin\chi, \qquad A_2 \sin\phi_2 = -E_0 \sin\beta \cos\chi$$

Or Stokes parameters by

$$I = E_0^2\,, \quad Q = E_0^2 \cos 2\beta \cos 2\chi$$
$$U = E_0^2 \cos 2\beta \sin 2\chi\,, \quad V = E_0^2 \sin 2\beta$$

- So $I^2 = Q^2 + U^2 + V^2$, double angles reflect the spin 2 field or headless vector nature of polarization

Polarization

Special cases

- If $\beta = 0, \pi/2, \pi$ then only one principal axis, ellipse collapses to a line and $V = 0 \rightarrow$ linear polarization oriented at angle χ

  If $\chi = 0, \pi/2, \pi$ then $I = \pm Q$ and $U = 0$

  If $\chi = \pi/4, 3\pi/4...$ then $I = \pm U$ and $Q = 0$ - so U is Q in a frame rotated by 45 degrees

- If $\beta = \pi/4, 3\pi/4$, then principal components have equal strength and E field rotates on a circle: $I = \pm V$ and $Q = U = 0 \rightarrow$ circular polarization

- $U/Q = \tan 2\chi$ defines angle of linear polarization and $V/I = \sin 2\beta$ defines degree of circular polarization

Natural Light

- A monochromatic plane wave is completely polarized $I^2 = Q^2 + U^2 + V^2$
- Polarization matrix is like a density matrix in quantum mechanics and allows for pure (coherent) states and mixed states
- Suppose the total $\mathbf{E}_{\rm tot}$ field is composed of different (frequency) components

$$\mathbf{E}_{\rm tot} = \sum_i \mathbf{E}_i$$

- Then components decorrelate in time average

$$\left\langle \mathbf{E}_{\rm tot} \mathbf{E}_{\rm tot}^\dagger \right\rangle = \sum_{ij} \left\langle \mathbf{E}_i \mathbf{E}_j^\dagger \right\rangle = \sum_i \left\langle \mathbf{E}_i \mathbf{E}_i^\dagger \right\rangle$$

Natural Light

- So Stokes parameters of incoherent contributions add

$$I = \sum_i I_i \quad Q = \sum_i Q_i \quad U = \sum_i U_i \quad V = \sum_i V_i$$

and since individual Q, U and V can have either sign:
$I^2 \geq Q^2 + U^2 + V^2$, all 4 Stokes parameters needed

Linear Polarization

- $Q \propto \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle$, $U \propto \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle$.
- Counterclockwise rotation of axes by $\theta = 45^\circ$

$$E_1 = (E_1' - E_2')/\sqrt{2}\,, \quad E_2 = (E_1' + E_2')/\sqrt{2}$$

- $U \propto \langle E_1' E_1'^* \rangle - \langle E_2' E_2'^* \rangle$, difference of intensities at 45° or Q'
- More generally, **P** transforms as a tensor under rotations and

$$Q' = \cos(2\theta) Q + \sin(2\theta) U$$
$$U' = -\sin(2\theta) Q + \cos(2\theta) U$$

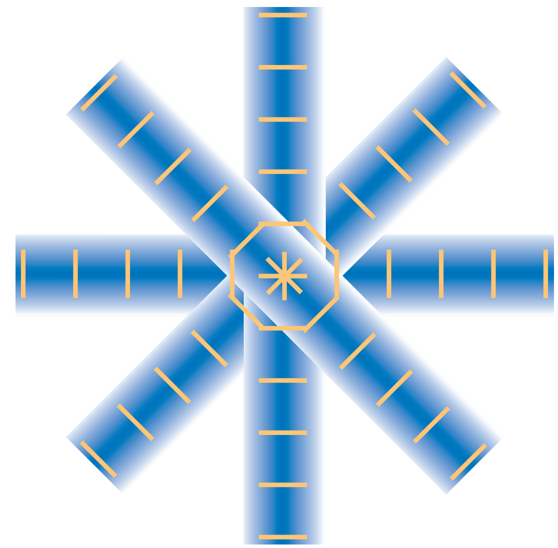
or

$$Q' \pm iU' = e^{\mp 2i\theta}[Q \pm iU]$$

acquires a phase under rotation and is a spin ± 2 object

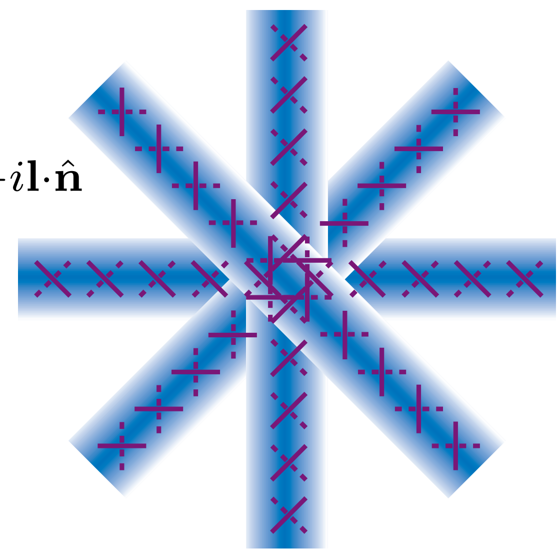
Coordinate Independent Representation

- Two directions: orientation of polarization and change in amplitude, i.e. Q and U in the basis of the Fourier wavevector (pointing with angle ϕ_l) for small sections of sky are called E and B components



$$E(\mathbf{l}) \pm iB(\mathbf{l}) = -\int d\hat{\mathbf{n}}[Q'(\hat{\mathbf{n}}) \pm iU'(\hat{\mathbf{n}})]e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}}$$

$$= -e^{\mp 2i\phi_l}\int d\hat{\mathbf{n}}[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})]e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}}$$



- For the B-mode to not vanish, the polarization must point in a direction not related to the wavevector - not possible for density fluctuations in linear theory
- Generalize to all-sky: eigenmodes of Laplace operator of tensor

Spin Harmonics

- Laplace Eigenfunctions

$$\nabla^2 {}_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1] = -[l(l+1)-4]_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1]$$

- Spin s spherical harmonics: orthogonal and complete

$$\int d\hat{\mathbf{n}}_s Y^*_{\ell m}(\hat{\mathbf{n}})_s Y_{\ell' m'}(\hat{\mathbf{n}}) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\sum_{\ell m} {}_sY^*_{\ell m}(\hat{\mathbf{n}})_s Y_{\ell m}(\hat{\mathbf{n}}') = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta')$$

where the ordinary spherical harmonics are $Y_{\ell m} = {}_0Y_{\ell m}$

- Given in terms of the rotation matrix

$${}_sY_{\ell m}(\beta\alpha) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell}_{-ms}(\alpha\beta 0)$$

Statistical Representation

- All-sky decomposition

$$[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})] = \sum_{\ell m} [E_{\ell m} \pm iB_{\ell m}]_{\pm 2}Y_{\ell m}(\hat{\mathbf{n}})$$

- Power spectra

$$\langle E^*_{\ell m} E_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{EE}$$

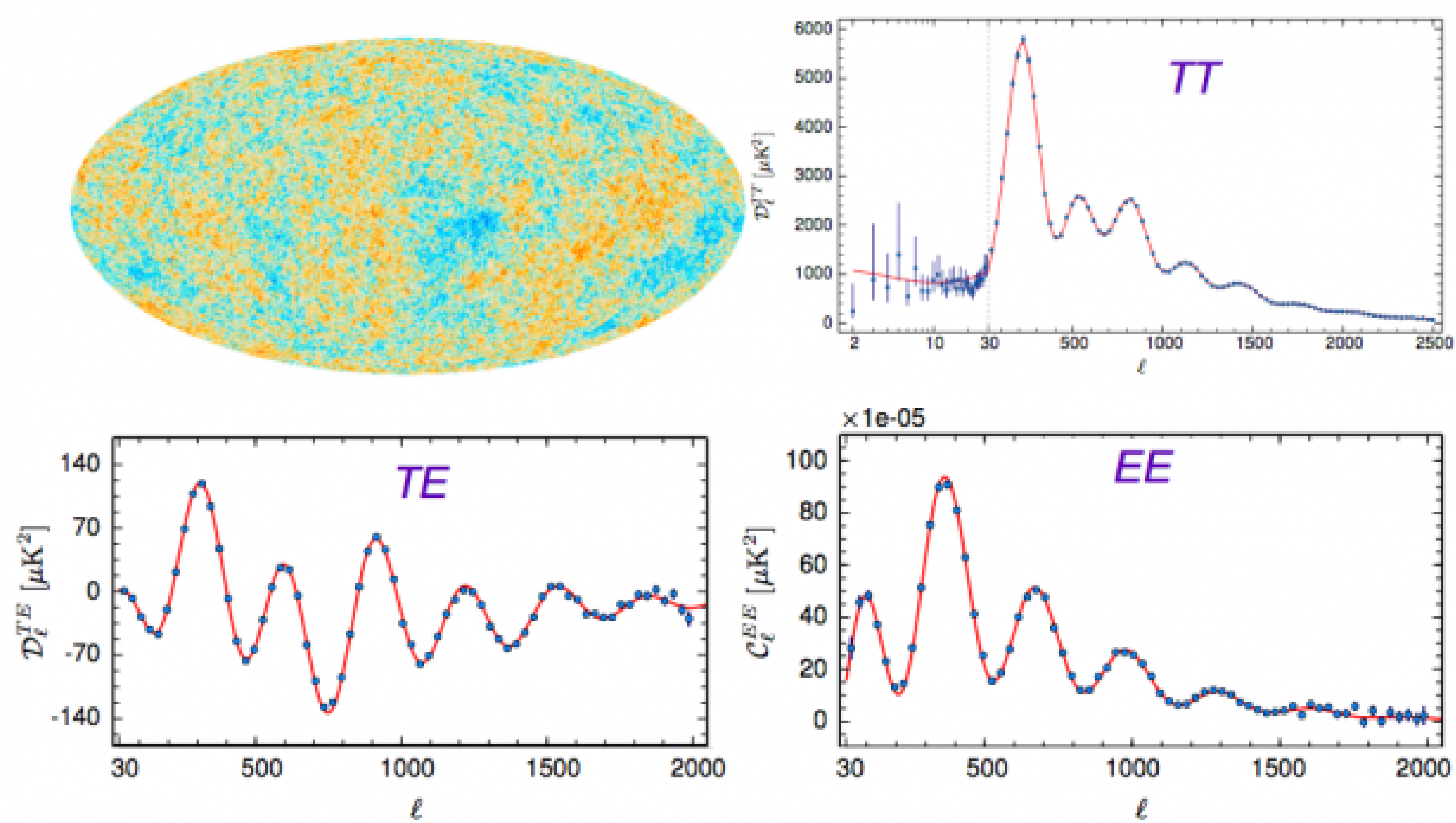
$$\langle B^*_{\ell m} B_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{BB}$$

- Cross correlation

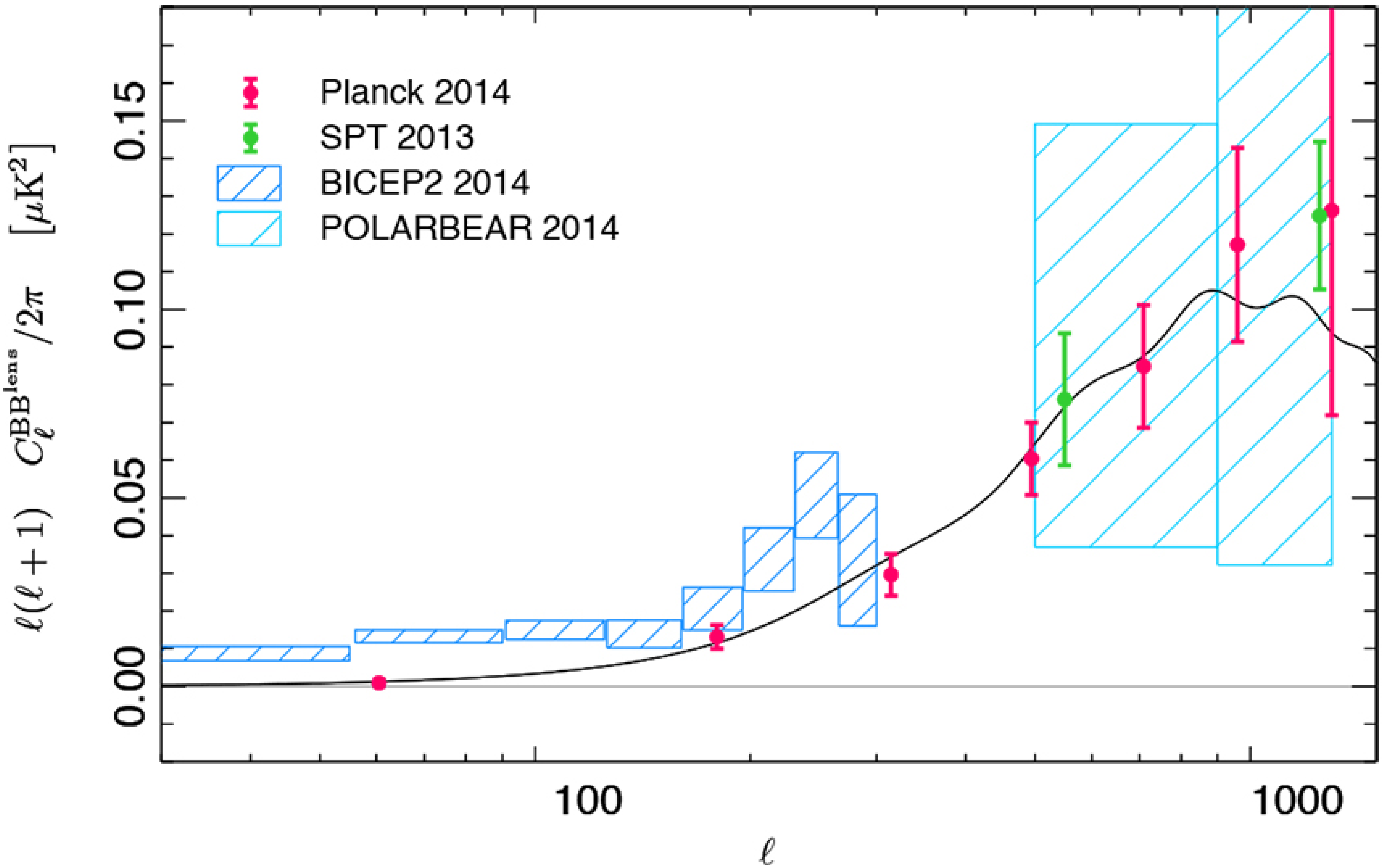
$$\langle \Theta^*_{\ell m} E_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{\Theta E}$$

others vanish if parity is conserved

Planck Power Spectrum

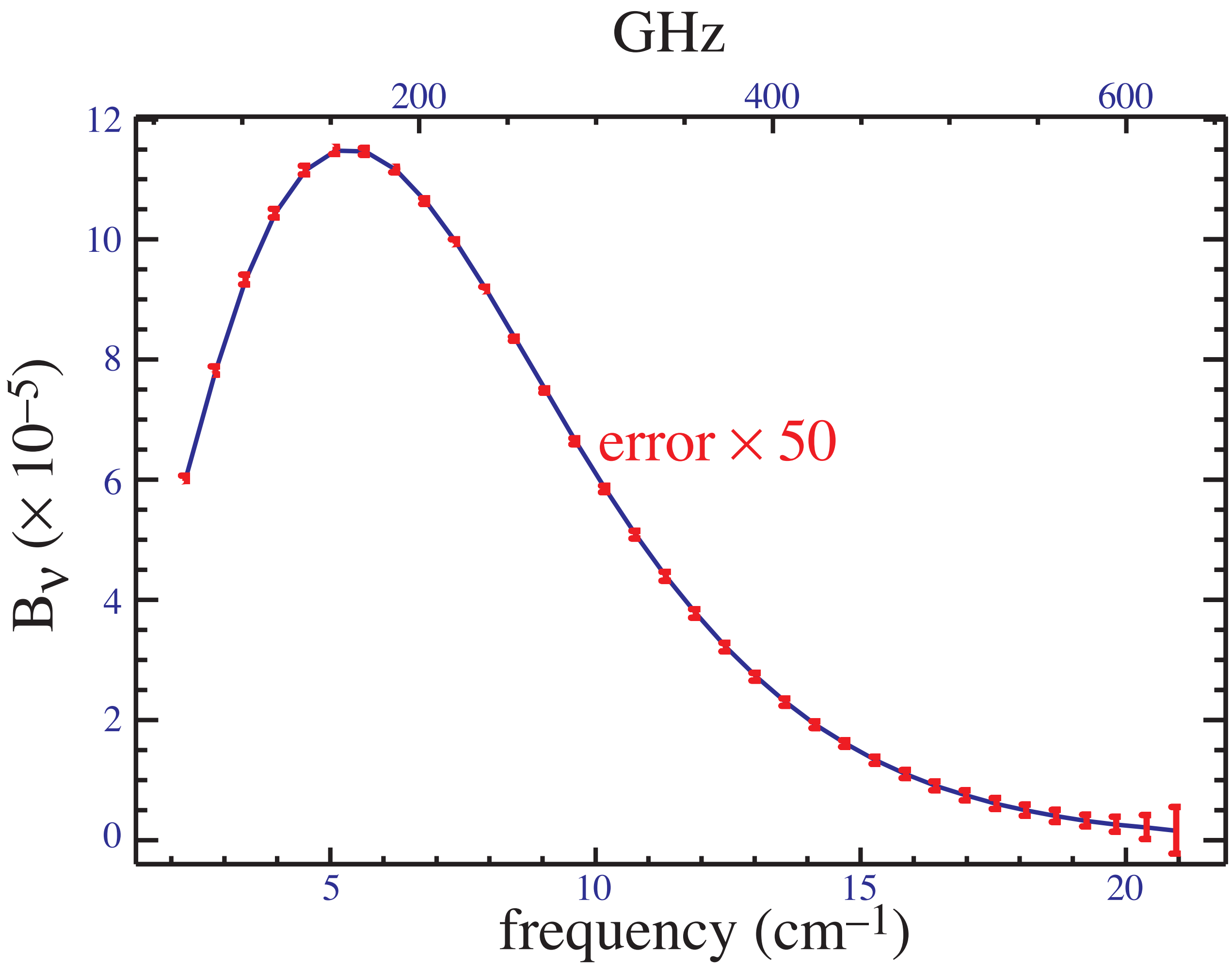


B-modes: Auto & Cross



CMB Blackbody

- COBE FIRAS revealed a blackbody spectrum at $T = 2.725$K (or cosmological density $\Omega_\gamma h^2 = 2.471 \times 10^{-5}$)



CMB Blackbody

- CMB is a (nearly) perfect blackbody characterized by a phase space distribution function

$$f = \frac{1}{e^{E/T} - 1}$$

where the temperature $T(\mathbf{x}, \hat{\mathbf{n}}, t)$ is observed at our position $\mathbf{x} = 0$ and time t_0 to be nearly isotropic with a mean temperature of $\bar{T} = 2.725\mathrm{K}$

- Our observable then is the temperature anisotropy

$$\Theta(\hat{\mathbf{n}}) \equiv \frac{T(0, \hat{\mathbf{n}}, t_0) - \bar{T}}{\bar{T}}$$

- Given that physical processes essentially put a band limit on this function it is useful to decompose it into a complete set of harmonic coefficients

Spherical Harmonics

- Laplace Eigenfunctions

$$\nabla^2 Y_\ell^m = -[l(l+1)] Y_\ell^m$$

- Orthogonal and complete

$$\int d\hat{\mathbf{n}} Y_\ell^{m*}(\hat{\mathbf{n}}) Y_{\ell'}^{m'}(\hat{\mathbf{n}}) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\sum_{\ell m} Y_\ell^{m*}(\hat{\mathbf{n}}) Y_\ell^m(\hat{\mathbf{n}}') = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta')$$

Generalizable to tensors on the sphere (polarization), modes on a curved FRW metric

- Conjugation

$$Y_\ell^{m*} = (-1)^m Y_\ell^{-m}$$

Multipole Moments

- Decompose into multipole moments

$$\Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_\ell^m(\hat{\mathbf{n}})$$

- So $\Theta_{\ell m}$ is complex but $\Theta(\hat{\mathbf{n}})$ real:

$$\begin{aligned}
\Theta^*(\hat{\mathbf{n}}) &= \sum_{\ell m} \Theta^*_{\ell m} Y_\ell^{m*}(\hat{\mathbf{n}}) \\
&= \sum_{\ell m} \Theta^*_{\ell m} (-1)^m Y_\ell^{-m}(\hat{\mathbf{n}}) \\
&= \Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_\ell^m(\hat{\mathbf{n}}) = \sum_{\ell -m} \Theta_{\ell -m} Y_\ell^{-m}(\hat{\mathbf{n}})
\end{aligned}$$

so m and $-m$ are not independent

$$\Theta^*_{\ell m} = (-1)^m \Theta_{\ell\, -m}$$

N-pt correlation

- Since the fluctuations are random and zero mean we are interested in characterizing the N-point correlation

$$\langle \Theta(\hat{\mathbf{n}}_1) \ldots \Theta(\hat{\mathbf{n}}_n) \rangle = \sum_{\ell_1 \ldots \ell_n} \sum_{m_1 \ldots m_n} \langle \Theta_{\ell_1 m_1} \ldots \Theta_{\ell_n m_n} \rangle Y_{\ell_1}^{m_1}(\hat{\mathbf{n}}_1) \ldots Y_{\ell_n}^{m_n}(\hat{\mathbf{n}}_n)$$

- Statistical isotropy implies that we should get the same result in a rotated frame

$$R[Y_\ell^m(\hat{\mathbf{n}})] = \sum_{m'} D^\ell_{m' m}(\alpha, \beta, \gamma) Y_\ell^{m'}(\hat{\mathbf{n}})$$

where α, β and γ are the Euler angles of the rotation and D is the Wigner function (note Y_ℓ^m is a D function)

$$\langle \Theta_{\ell_1 m_1} \ldots \Theta_{\ell_n m_n} \rangle = \sum_{m_1' \ldots m_n'} \langle \Theta_{\ell_1 m_1'} \ldots \Theta_{\ell_n m_n'} \rangle D^{\ell_1}_{m_1 m_1'} \ldots D^{\ell_n}_{m_n m_n'}$$

N-pt correlation

- For any N-point function, combine rotation matrices (group multiplication; angular momentum addition) and orthogonality

$$\sum_m (-1)^{m_2-m} D^{\ell_1}_{m_1 m} D^{\ell_1}_{-m_2 -m} = \delta_{m_1 m_2}$$

- The simplest case is the 2pt function:

$$\langle \Theta_{\ell_1 m_1} \Theta_{\ell_2 m_2} \rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 -m_2} (-1)^{m_1} C_{\ell_1}$$

where C_ℓ is the power spectrum. Check

$$\begin{aligned} &= \sum_{m_1' m_2'} \delta_{\ell_1 \ell_2} \delta_{m_1' -m_2'} (-1)^{m_1'} C_{\ell_1} D^{\ell_1}_{m_1 m_1'} D^{\ell_2}_{m_2 m_2'} \\ &= \delta_{\ell_1 \ell_2} C_{\ell_1} \sum_{m_1'} (-1)^{m_1'} D^{\ell_1}_{m_1 m_1'} D^{\ell_2}_{m_2 -m_1'} = \delta_{\ell_1 \ell_2} \delta_{m_1 -m_2} (-1)^{m_1} C_{\ell_1} \end{aligned}$$

N-pt correlation

- Using the reality of the field

$$\langle \Theta^*_{\ell_1 m_1} \Theta_{\ell_2 m_2} \rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} C_{\ell_1} \, .$$

- If the statistics were Gaussian then all the N-point functions would be defined in terms of the products of two-point contractions, e.g.

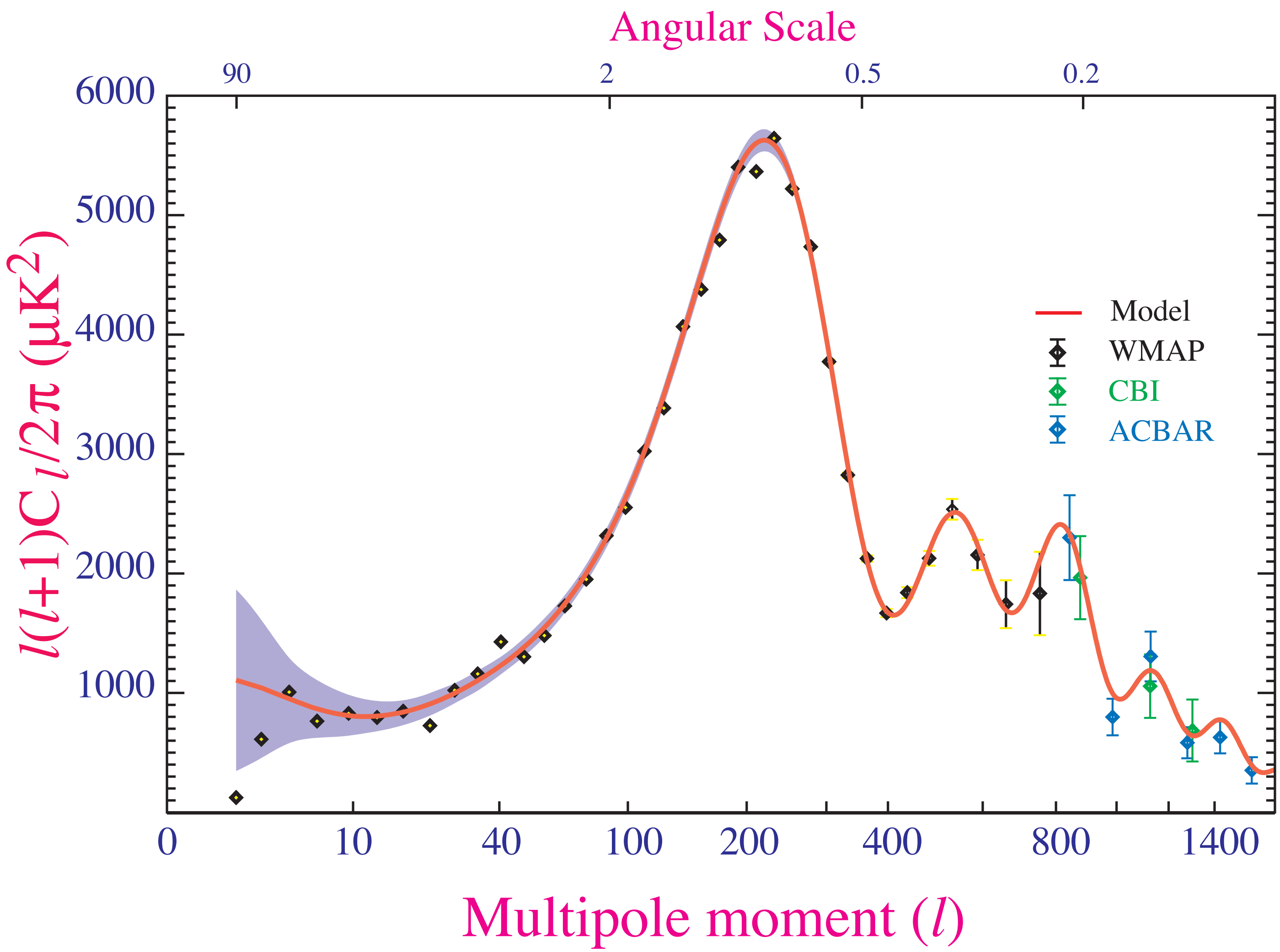
$$\langle \Theta_{\ell_1 m_1} \Theta_{\ell_2 m_2} \Theta_{\ell_3 m_3} \Theta_{\ell_4 m_4} \rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} \delta_{\ell_3 \ell_4} \delta_{m_3 m_4} C_{\ell_1} C_{\ell_3} + \text{perm.}$$

- More generally we can define the isotropy condition beyond Gaussianity, e.g. the bispectrum

$$\langle \Theta_{\ell_1 m_1} \ldots \Theta_{\ell_3 m_3} \rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1 \ell_2 \ell_3}$$

CMB Temperature Fluctuations

- Angular Power Spectrum



Why $\ell^2 C_\ell/2\pi$?

- Variance of the temperature fluctuation field

$$
\begin{aligned}
\langle \Theta(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}})\rangle &= \sum_{\ell m}\sum_{\ell' m'} \langle \Theta_{\ell m}\Theta^*_{\ell' m'}\rangle Y_\ell^m(\hat{\mathbf{n}}) Y_{\ell'}^{m'*}(\hat{\mathbf{n}}) \\
&= \sum_\ell C_\ell \sum_m Y_\ell^m(\hat{\mathbf{n}}) Y_\ell^{m*}(\hat{\mathbf{n}}) \\
&= \sum_\ell \frac{2\ell+1}{4\pi} C_\ell
\end{aligned}
$$

  via the angle addition formula for spherical harmonics

- For some range $\Delta\ell \approx \ell$ the contribution to the variance is

$$
\langle \Theta(\hat{\mathbf{n}})\Theta(\hat{\mathbf{n}})\rangle_{\ell\pm\Delta\ell/2} \approx \Delta\ell \frac{2\ell+1}{4\pi} C_\ell \approx \frac{\ell^2}{2\pi} C_\ell
$$

- Conventional to use $\ell(\ell+1)/2\pi$ for reasons below

Cosmic Variance

- We only have access to our sky, not the ensemble average
- There are $2\ell + 1$ m-modes of given ℓ mode, so average

$$\hat{C}_\ell = \frac{1}{2\ell + 1} \sum_m \Theta^*_{\ell m} \Theta_{\ell m}$$

- $\langle \hat{C}_\ell \rangle = C_\ell$ but now there is a cosmic variance

$$\sigma^2_{C_\ell} = \frac{\langle (\hat{C}_\ell - C_\ell)(\hat{C}_\ell - C_\ell) \rangle}{C_\ell^2} = \frac{\langle \hat{C}_\ell \hat{C}_\ell \rangle - C_\ell^2}{C_\ell^2}$$

- For Gaussian statistics

$$\begin{aligned} \sigma^2_{C_\ell} &= \frac{1}{(2\ell + 1)^2 C_\ell^2} \langle \sum_{mm'} \Theta^*_{\ell m} \Theta_{\ell m} \Theta^*_{\ell m'} \Theta_{\ell m'} \rangle - 1 \\ &= \frac{1}{(2\ell + 1)^2} \sum_{mm'} (\delta_{mm'} + \delta_{m-m'}) = \frac{2}{2\ell + 1} \end{aligned}$$

Cosmic Variance

- Note that the distribution of $\hat{C}_\ell$ is that of a sum of squares of Gaussian variates
- Distributed as a χ^2 of $2\ell + 1$ degrees of freedom
- Approaches a Gaussian for $2\ell + 1 \to \infty$ (central limit theorem)
- Anomalously low quadrupole is not that unlikely
- σ_{C_ℓ} is a useful quantification of errors at high ℓ
- Suppose C_ℓ depends on a set of cosmological parameters c_i then we can estimate errors of c_i measurements by error propagation

$$
\begin{aligned}
F_{ij} &= \mathrm{Cov}^{-1}(c_i, c_j) = \sum_{\ell\ell'} \frac{\partial C_\ell}{\partial c_i} \mathrm{Cov}^{-1}(\mathrm{C}_\ell, \mathrm{C}_{\ell'}) \frac{\partial C_{\ell'}}{\partial c_j} \\
&= \sum_\ell \frac{(2\ell + 1)}{2C_\ell^2} \frac{\partial C_\ell}{\partial c_i} \frac{\partial C_\ell}{\partial c_j}
\end{aligned}
$$

Idealized Statistical Errors

- Take a noisy estimator of the multipoles in the map

$$\hat{\Theta}_{\ell m} = \Theta_{\ell m} + N_{\ell m}$$

and take the noise to be statistically isotropic

$$\langle N^*_{\ell m} N_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell^{NN}$$

- Construct an unbiased estimator of the power spectrum $\langle \hat{C}_\ell \rangle = C_\ell$

$$\hat{C}_\ell = \frac{1}{2\ell + 1} \sum_{m=-l}^{l} \hat{\Theta}^*_{\ell m} \hat{\Theta}_{\ell m} - C_\ell^{NN}$$

- Covariance in estimator

$$\mathrm{Cov}(C_\ell, C_{\ell'}) = \frac{2}{2\ell + 1} (C_\ell + C_\ell^{NN})^2 \delta_{\ell\ell'}$$

Incomplete Sky

- On a small section of sky, the number of independent modes of a given ℓ is no longer $2\ell + 1$
- As in Fourier analysis, there are two limitations: the lowest ℓ mode that can be measured is the wavelength that fits in angular patch θ

$$\ell_{\rm min} = \frac{2\pi}{\theta};$$

  modes separated by $\Delta\ell < \ell_{\rm min}$ cannot be measured independently
- Estimates of C_ℓ covary on a scale imposed by $\Delta\ell < \ell_{\rm min}$
- Crude approximation: account only for the loss of independent modes by rescaling the errors rather than introducing covariance

$$\mathrm{Cov}(C_\ell, C_{\ell'}) = \frac{2}{(2\ell+1) f_{\rm sky}} (C_\ell + C_\ell^{NN})^2 \delta_{\ell\ell'}$$

Time Ordered Data

- Beyond idealizations like $|\Theta_{\ell m}|^2$ type C_ℓ estimators and $f_{\rm sky}$ mode counting, basic aspects of data analysis are useful even for theorists
- Starting point is a string of "time ordered" data coming out of the instrument (post removal of systematic errors, data cuts)
- Begin with a model of the time ordered data as (implicit summation or matrix operation)

$$\mathbf{d} = \mathbf{P}\boldsymbol{\Theta} + \mathbf{n}$$

  where the elements of the vector Θ_i denotes pixelized positions indexed by i and the element of the data d_t is a time ordered stream indexed by t.
- Noise n_t is drawn from distribution with known power spectrum

$$\langle n_t n_{t'} \rangle = C_{d,tt'}$$

Design Matrix

- The design, pointing or projection matrix $\mathbf{P}$ is the mapping between pixel space and the time ordered data
- Simplest incarnation: row with all zeros except one column which just says what point in the sky the telescope is pointing at that time

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 1 & \ldots & 0 \end{pmatrix}$$

- If each pixel were only measured once in this way then the estimator of the map would just be the inverse of $\mathbf{P}$
- More generally encorporates differencing, beam, rotation (for polarization) and unequal coverage of pixels

Maximum Likelihood Mapmaking

- What is the best estimator of the underlying map Θ_i?
- Likelihood function: the probability of getting the data given the theory $\mathcal{L}_{\rm theory}({\rm data}) \equiv P[{\rm data}|{\rm theory}]$. In this case, the *theory* is the vector of pixels $\mathbf{\Theta}$.

$$\mathcal{L}_{\Theta}(\mathbf{d}) = \frac{1}{(2\pi)^{N_t/2}\sqrt{\det \mathbf{C}_d}} \exp\left[-\frac{1}{2}\left(\mathbf{d}-\mathbf{P\Theta}\right)^t \mathbf{C}_d^{-1}\left(\mathbf{d}-\mathbf{P\Theta}\right)\right] .$$

- Bayes theorem says that $P[\mathbf{\Theta}|\mathbf{d}]$, the probability that the temperatures are equal to $\mathbf{\Theta}$ given the data, is proportional to the likelihood function times a *prior* $P(\mathbf{\Theta})$, taken to be uniform

$$P[\mathbf{\Theta}|\mathbf{d}] \propto P[\mathbf{d}|\mathbf{\Theta}] \equiv \mathcal{L}_{\mathbf{\Theta}}(\mathbf{d})$$

Maximum Likelihood Mapmaking

- Maximizing the likelihood of $\boldsymbol{\Theta}$ is simple since the log-likelihood is quadratic – it is equivalent to minimizing the variance of the estimator
- Differentiating the argument of the exponential with respect to $\boldsymbol{\Theta}$ and setting to zero leads immediately to the estimator

$$\begin{aligned}
(\mathbf{P}^{\rm t}\mathbf{C}_d^{-1}\mathbf{P})\hat{\boldsymbol{\Theta}} &= \mathbf{P}^{\rm t}\mathbf{C}_d^{-1}\mathbf{d} \\
\hat{\boldsymbol{\Theta}} &= (\mathbf{P}^{\rm t}\mathbf{C}_d^{-1}\mathbf{P})^{-1}\mathbf{P}^{\rm t}\mathbf{C}_d^{-1}\mathbf{d}\,,
\end{aligned}$$

which is unbiased

$$\langle\hat{\boldsymbol{\Theta}}\rangle = (\mathbf{P}^{\rm t}\mathbf{C}_d^{-1}\mathbf{P})^{-1}\mathbf{P}^{\rm t}\mathbf{C}_d^{-1}\mathbf{P}\boldsymbol{\Theta} = \boldsymbol{\Theta}$$

Maximum Likelihood Mapmaking

- And has the covariance

$$\begin{aligned} \mathbf{C}_N &\equiv \langle \hat{\mathbf{\Theta}} \mathbf{\Theta}^t \rangle - \hat{\mathbf{\Theta}} \mathbf{\Theta}^t \\ &= (\mathbf{P}^{\rm t} \mathbf{C}_d^{-1} \mathbf{P})^{-1} \mathbf{P}^{\rm t} \mathbf{C}_d^{-1} \langle \mathbf{d} \mathbf{d}^{\rm t} \rangle \mathbf{C}_d^{-{\rm t}} \mathbf{P} (\mathbf{P}^{\rm t} \mathbf{C}_d^{-1} \mathbf{P})^{-{\rm t}} - \hat{\mathbf{\Theta}} \mathbf{\Theta}^t \\ &= (\mathbf{P}^{\rm t} \mathbf{C}_d^{-1} \mathbf{P})^{-1} \mathbf{P}^{\rm t} \mathbf{C}_d^{-{\rm t}} \mathbf{P} (\mathbf{P}^{\rm t} \mathbf{C}_d^{-1} \mathbf{P})^{-{\rm t}} \\ &= (\mathbf{P}^{\rm t} \mathbf{C}_d^{-1} \mathbf{P})^{-1} \end{aligned}$$

  The estimator can be rewritten using the covariance matrix as a renormalization that ensures an unbiased estimator

$$\hat{\mathbf{\Theta}} = \mathbf{C}_N \mathbf{P} \mathbf{C}_d^{-1} \mathbf{d} \,,$$

- Given the large dimension of the time ordered data, direct matrix manipulation is unfeasible. A key simplifying assumption is the stationarity of the noise, that $C_{d,tt'}$ depends only on $t - t'$ (temporal statistical homogeneity)

Foregrounds

- Maximum likelihood mapmaking can be applied to the time streams of multiple observations frequencies N_ν and hence obtain multiple maps
- A cleaned CMB map can be obtained by modeling the maps as

$$\hat{\Theta}_i^\nu = A_i^\nu \Theta_i + n_i^\nu + f_i^\nu$$

where $A_i^\nu = 1$ if all the maps are at the same resolution (otherwise, embed the beam as in the pointing matrix; f_i^ν is the foreground model - e.g. a set of sky maps and a spectrum for each foreground, or more generally including a covariance matrix between frequencies due to varying spectral index

- 5 foregrounds: synchrotron, free-free, radio pt sources, at low frequencies and dust and IR pt sources at high frequencies.

Pixel Likelihood Function

- The next step in the chain of inference is to go from the map to the power spectrum
- In the most idealized form (no beam) we model

$$\Theta_i = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m}(\mathbf{n}_i)$$

and using the angle addition formula

$$\sum_m Y^*_{\ell m}(\mathbf{n}_i) Y_{\ell m}(\mathbf{n}_j) = \frac{2\ell+1}{4\pi} P_\ell(\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j)$$

with averages now including realizations of the signal

$$\langle \hat\Theta_i \hat\Theta_j \rangle \equiv C_{\Theta,ij} = C_{N,ij} + C_{S,ij}$$

Pixel Likelihood Function

- Pixel covariance matrix for the signal characterizes the sample variance of Θ_i through the power spectrum C_ℓ

$$C_{S,ij} \equiv \langle \Theta_i \Theta_j \rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell P_\ell(\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j)$$

- More generally the sky map is convolved with a beam and so the power spectrum is multiplied by the square of the beam transform
- From the pixel likelihood function we can now directly use Bayes' theorem to get the posterior probability of cosmological parameters c upon which the power spectrum depends

$$\mathcal{L}_{\mathbf{c}}(\boldsymbol{\Theta}) = \frac{1}{(2\pi)^{N_p/2}\sqrt{\det \mathbf{C}_\Theta}} \exp\left(-\frac{1}{2} \boldsymbol{\Theta}^{\rm t} \mathbf{C}_\Theta^{-1} \boldsymbol{\Theta} \right)$$

where N_p is the number of pixels in the map.

Pixel Likelihood Function

- Generalization of the Fisher matrix, curvature of the log Likelihood function

$$F_{ab} \equiv -\langle \frac{\partial^2 \ln \mathcal{L}_{\mathbf{c}}(\boldsymbol{\Theta})}{\partial c_a \partial c_b} \rangle$$

- Cramer-Rao theorem says that $\mathbf{F}^{-1}$ gives the minimum variance for an unbiased estimator of $\mathbf{c}$.
- Correctly propagates effects of pixel weights, noise - generalizes straightforwardly to polarization (E, B mixing etc)

Power Spectrum

- It is computationally convenient and sufficient at high ℓ to divide this into two steps: estimate the power spectrum $\hat{C}_\ell$ and approximate the likelihood function for $\hat{C}_\ell$ as the data and $C_\ell(c)$ as the model.
- In principle we can just use Bayes' theorem to get the maximum likelihood estimator $\hat{C}_\ell$ and the joint posterior probability distribution or covariance
- Although the pixel likelihood is Gaussian in the anisotropies Θ_i it is not in C_ℓ and so the "mapmaking" procedure above does not work

Power Spectrum

- MASTER approach is to use harmonic transforms on the map, mask and all
- Masked pixels multiply the map in real space and convolve the multipoles in harmonic space - so these pseudo-C_ℓ's are convolutions on the true C_ℓ spectrum
- Invert the convolution to form an unbiased estimator and propagate the noise and approximate the $\mathcal{L}_{C_\ell}(\hat{C}_\ell)$
- Now we can use Bayes' theorem with C_ℓ parameterized by cosmological parameters **c** to find the joint posterior distribution of **c**
- Still computationally expensive to integrate likelihood over a multidimensional cosmological parameter space

MCMC

- Monte Carlo Markov Chain (MCMC)
- Start with a set of cosmological parameters $\mathbf{c}^m$, compute likelihood
- Take a random step in parameter space to $\mathbf{c}^{m+1}$ of size drawn from a multivariate Gaussian (a guess at the parameter covariance matrix) $\mathbf{C}_c$ (e.g. from the crude Fisher approximation or the covariance of a previous short chain run). Compute likelihood.
- Draw a random number between 0,1 and if the likelihood ratio exceeds this value take the step (add to Markov chain); if not then do not take the step (add the original point to the Markov chain). Repeat.
- Given Bayes' theorem the chain is then a sampling of the joint posterior probability density of the parameters

Parameter Errors

- Can compute any statistic based on the probability distribution of parameters
- For example, compute the mean and variance of a given parameter

$$\bar{c}_i = \frac{1}{N_M} \sum_{m=1}^{N_M} c_i^m$$

$$\sigma^2(c_i) = \frac{1}{N_M - 1} \sum_{m=1}^{N_M} (c_i^m - \bar{c}_i)^2$$

- Trick is in assuring burn in (not sensitive to initial point), step size, and convergence
- Usually requires running multiple chains. Typically tens of thousands of elements per chain.

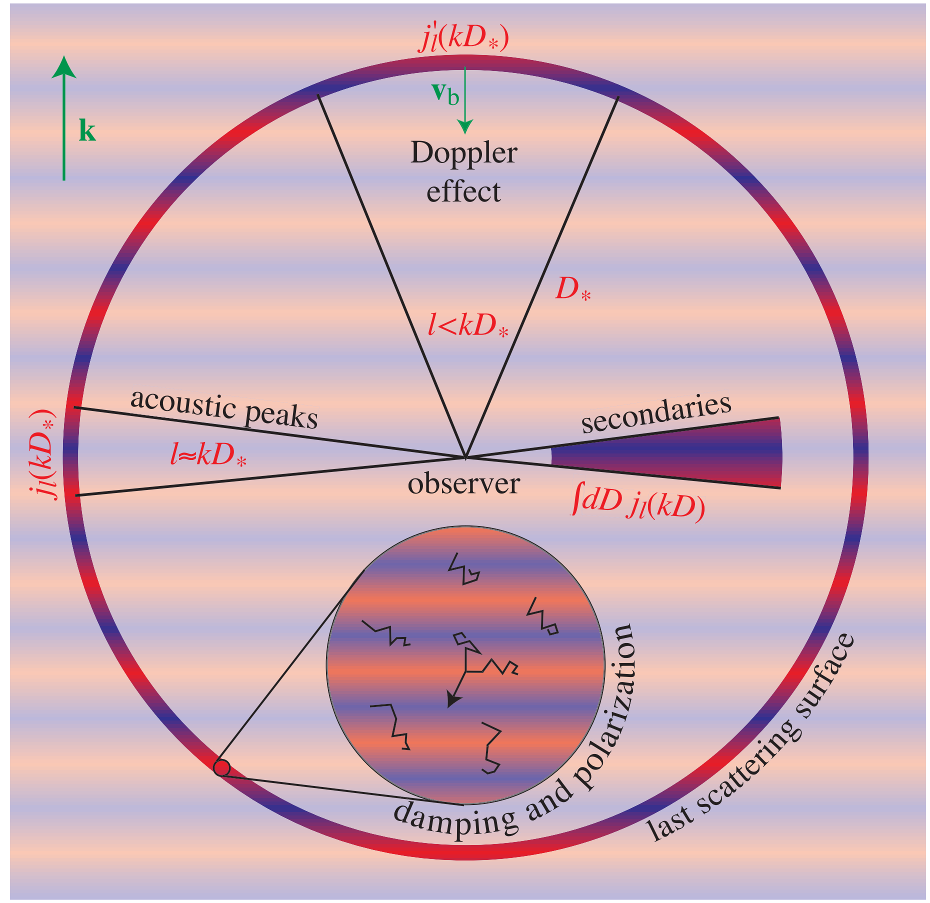
Inhomogeneity vs Anisotropy

- Θ is a function of position as well as direction but we only have access to our position
- Light travels at the speed of light so the radiation we receive in direction $\hat{\mathbf{n}}$ was $(\eta_0 - \eta)\hat{\mathbf{n}}$ at conformal time η
- Inhomogeneity at a distance appears as an anisotopy to the observer
- We need to transport the radiation from the initial conditions to the observer
- This is done with the Boltzmann or radiative transfer equation
- In the absence of scattering, emission or absorption the Boltzmann equation is simply

$$\frac{Df}{Dt} = 0$$

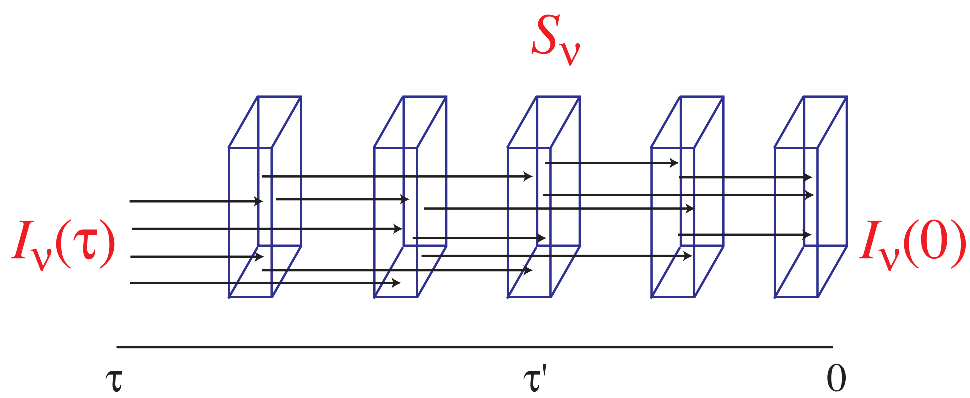
Last Scattering

- Angular distribution of radiation is the 3D temperature field projected onto a shell - surface of last scattering
- Shell radius is distance from the observer to recombination: called the last scattering surface



- Take the radiation distribution at last scattering to also be described by an isotropic temperature fluctuation field $\Theta(\mathbf{x})$

Integral Solution to Radiative Transfer



- Formal solution for specific intensity $I_\nu = 2h\nu^3 f/c^2$

$$I_\nu(0) = I_\nu(\tau)e^{-\tau} + \int_0^\tau d\tau' S_\nu(\tau')e^{-\tau'}$$

- Specific intensity I_ν attenuated by absorption and replaced by source function, attenuated by absorption from foreground matter
- Θ satisfies the same relation for a blackbody

Angular Power Spectrum

- Take recombination to be instantaneous: $d\tau e^{-\tau} = dD\delta(D - D_*)$ and the source to be the local temperature inhomogeneity

$$\Theta(\hat{\mathbf{n}}) = \int dD\, \Theta(\mathbf{x})\delta(D - D_*)$$

  where D is the comoving distance and D_* denotes recombination.

- Describe the temperature field by its Fourier moments

$$\Theta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

- Note that Fourier moments $\Theta(\mathbf{k})$ have units of volume k^{-3}
- 2 point statistics of the real-space field are translationally and rotationally invariant
- Described by power spectrum

Spatial Power Spectrum

- Translational invariance

$$\langle \Theta(\mathbf{x}')\Theta(\mathbf{x})\rangle = \langle \Theta(\mathbf{x}'+\mathbf{d})\Theta(\mathbf{x}+\mathbf{d})\rangle$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\langle \Theta^*(\mathbf{k}')\Theta(\mathbf{k})\rangle e^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{x}'}$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\langle \Theta^*(\mathbf{k}')\Theta(\mathbf{k})\rangle e^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{x}'+i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{d}}$$

So two point function requires $\delta(\mathbf{k}-\mathbf{k}')$; rotational invariance says coefficient depends only on magnitude of k not its direction

$$\langle \Theta(\mathbf{k})^*\Theta(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P_T(k)$$

Note that $\Theta(\mathbf{k})$, $\delta(\mathbf{k}-\mathbf{k}')$ have units of volume and so P_T must have units of volume

Dimensionless Power Spectrum

- Variance

$$\sigma_\Theta^2 \equiv \langle \Theta(\mathbf{x})\Theta(\mathbf{x})\rangle = \int \frac{d^3k}{(2\pi)^3} P_T(k)$$

$$= \int \frac{k^2 dk}{2\pi^2} \int \frac{d\Omega}{4\pi} P_T(k)$$

$$= \int d\ln k \frac{k^3}{2\pi^2} P_T(k)$$

- Define power per logarithmic interval

$$\Delta_T^2(k) \equiv \frac{k^3 P_T(k)}{2\pi^2}$$

- This quantity is dimensionless.

Angular Power Spectrum

- Temperature field

$$\Theta(\hat{\mathbf{n}}) = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) e^{i\mathbf{k}\cdot D_* \hat{\mathbf{n}}}$$

- Multipole moments $\Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m}$
- Expand out plane wave in spherical coordinates

$$e^{i\mathbf{k}D_*\cdot\hat{\mathbf{n}}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kD_*) Y^*_{\ell m}(\mathbf{k}) Y_{\ell m}(\hat{\mathbf{n}})$$

- Angular moment

$$\Theta_{\ell m} = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) 4\pi i^\ell j_\ell(kD_*) Y^*_{\ell m}(\mathbf{k})$$

Angular Power Spectrum

- Power spectrum

$$\langle \Theta^*_{\ell m} \Theta_{\ell' m'} \rangle = \int \frac{d^3k}{(2\pi)^3} (4\pi)^2 i^{\ell-\ell'} j_\ell(kD_*) j_{\ell'}(kD_*) Y_{\ell m}(\mathbf{k}) Y^*_{\ell' m'}(\mathbf{k}) P_T(k)$$

$$= \delta_{\ell\ell'} \delta_{mm'} 4\pi \int d\ln k \, j^2_\ell(kD_*) \Delta^2_T(k)$$

with $\int_0^\infty j^2_\ell(x) d\ln x = 1/(2\ell(\ell+1))$, slowly varying Δ^2_T

- Angular power spectrum:

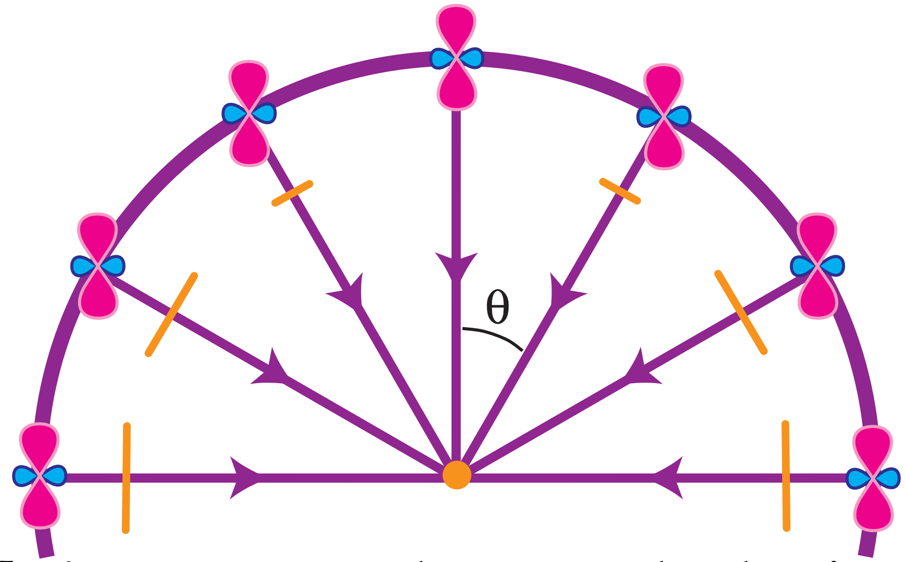
$$C_\ell = \frac{4\pi \Delta^2_T(\ell/D_*)}{2\ell(\ell+1)} = \frac{2\pi}{\ell(\ell+1)} \Delta^2_T(\ell/D_*)$$

- Not surprisingly, a relationship between $\ell^2 C_\ell / 2\pi$ and Δ^2_T at $\ell \gg 1$. By convention use $\ell(\ell+1)$ to make relationship exact

- This is a property of a thin-shell isotropic source, now generalize.

Generalized Source

- For example,
  if the emission surface
  is moving with respect
  to the observer then
  radiation has an intrinsic
  dipole pattern at emission



- More generally, we know the Y_ℓ^m's are a complete angular basis and plane waves are complete spatial basis
- Local source distribution decomposed into plane-wave modulated multipole moments

$$S_\ell^{(m)}(-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k}\cdot\mathbf{x})$$

where prefactor is for convenience when fixing $\hat{\mathbf{z}} = \hat{\mathbf{k}}$

Generalized Source

- So general solution is for a single source shell is

$$\Theta(\hat{\mathbf{n}}) = \sum_{\ell m} S_\ell^{(m)} (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot D_* \hat{\mathbf{n}})$$

and for a source that is a function of distance

$$\Theta(\hat{\mathbf{n}}) = \int dD e^{-\tau} \sum_{\ell m} S_\ell^{(m)}(D) (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot D\hat{\mathbf{n}})$$

- Note that unlike the isotropic source, we have two pieces that depend on $\hat{\mathbf{n}}$
- Observer sees the total angular structure

$$Y_\ell^m(\hat{\mathbf{n}}) e^{i\mathbf{k}D_* \cdot \hat{\mathbf{n}}} = 4\pi \sum_{\ell' m'} i^{\ell'} j_{\ell'}(kD_*) Y_{\ell'}^{m' *}(\mathbf{k}) Y_{\ell'}^{m'}(\hat{\mathbf{n}}) Y_\ell^m(\hat{\mathbf{n}})$$

Generalized Source

- We extract the observed multipoles by the addition of angular momentum $Y_{\ell'}^{m'}(\hat{\mathbf{n}})Y_{\ell}^{m}(\hat{\mathbf{n}}) \to Y_L^M(\hat{\mathbf{n}})$
- Radial functions become linear sums over j_ℓ with the recoupling (Clebsch-Gordan) coefficients
- These radial weight functions carry important information about how spatial fluctuations project onto angular fluctuations - or the sharpness of the angular transfer functions
- Same is true of polarization - source is Thomson scattering
- Polarization has an intrinsic quadrupolar distribution, recoupled by orbital angular momentum into fine scale polarization anisotropy
- Formal integral solution to the Boltzmann or radiative transfer equation
- Source functions also follow from the Boltzmann equation

Polarization Basis

- Define the angularly dependent Stokes perturbation

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta), \quad Q(\mathbf{x}, \hat{\mathbf{n}}, \eta), \quad U(\mathbf{x}, \hat{\mathbf{n}}, \eta)$$

- Decompose into normal modes: plane waves for spatial part and spherical harmonics for angular part

$$G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k}\cdot\mathbf{x})$$

$${}_{\pm 2}G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} {}_{\pm 2}Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k}\cdot\mathbf{x})$$

- In a spatially curved universe generalize the plane wave part
- For a single $\mathbf{k}$ mode, choose a coordinate system $\hat{\mathbf{z}} = \hat{\mathbf{k}}$

Normal Modes

- Temperature and polarization fields

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} \Theta_\ell^{(m)} G_\ell^m$$

$$[Q \pm iU](\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} [E_\ell^{(m)} \pm iB_\ell^{(m)}]_{\pm 2} G_\ell^m$$

- For each $\mathbf{k}$ mode, work in coordinates where $\mathbf{k} \parallel \mathbf{z}$ and so $m = 0$ represents scalar modes, $m = \pm 1$ vector modes, $m = \pm 2$ tensor modes, $|m| > 2$ vanishes. Since modes add incoherently and $Q \pm iU$ is invariant up to a phase, rotation back to a fixed coordinate system is trivial.

Liouville Equation

- In absence of scattering, the phase space distribution of photons in each polarization state a is conserved along the propagation path
- Rewrite variables in terms of the photon propagation direction $\mathbf{q} = q\hat{\mathbf{n}}$, so $f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta)$ and

$$\frac{D}{D\eta} f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta) = 0 = \left(\frac{\partial}{\partial \eta} + \frac{d\mathbf{x}}{d\eta} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{d\hat{\mathbf{n}}}{d\eta} \cdot \frac{\partial}{\partial \hat{\mathbf{n}}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a$$

- For simplicity, assume spatially flat universe $K = 0$ then $d\hat{\mathbf{n}}/d\eta = 0$ and $d\mathbf{x} = \hat{\mathbf{n}} d\eta$

$$\dot{f}_a + \hat{\mathbf{n}} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0$$

- The spatial gradient describes the conversion from inhomogeneity to anisotropy and the $\dot{q}$ term the gravitational sources.

Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies
- Spatial gradient term hits plane wave:

$$\hat{\mathbf{n}} \cdot \nabla e^{i\mathbf{k}\cdot\mathbf{x}} = i\hat{\mathbf{n}} \cdot \mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} = i\sqrt{\frac{4\pi}{3}} k Y_1^0(\hat{\mathbf{n}}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

- Dipole term adds to angular dependence through the addition of angular momentum

$$\sqrt{\frac{4\pi}{3}} Y_1^0 Y_\ell^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell+1)(2\ell-1)}} Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell+1)(2\ell+3)}} Y_{\ell+1}^m$$

where $\kappa_\ell^m = \sqrt{\ell^2 - m^2}$ is given by Clebsch-Gordon coefficients.

Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

$$\dot{\Theta}_\ell^{(m)} = k \left[\frac{\kappa_\ell^m}{2\ell+1} \Theta_{\ell-1}^{(m)} - \frac{\kappa_{\ell+1}^m}{2\ell+3} \Theta_{\ell+1}^{(m)} \right] - \dot{\tau} \Theta_\ell^{(m)} + S_\ell^{(m)}$$

  where $S_\ell^{(m)}$ are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic $\ell = 0$ temperature perturbation will eventually become a high order anisotropy by "free streaming" or simple projection

- Original CMB codes solved the full hierarchy equations out to the ℓ of interest.

Integral Solution

- Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface
- In general, the solution describes the decomposition of the source $S_\ell^{(m)}$ with its local angular dependence as seen at a distance D.
- Proceed by decomposing the angular dependence of the plane wave

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} j_\ell(kD) Y_\ell^0(\hat{\mathbf{n}})$$

- Recouple to the local angular dependence of G_ℓ^m

$$G_{\ell_s}^m = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} \alpha_{\ell_s \ell}^{(m)}(kD) Y_\ell^m(\hat{\mathbf{n}})$$

Integral Solution

- Projection kernels:

$$\alpha^{(m=0)}_{\ell_s=0\ell} \equiv j_\ell \qquad \alpha^{(m=0)}_{\ell_s=1\ell} \equiv j'_\ell$$

- Integral solution:

$$\frac{\Theta_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S^{(m)}_{\ell_s} \alpha^{(m)}_{\ell_s \ell}(k(\eta_0-\eta))$$

- Power spectrum:

$$C_\ell = 4\pi \int \frac{dk}{k} \frac{k^3}{2\pi^2} \sum_m \frac{\langle \Theta_\ell^{(m)*} \Theta_\ell^{(m)} \rangle}{(2\ell+1)^2}$$

- Integration over an oscillatory radial source with finite width - suppression of wavelengths that are shorter than width leads to reduction in power by $k\Delta\eta/\ell$ in the "Limber approximation"

Polarization Hierarchy

- In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hierarchy:

$$\dot{E}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell+1)}B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}E_{\ell+1}^{(m)}\right] - \dot{\tau}E_\ell^{(m)} + \mathcal{E}_\ell^{(m)}$$

$$\dot{B}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell+1)}E_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}B_{\ell+1}^{(m)}\right] - \dot{\tau}B_\ell^{(m)} + \mathcal{B}_\ell^{(m)}$$

  where ${}_2\kappa_\ell^m = \sqrt{(\ell^2 - m^2)(\ell^2-4)/\ell^2}$ is given by the Clebsch-Gordon coefficients and $\mathcal{E}$, $\mathcal{B}$ are the sources (scattering only).

- Note that for vectors and tensors $|m| > 0$ and B modes may be generated from E modes by projection. Cosmologically $\mathcal{B}_\ell^{(m)} = 0$

Polarization Integral Solution

- Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

$$\frac{E_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \epsilon_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

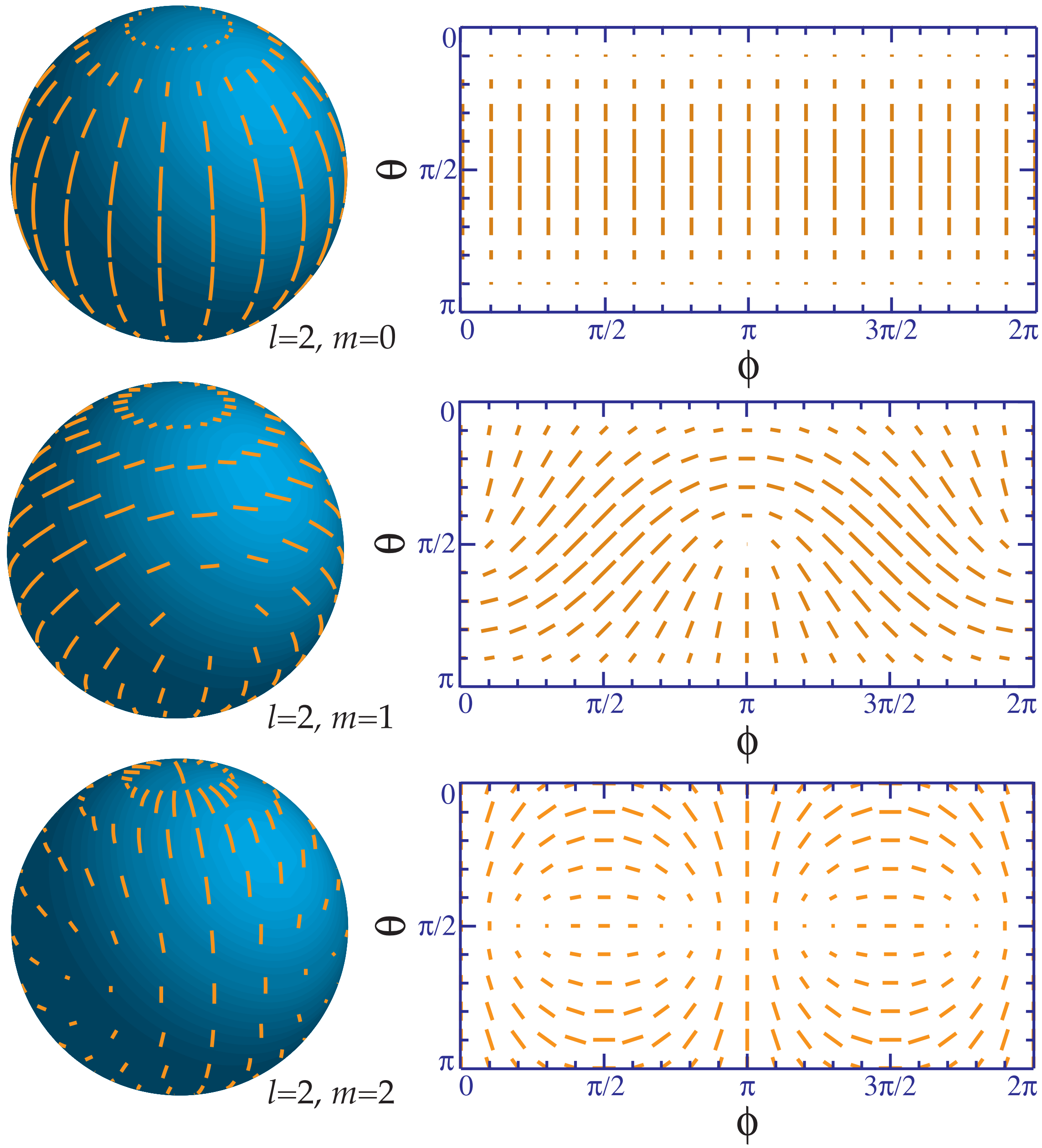
$$\frac{B_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \beta_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

- Power spectrum $XY = \Theta\Theta, \Theta E, EE, BB$:

$$C_\ell^{XY} = 4\pi \int \frac{dk}{k} \frac{k^3}{2\pi^2} \sum_m \frac{\langle X_\ell^{(m)*} Y_\ell^{(m)} \rangle}{(2\ell+1)^2}$$

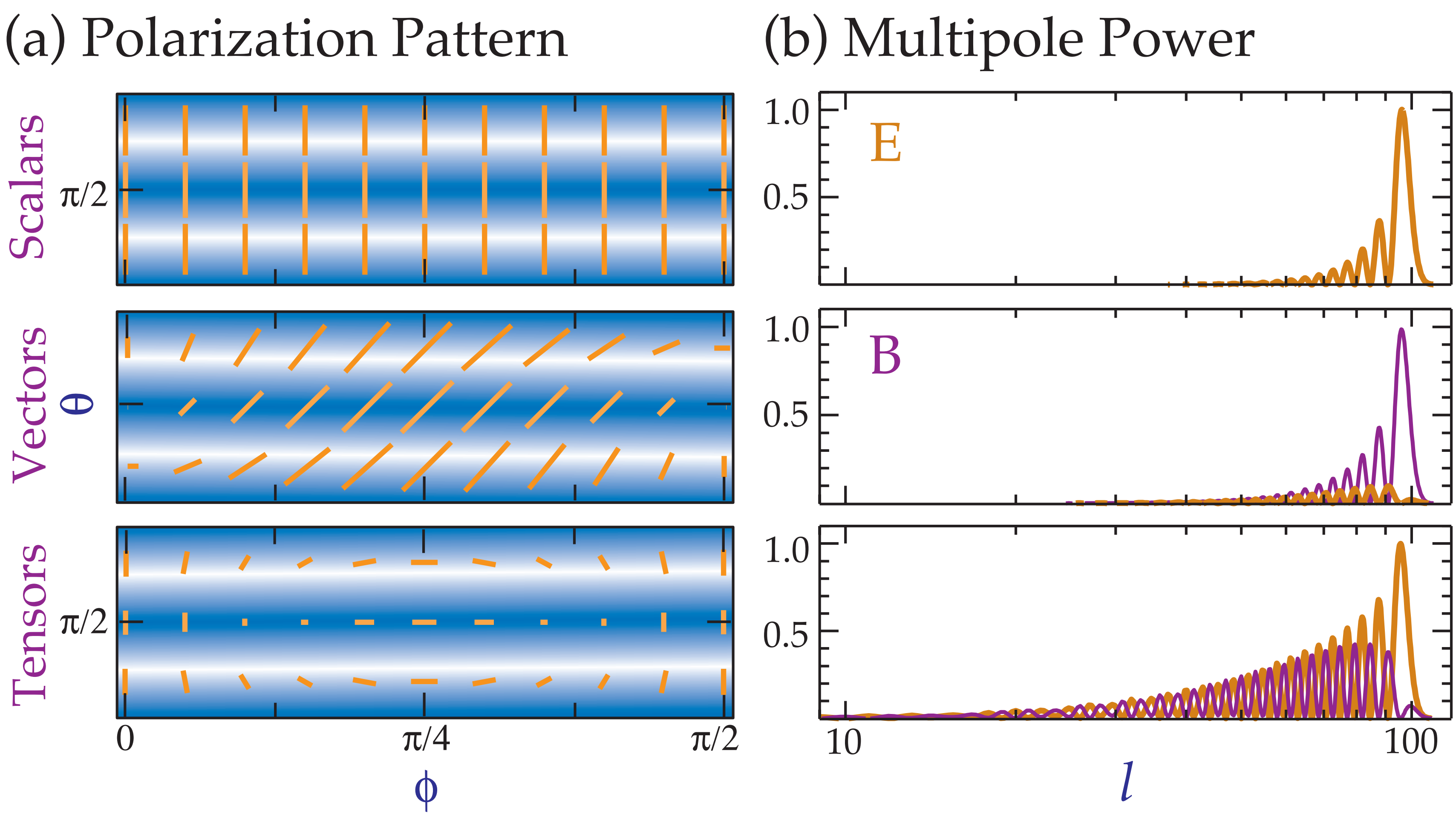
- We shall see that the only sources of temperature anisotropy are $\ell = 0, 1, 2$ and polarization anisotropy $\ell = 2$
- In the basis of $\hat{\mathbf{z}} = \hat{\mathbf{k}}$ there are only $m = 0, \pm 1, \pm 2$ or scalar, vector and tensor components

Polarization Sources



Polarization Transfer

- A polarization source function with $\ell = 2$, modulated with plane wave orbital angular momentum
- Scalars have no B mode contribution, vectors mostly B and tensor comparable B and E



Polarization Transfer

- Radial mode functions characterize the projection from $k \to \ell$ or inhomogeneity to anisotropy
- Compared to the scalar T monopole source:

  scalar T dipole source very broad

  tensor T quadrupole, sharper

  scalar E polarization, sharper

  tensor E polarization, broad

  tensor B polarization, very broad
- These properties determine whether features in the k-mode spectrum, e.g. acoustic oscillations, intrinsic structure, survive in the anisotropy