Ast 448

Set 2: Beyond the Primary Temperature Anisotropy

(Polarization, Inflation, Lensing, Boltzmann/Radiation-Transfer Eqn, Spectral Distortions...)

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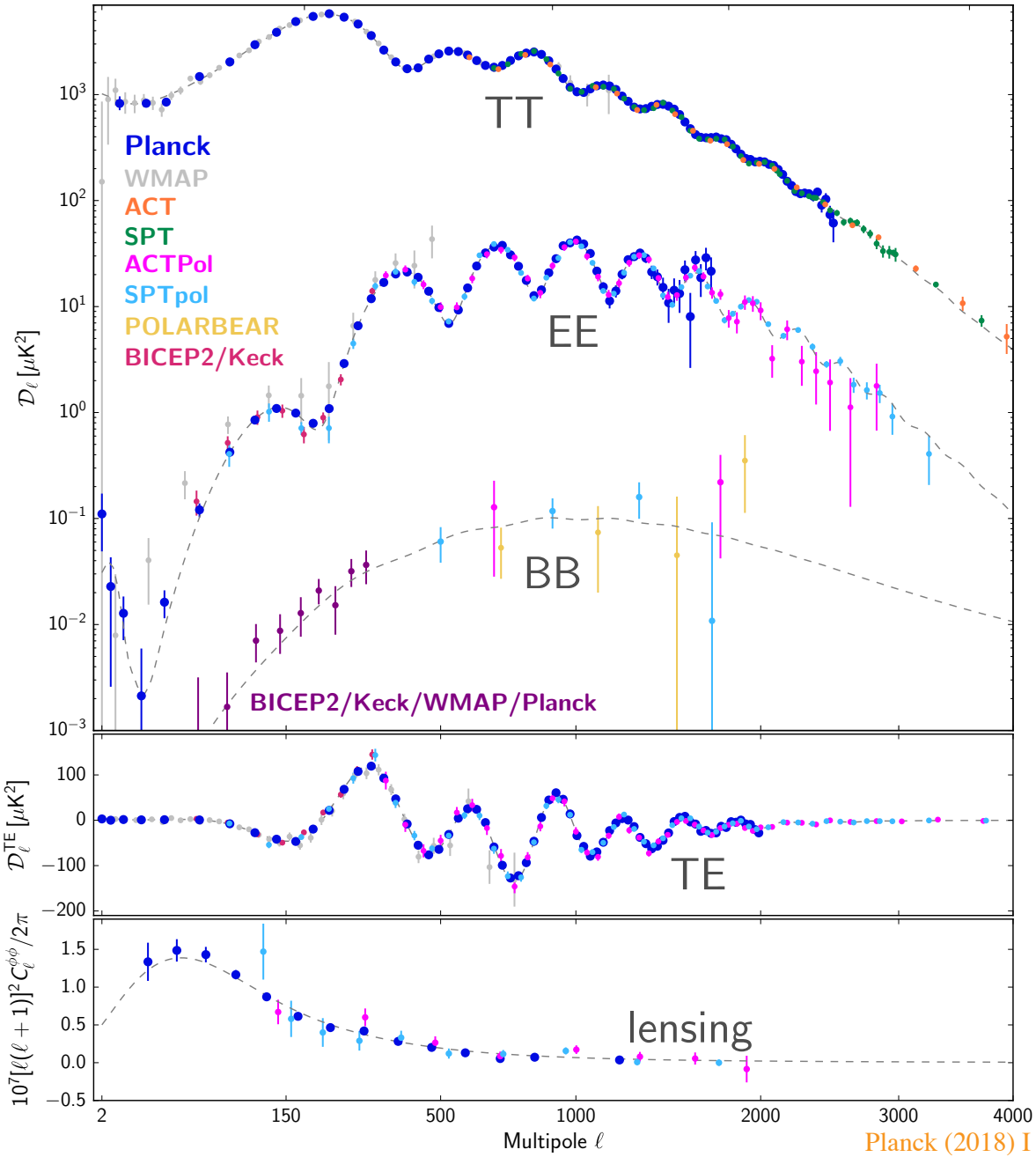
CMB Power Spectra

- Power spectra of CMB
  - temperature
  - polarization
  - lensing



Planck (2018) I

Stokes Parameters: A Brief Review

- Specific intensity is related to quadratic combinations of the electric field.
- Define intensity matrix (time averaged over oscillations) $\langle \mathbf{E}\,\mathbf{E}^\dagger \rangle$
- Hermitian matrix can be decomposed into Pauli matrices

$$\mathbf{P} = \langle \mathbf{E}\,\mathbf{E}^\dagger \rangle = \frac{1}{2}\left(I\boldsymbol{\sigma}_0 + Q\,\boldsymbol{\sigma}_3 + U\,\boldsymbol{\sigma}_1 - V\,\boldsymbol{\sigma}_2\right) ,$$

where

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Stokes parameters recovered as $\mathrm{Tr}(\sigma_i \mathbf{P})$
- CMB specific: blackbody so units of brightness temperature for Stokes parameters $I \to \Theta$

Stokes Parameters

- Consider a general plane wave solution

$$\mathbf{E}(t,z) = E_1(t,z)\hat{\mathbf{e}}_1 + E_2(t,z)\hat{\mathbf{e}}_2$$
$$E_1(t,z) = A_1 e^{i\phi_1} e^{i(kz-\omega t)}$$
$$E_2(t,z) = A_2 e^{i\phi_2} e^{i(kz-\omega t)}$$

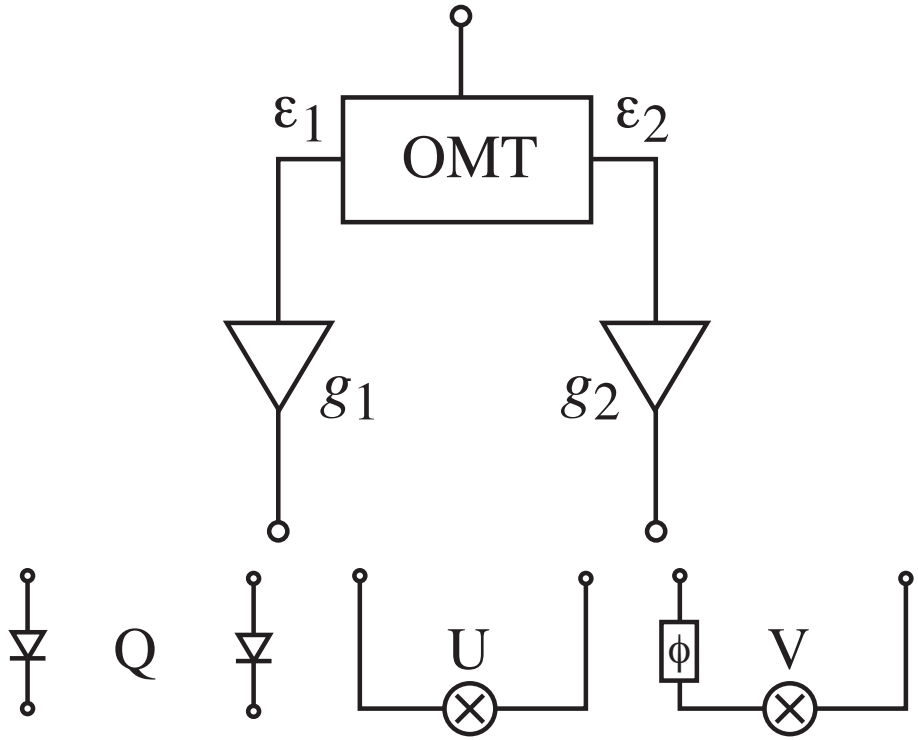
- Explicitly:

$$I = \langle E_1 E_1^* + E_2 E_2^* \rangle = A_1^2 + A_2^2$$
$$Q = \langle E_1 E_1^* - E_2 E_2^* \rangle = A_1^2 - A_2^2$$
$$U = \langle E_1 E_2^* + E_2 E_1^* \rangle = 2A_1 A_2 \cos(\phi_2 - \phi_1)$$
$$V = -i \langle E_1 E_2^* - E_2 E_1^* \rangle = 2A_1 A_2 \sin(\phi_2 - \phi_1)$$

so that the Stokes parameters define the state up to an unobservable overall phase of the wave

Detection

- This suggests that abstractly there are two different ways to detect polarization: separate and difference orthogonal modes (bolometers I, Q) or correlate the separated components (U, V).



- In the correlator example the natural output would be U but one can recover V by introducing a phase lag $\phi = \pi/2$ on one arm, and Q by having the OMT pick out directions rotated by $\pi/4$.
- Likewise, in the bolometer example, one can rotate the polarizer and also introduce a coherent front end to change V to U.

Detection

- Techniques also differ in the systematics that can convert unpolarized sky to fake polarization
- Differencing detectors are sensitive to relative gain fluctuations
- Correlation detectors are sensitive to cross coupling between the arms
- More generally, the intended block diagram and systematic problems map components of the polarization matrix onto others and are kept track of through "Jones" or instrumental response matrices $\mathbf{E}_{\rm det} = \mathbf{J}\mathbf{E}_{\rm in}$

$$\mathbf{P}_{\rm det} = \mathbf{J}\mathbf{P}_{\rm in}\mathbf{J}^{\dagger}$$

where the end result is either a differencing or a correlation of the $\mathbf{P}_{\rm det}$.

Polarization

- Radiation field involves a directed quantity, the electric field vector, which defines the polarization
- Consider a general plane wave solution

$$\mathbf{E}(t,z) = E_1(t,z)\hat{\mathbf{e}}_1 + E_2(t,z)\hat{\mathbf{e}}_2$$
$$E_1(t,z) = \mathrm{Re}A_1 e^{i\phi_1} e^{i(kz-\omega t)}$$
$$E_2(t,z) = \mathrm{Re}A_2 e^{i\phi_2} e^{i(kz-\omega t)}$$

or at $z = 0$ the field vector traces out an ellipse

$$\mathbf{E}(t,0) = A_1\cos(\omega t - \phi_1)\hat{\mathbf{e}}_1 + A_2\cos(\omega t - \phi_2)\hat{\mathbf{e}}_2$$

with principal axes defined by

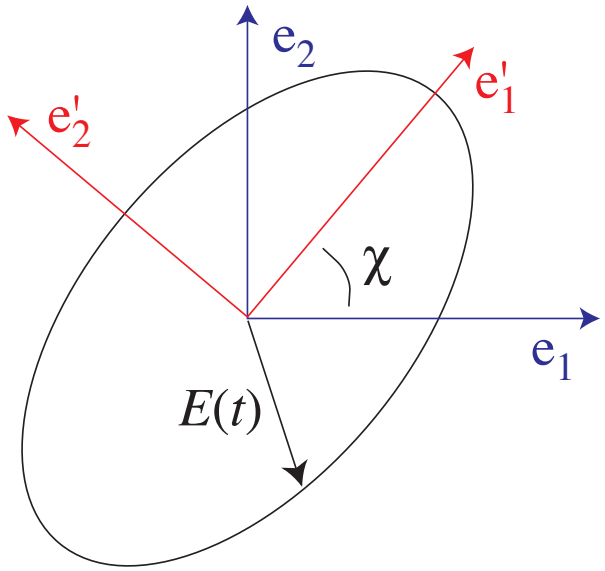
$$\mathbf{E}(t,0) = A_1'\cos(\omega t)\hat{\mathbf{e}}_1' - A_2'\sin(\omega t)\hat{\mathbf{e}}_2'$$

so as to trace out a clockwise rotation for $A_1', A_2' > 0$

Polarization

- Define polarization angle

$$\hat{\mathbf{e}}'_1 = \cos\chi\hat{\mathbf{e}}_1 + \sin\chi\hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}'_2 = -\sin\chi\hat{\mathbf{e}}_1 + \cos\chi\hat{\mathbf{e}}_2$$



- Match

$$\begin{aligned}\mathbf{E}(t,0) &= A'_1\cos\omega t[\cos\chi\hat{\mathbf{e}}_1 + \sin\chi\hat{\mathbf{e}}_2] \\ &\quad - A'_2\cos\omega t[-\sin\chi\hat{\mathbf{e}}_1 + \cos\chi\hat{\mathbf{e}}_2] \\ &= A_1[\cos\phi_1\cos\omega t + \sin\phi_1\sin\omega t]\hat{\mathbf{e}}_1 \\ &\quad + A_2[\cos\phi_2\cos\omega t + \sin\phi_2\sin\omega t]\hat{\mathbf{e}}_2\end{aligned}$$

Polarization

- Define relative strength of two principal states

$$A_1' = E_0 \cos\beta \quad A_2' = E_0 \sin\beta$$

- Characterize the polarization by two angles

$$A_1 \cos\phi_1 = E_0 \cos\beta \cos\chi, \qquad A_1 \sin\phi_1 = E_0 \sin\beta \sin\chi,$$
$$A_2 \cos\phi_2 = E_0 \cos\beta \sin\chi, \qquad A_2 \sin\phi_2 = -E_0 \sin\beta \cos\chi$$

Or Stokes parameters by

$$I = E_0^2\,, \quad Q = E_0^2 \cos 2\beta \cos 2\chi$$
$$U = E_0^2 \cos 2\beta \sin 2\chi\,, \quad V = E_0^2 \sin 2\beta$$

- So $I^2 = Q^2 + U^2 + V^2$, double angles reflect the spin 2 field or headless vector nature of polarization

Polarization

Special cases

- If $\beta = 0, \pi/2, \pi$ then only one principal axis, ellipse collapses to a line and $V = 0 \rightarrow$ linear polarization oriented at angle χ

  If $\chi = 0, \pi/2, \pi$ then $I = \pm Q$ and $U = 0$

  If $\chi = \pi/4, 3\pi/4 ...$ then $I = \pm U$ and $Q = 0$ – so U is Q in a frame rotated by 45 degrees

- If $\beta = \pi/4, 3\pi/4$, then principal components have equal strength and E field rotates on a circle: $I = \pm V$ and $Q = U = 0 \rightarrow$ circular polarization

- $U/Q = \tan 2\chi$ defines angle of linear polarization and $V/I = \sin 2\beta$ defines degree of circular polarization

Natural Light

- A monochromatic plane wave is completely polarized
  $I^2 = Q^2 + U^2 + V^2$
- Polarization matrix is like a density matrix in quantum mechanics and allows for pure (coherent) states and mixed states
- Suppose the total $\mathbf{E}_{\text{tot}}$ field is composed of different (frequency) components

$$\mathbf{E}_{\text{tot}} = \sum_i \mathbf{E}_i$$

- Then components decorrelate in time average

$$\left\langle \mathbf{E}_{\text{tot}} \mathbf{E}_{\text{tot}}^\dagger \right\rangle = \sum_{ij} \left\langle \mathbf{E}_i \mathbf{E}_j^\dagger \right\rangle = \sum_i \left\langle \mathbf{E}_i \mathbf{E}_i^\dagger \right\rangle$$

Natural Light

- So Stokes parameters of incoherent contributions add

$$I = \sum_i I_i \quad Q = \sum_i Q_i \quad U = \sum_i U_i \quad V = \sum_i V_i$$

and since individual Q, U and V can have either sign:
$I^2 \geq Q^2 + U^2 + V^2$, all 4 Stokes parameters needed

Linear Polarization

- $Q \propto \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle$, $U \propto \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle$.
- Counterclockwise rotation of axes by $\theta = 45^\circ$

$$E_1 = (E_1' - E_2')/\sqrt{2}\,, \quad E_2 = (E_1' + E_2')/\sqrt{2}$$

- $U \propto \langle E_1' E_1'^* \rangle - \langle E_2' E_2'^* \rangle$, difference of intensities at 45° or Q'
- More generally, $\mathbf{P}$ transforms as a tensor under rotations and

$$Q' = \cos(2\theta) Q + \sin(2\theta) U$$
$$U' = -\sin(2\theta) Q + \cos(2\theta) U$$

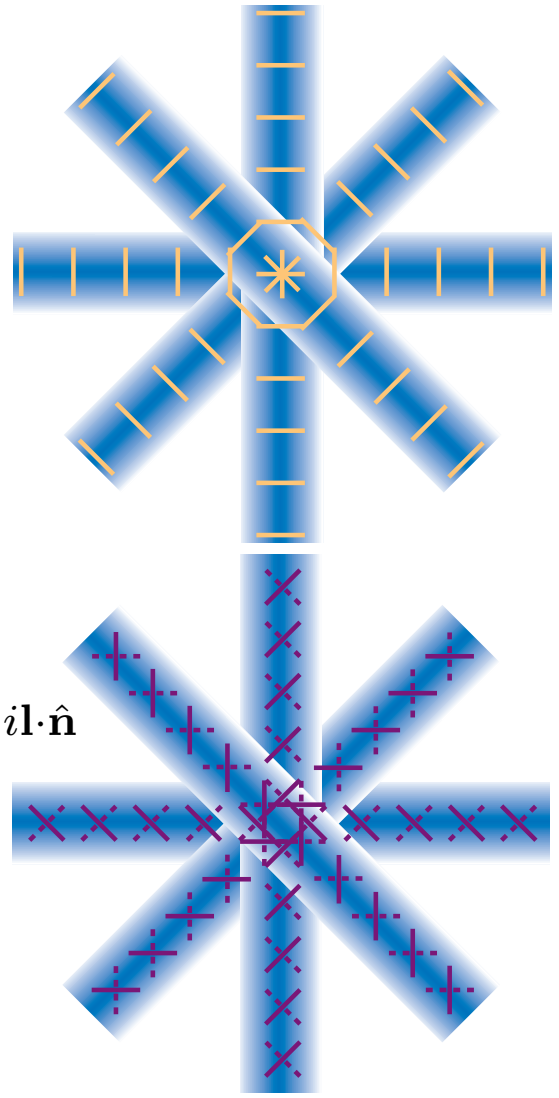
or

$$Q' \pm iU' = e^{\mp 2i\theta}[Q \pm iU]$$

acquires a phase under rotation and is a spin ± 2 object

Coordinate Independent Representation

- Two directions: orientation of polarization and change in amplitude, i.e. Q and U in the basis of the Fourier wavevector (pointing with angle ϕ_l) for small sections of sky are called E and B components



$$E(\mathbf{l}) \pm iB(\mathbf{l}) = -\int d\hat{\mathbf{n}}[Q'(\hat{\mathbf{n}}) \pm iU'(\hat{\mathbf{n}})]e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}}$$

$$= -e^{\mp 2i\phi_l}\int d\hat{\mathbf{n}}[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})]e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}}$$

- For the B-mode to not vanish, the polarization must point in a direction not related to the wavevector - not possible for density fluctuations in linear theory
- Generalize to all-sky: eigenmodes of Laplace operator of tensor

Spin Harmonics

- Laplace Eigenfunctions

$$\nabla^2 {}_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1] = -[l(l+1) - 4]_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1]$$

- Spin s spherical harmonics: orthogonal and complete

$$\int d\hat{\mathbf{n}}\, {}_sY^*_{\ell m}(\hat{\mathbf{n}})\, {}_sY_{\ell' m'}(\hat{\mathbf{n}}) = \delta_{\ell\ell'}\delta_{mm'}$$

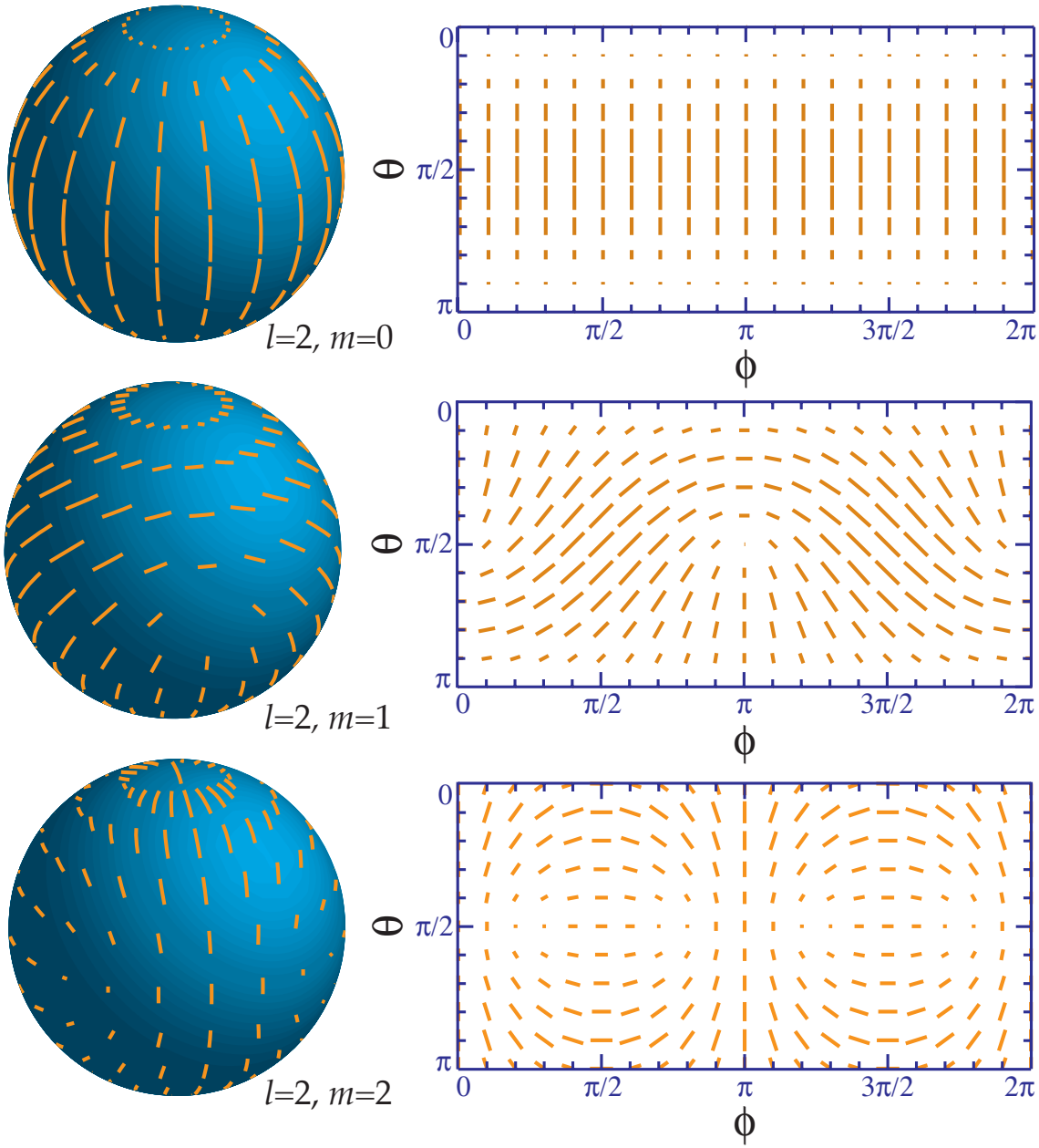
$$\sum_{\ell m} {}_sY^*_{\ell m}(\hat{\mathbf{n}})\, {}_sY_{\ell m}(\hat{\mathbf{n}}') = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta')$$

  where the ordinary spherical harmonics are $Y_{\ell m} = {}_0Y_{\ell m}$

- Given in terms of the rotation matrix

$${}_sY_{\ell m}(\beta\alpha) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell}_{-ms}(\alpha\beta 0)$$

Polarization Multipoles



Statistical Representation

- All-sky decomposition

$$[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})] = \sum_{\ell m} [E_{\ell m} \pm iB_{\ell m}]_{\pm 2} Y_{\ell m}(\hat{\mathbf{n}})$$

- Power spectra

$$\langle E^*_{\ell m} E_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{EE}$$
$$\langle B^*_{\ell m} B_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{BB}$$

- Cross correlation

$$\langle \Theta^*_{\ell m} E_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{\Theta E}$$

others vanish if parity is conserved

Thomson Scattering

- Polarization state of radiation in direction $\hat{\mathbf{n}}$ described by the intensity matrix $\left\langle E_i(\hat{\mathbf{n}}) E_j^*(\hat{\mathbf{n}}) \right\rangle$, where $\mathbf{E}$ is the electric field vector and the brackets denote time averaging.
- Differential cross section

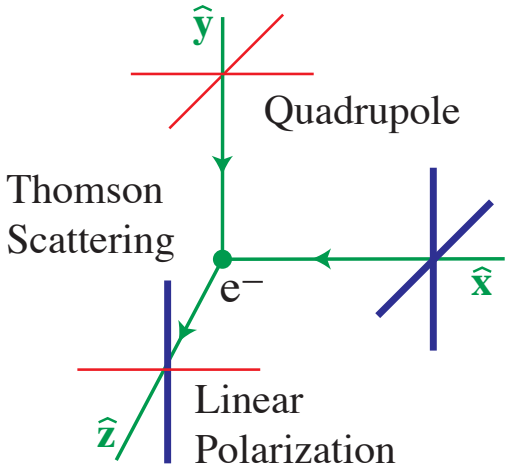
$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \,,$$

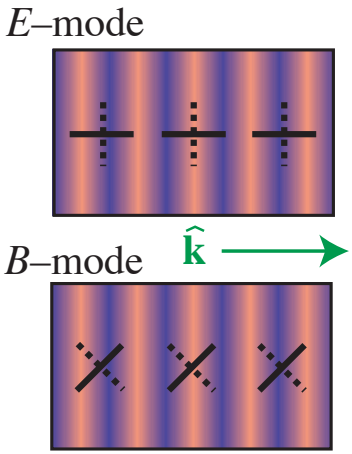
  where $\sigma_T = 8\pi\alpha^2/3m_e$ is the Thomson cross section, $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.
- Summed over angle and incoming polarization

$$\sum_{i=1,2} \int d\hat{\mathbf{n}}' \frac{d\sigma}{d\Omega} = \sigma_T$$

Polarization Generation

- Heuristic:
  incoming radiation shakes
  an electron in direction
  of electric field vector $\hat{\mathbf{E}}'$



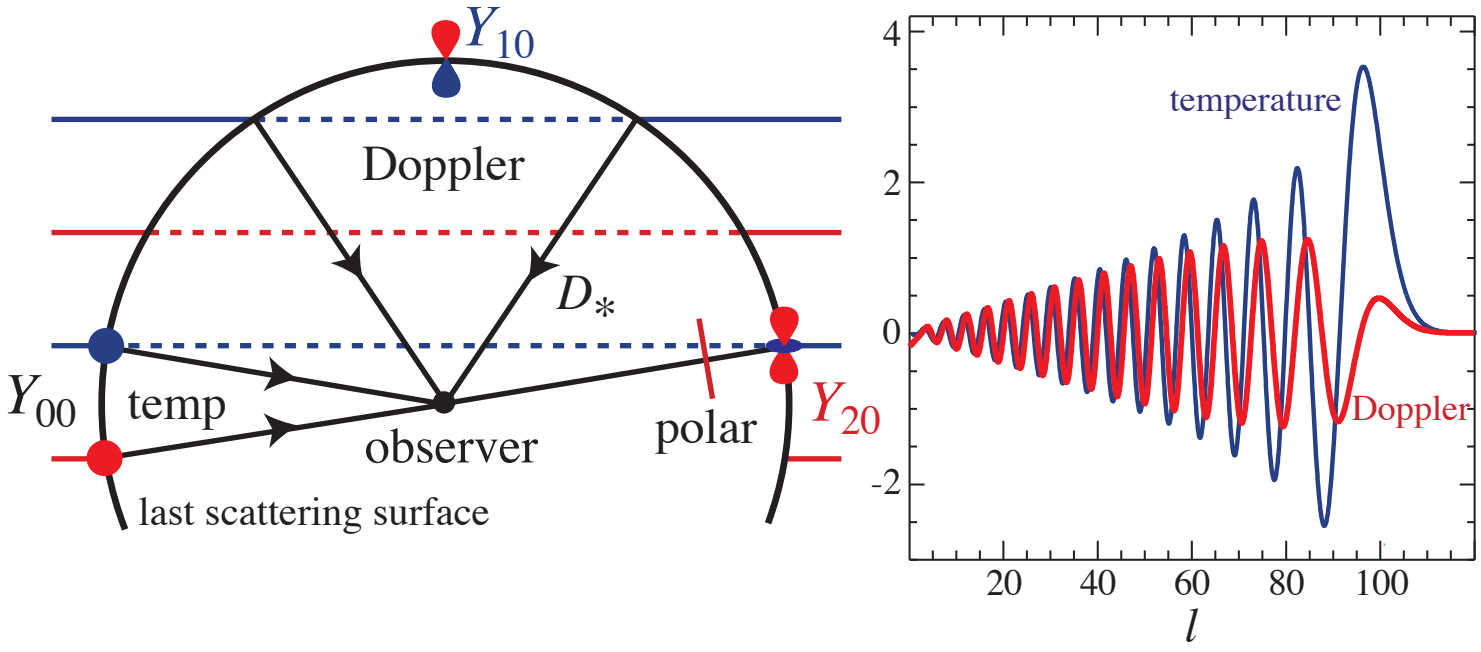


- Radiates photon with
  polarization also in direction $\hat{\mathbf{E}}'$
- But photon cannot be longitudinally polarized so that scattering into $90°$ can only pass one polarization
- Linearly polarized radiation like polarization by reflection
- Unlike reflection of sunlight, incoming radiation is nearly isotropic
- Missing from direction orthogonal to original incoming direction
- Only quadrupole anisotropy generates polarization by Thomson scattering

Polarized Radiative Transfer

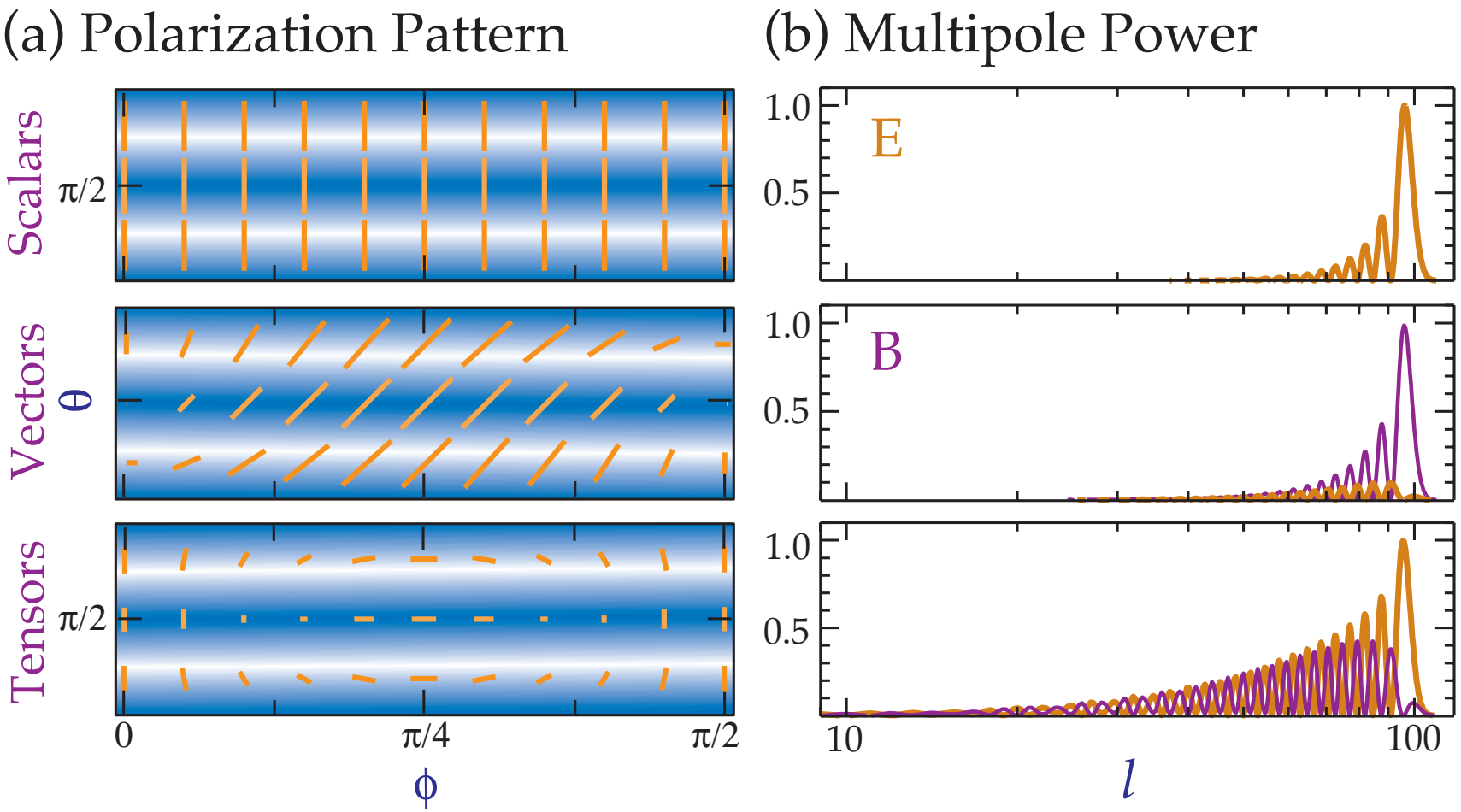
- Source of linear polarization is the radiation quadrupole
- Quadrupolar structure transferred through plane wave or orbital angular momentum onto polarization anisotropy
- Recall monopole and dipole emission structure – same procedure except couple to $s = \pm 2, \ell = 2$:

$$ {}_{\pm 2}Y_{2m} Y_{\ell 0} \rightarrow {}_{\pm 2}Y_{(\ell-2)m} \cdots {}_{\pm 2}Y_{(\ell+2)m} $$



Polarization Transfer

- A polarization source function with $\ell = 2$, modulated with plane wave orbital angular momentum
- Scalars have no B mode contribution, vectors mostly B and tensor comparable B and E



Acoustic Polarization

- Break down of tight-coupling leads to quadrupole anisotropy of

$$\pi_\gamma \approx \frac{k}{\dot{\tau}} v_\gamma$$

- Scaling $k_D = (\dot{\tau}/\eta_*)^{1/2} \rightarrow \dot{\tau} = k_D^2 \eta_*$
- Know: $k_D s_* \approx k_D \eta_* \approx 10$
- So:

$$\pi_\gamma \approx \frac{k}{k_D} \frac{1}{10} v_\gamma$$

$$\Delta_P \approx \frac{\ell}{\ell_D} \frac{1}{10} \Delta_T$$

Acoustic Polarization

- Gradient of velocity is along direction of wavevector, so polarization is pure E-mode
- Velocity is 90° out of phase with temperature – turning points of oscillator are zero points of velocity:

$$\Theta + \Psi \propto \cos(ks); \quad v_\gamma \propto \sin(ks)$$

- Polarization peaks are at troughs of temperature power

Cross Correlation

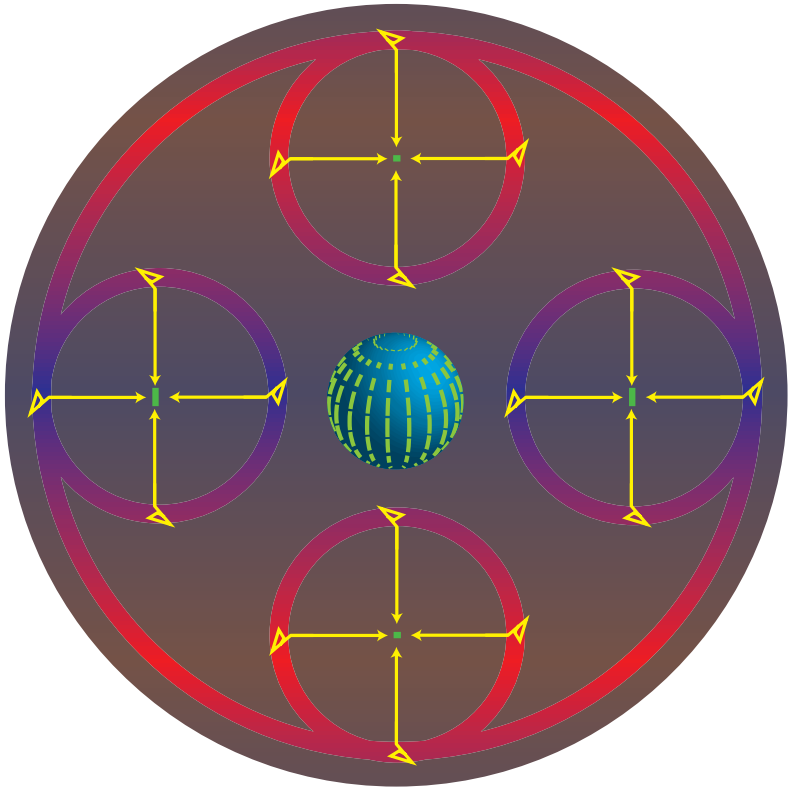
- Cross correlation of temperature and polarization

$$(\Theta + \Psi)(v_\gamma) \propto \cos(ks)\sin(ks) \propto \sin(2ks)$$

- Oscillation at twice the frequency
- Correlation: radial or tangential around hot spots
- Partial correlation: easier to measure if polarization data is noisy, harder to measure if polarization data is high S/N or if bands do not resolve oscillations
- Good check for systematics and foregrounds
- Comparison of temperature and polarization is proof against features in initial conditions mimicking acoustic features

Reionization

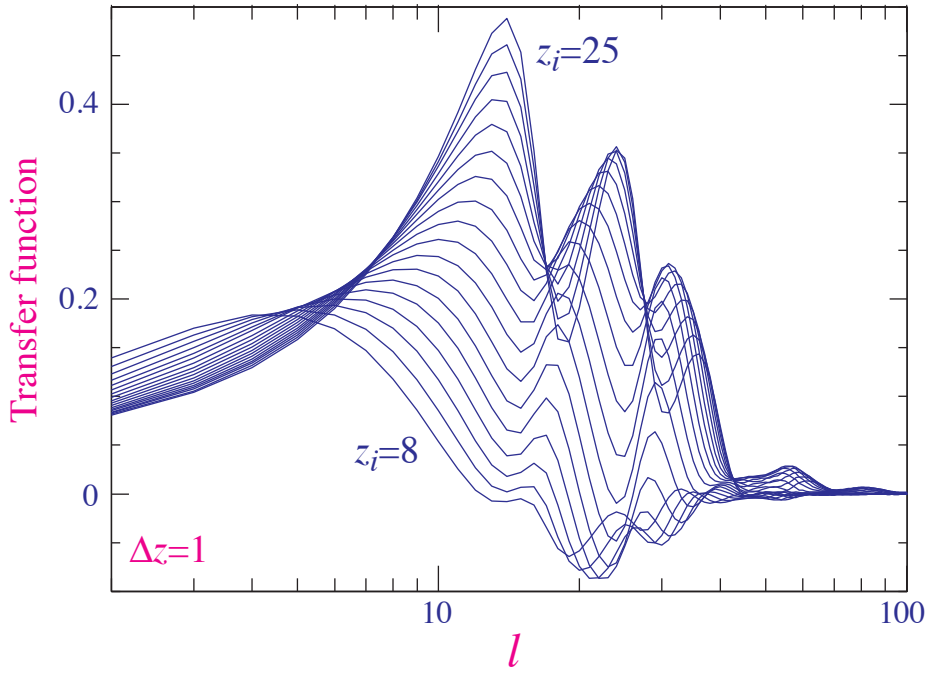
- Reionization causes rescattering of radiation
- Suppresses temperature anisotopy as $e^{-\tau}$ and changes interpretation of amplitude to $A_s e^{-2\tau}$
- Electron sees temperature anisotropy on its recombination surface



- For wavelengths that are comparable to the horizon at reionization, a quadrupole moment
- Rescatters to a linear polarization that is correlated with the Sachs-Wolfe temperature anisotropy

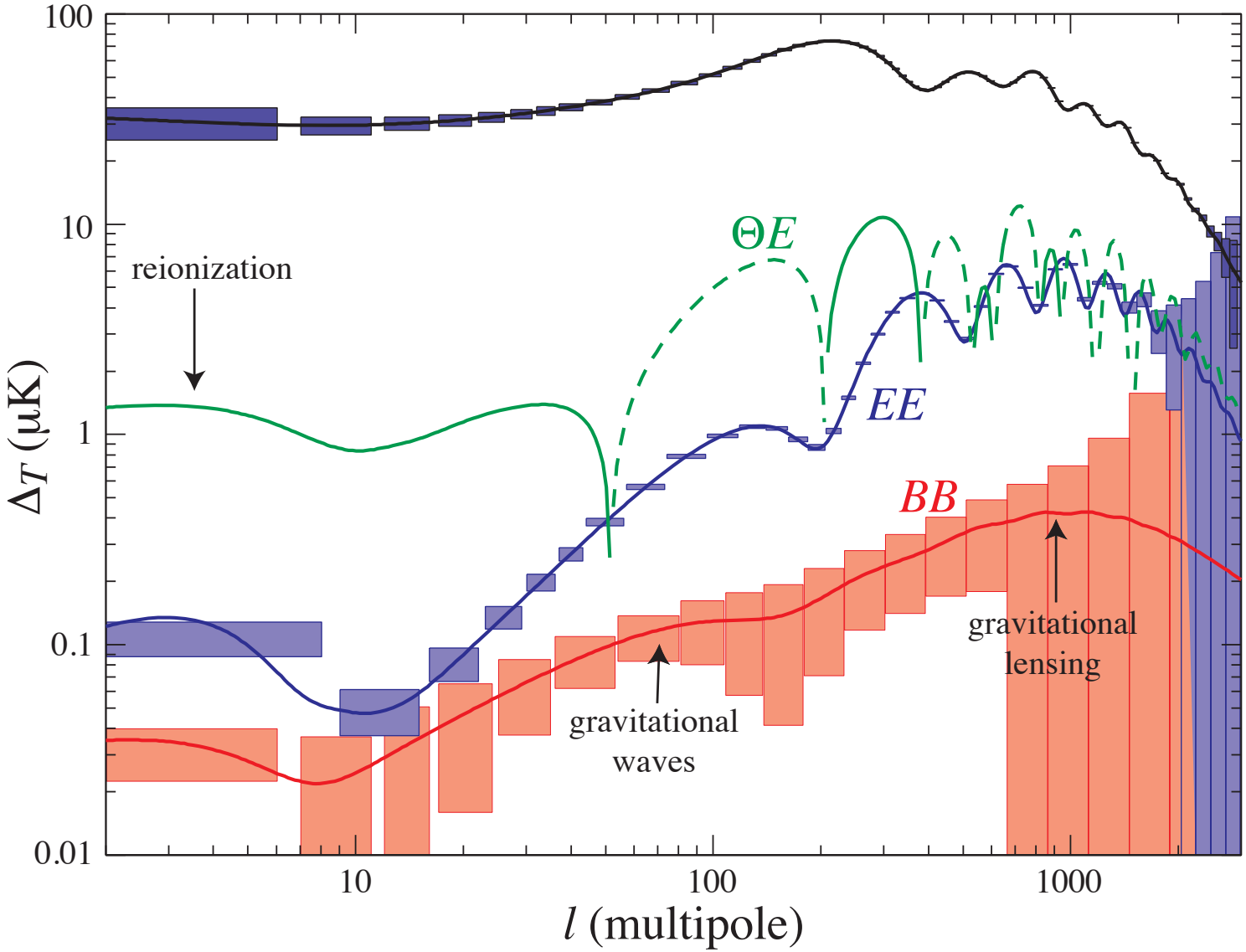
Reionization

- Amplitude of C_ℓ^{EE} depends mainly on τ
- Shape of C_ℓ^{EE} depends on reionization history
- Horizon at earlier epochs subtends a smaller angle, higher multipole peak



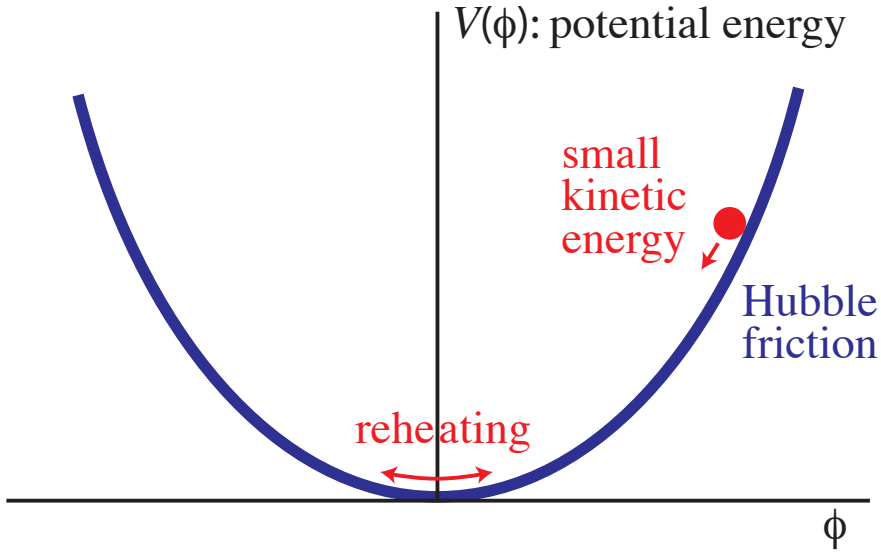
- Precision measurements can constrain the reionization history to be either low or high z dominated

Polarized Landscape



Inflation: Acceleration from Scalar Field

- Unlike a true cosmological constant, the period of exponential expansion must end to produce the hot big bang phase



- A cosmological constant is like potential energy - so imagine a ball rolling slowly in into a valley eventually converting potential into kinetic energy
- Technically, this is a scalar field: where the position on the hill is ϕ and the height of the potential is $V(\phi)$
- In spacetime $\phi(\mathbf{x}, t)$ is a function of position: different spacetime points can be at different field positions

Horizon Problem

- The horizon in a decelerating universe scales as $\eta \propto a^{(1+3w)/2}$, $w > -1/3$. For example in a matter dominated universe

$$\eta \propto a^{1/2}$$

- CMB decoupled at $a_* = 10^{-3}$ so subtends an angle on the sky

$$\frac{\eta_*}{\eta_0} = a_*^{1/2} \approx 0.03 \approx 2^\circ$$

- So why is the CMB sky isotropic to 10^{-5} in temperature if it is composed of $\sim 10^4$ causally disconnected regions
- If smooth by fiat, why are there 10^{-5} fluctuations correlated on superhorizon scales

Flatness & Relic Problems

- Flatness problem: why is the radius of curvature larger than the observable universe. (Before the CMB determinations, why is it at least comparable to observable universe $|\Omega_K| \lesssim \Omega_m$)
- Also phrased as a coincidence problem: since $\rho_K \propto a^{-2}$ and $\rho_m \propto a^{-3}$, why would they be comparable today – modern version is dark energy coincidence $\rho_\Lambda = \text{const.}$
- Relic problem – why don't relics like monopoles dominate the energy density

- Inflation is a theory that solves all three problems at once and also supplies a source for density perturbations

Accelerating Expansion

- In a matter or radiation dominated universe, the horizon grows as a power law in a so that there is no way to establish causal contact on a scale longer than the inverse Hubble length ($1/aH$, comoving coordinates) at any given time: general for a decelerating universe

$$\eta = \int d\ln a \frac{1}{aH(a)}$$

- $H^2 \propto \rho \propto a^{-3(1+w)}$, $aH \propto a^{-(1+3w)/2}$, critical value of $w = -1/3$, the division between acceleration and deceleration determines whether as the universe expands comoving observers leave or come into causal contact
- Recall this is our fate in the current accelerating expansion – observers that were once in causal contact will no longer be able to communicate with each other due to the rapid expansion

Causal Contact

- True horizon always grows meaning that one always sees more and more of the universe. But the comoving Hubble length decreases: the difference in conformal time, the distance a photon can travel between two epochs denoted by the scale factor decreases.
  Regions that were in causal contact, leave causal contact.
- Horizon problem solved if the universe was in an acceleration phase up to η_i and the conformal time since then is shorter than the total conformal age

$$\eta_0 \gg \eta_0 - \eta_i$$

$$\text{total distance} \gg \text{distance traveled since inflation}$$

$$\text{apparent horizon}$$

Flatness & Relic

- Comoving radius of curvature is constant and can even be small compared to the full horizon $R \ll \eta_0$ yet still $\eta_0 \gg R \gg \eta_0 - \eta_i$
- In physical coordinates, the rapid expansion of the universe makes the current observable universe much smaller than the curvature scale
- Likewise, the number density of relics formed before the accelerating (or inflationary) epoch is diluted to make them rare in the current observable volume
- Common to place the zero point of (conformal) time at the end of inflation $\tilde{\eta} \equiv \eta - \eta_i$. Here conformal time is negative during inflation and size reflects the distance a photon can travel from that epoch to the end of inflation. To avoid confusion with the original zero point $\eta(a = 0) = 0$ let's call this $\tilde{\eta}$.

Sufficient Inflation

- If the accelerating component has equation of state $w = -1$, $\rho =$ const., $H = H_i$ const. so that $a \propto \exp(Ht)$

$$\tilde{\eta} = \int_{a_i}^{a} d\ln a \frac{1}{aH} = -\frac{1}{aH_i}\Big|_{a_i}^{a}$$
$$\approx -\frac{1}{aH_i} \quad (a_i \gg a)$$

- In particular, the current horizon scale $H_0\tilde{\eta}_0 \approx 1$ exited the horizon during inflation at

$$\tilde{\eta}_0 \approx H_0^{-1} = \frac{1}{a_H H_i}$$
$$a_H = \frac{H_0}{H_i}$$

Sufficient Inflation

- Given some energy scale for inflation that defines H_i, this tells us what the scale factor a_H was when the current horizon left the horizon during inflation
- If we knew what the scale factor a_i was at the end of inflation, we could figure out the number of efolds $N = \ln(a_i/a_H)$ between these two epochs
- A rough way to characterize this is to quote it in terms of an effective temperature $T \propto T_{\rm CMB} a^{-1}$ at the end of inflation

$$\ln \frac{a_i}{a_H} = \ln \frac{T_{\rm CMB}}{T_i} \frac{H_i}{H_0} = 65 + 2 \ln \left(\frac{\rho_i^{1/4}}{10^{14}{\rm GeV}} \right) - \ln \left(\frac{T_i}{10^{10}{\rm GeV}} \right)$$

- So inflation lasted at least ~ 60efolds - a more detailed calculation would involve the epoch of reheating and g_* factors, so $T_i \neq T_{\rm reheat}$

Scalar Fields

- Inflation ends when the field rolls sufficiently down the potential that its kinetic energy becomes comparable to its potential energy
- The field then oscillates at the bottom of the potential and small couplings to standard model particles "reheats" the universe converting the inflaton energy into particles
- Due to the uncertainty principle in quantum mechanics, the field cannot remain perfectly unperturbed
- The small field fluctuations mean that inflation ends at a slightly different time at different points in space - leaving fluctuations in the scale factor, which are curvature or gravitational potential fluctuations
- Gravitational attraction into these potential wells forms all of the structure in the universe

Scalar Fields

- Mathematically, the scalar field obeys the Klein-Gordon equation in an expanding universe $\Box\phi = dV/d\phi \equiv V'$

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + V' = 0$$

  where $V' = dV/d\phi$ is the slope of the potential - the first and third term look like the equations of motion of a ball rolling down a hill - acceleration = gradient of potential

- The second $d\phi/dt$ term is a friction term provided by the expansion - "Hubble friction" - just like particle numbers and energy density dilute with the expansion, so too does the kinetic energy of the scalar field.

Scalar Fields

- Kinetic energy is

$$\rho_{\rm kinetic} = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2$$

so, without the V' forcing term, how does the energy density decay?

- Transform to $\ln a = N$ assuming $H \approx$ const.

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} = 0 \rightarrow \frac{d\phi}{dt} \propto a^{-3}$$

KE decays as $\rho_{\rm kinetic} \propto a^{-6} = a^{-3(1+w_{\rm kinetic})}$, or $w_{\rm kinetic} = +1$

- Compare with the potential energy at fixed field position $w_{\rm potential} = -1$
- Total pressure: $p_\phi = \rho_{\rm kinetic} - \rho_{\rm potential} = 1/2(d\phi/dt)^2 - V$

Scalar Fields

- As the field rolls it slowly loses total energy to friction, which defines the slow roll parameter

$$\epsilon_H = -\frac{d\ln H}{d\ln a} = \frac{3}{2}(1+w_\phi)$$

- Requirement that inflation last for the sufficient $\sim$60 efolds requires that $\epsilon_H \lesssim 1/60 \ll 1$
- This requirement also means that ϵ_H must also be slowly varying so as not to grow much during these 60 efolds

$$\delta_1 = \frac{1}{2}\frac{d\ln\epsilon_H}{d\ln a} - \epsilon_H$$

with $|\delta_1| \ll 1$ (its defined this way since it also determines how close the roll is to friction dominated $3Hd\phi/dt \approx -V'$)

Perturbation Generation

- Horizon scale $1/H$ during inflation acts like event horizon - things that are separated by more than this distance leave causal contact
- Result of treating field fluctuations as a quantum simple harmonic oscillator: uncertainty principle leads to inevitable fluctuations
- Fluctuations freeze in when the comoving wavelength $\lambda = 2\pi/k$ becomes larger than the comoving horizon $1/aH$, so that parts of the fluctuation are no longer in causal contact with itself, i.e. when $k \approx aH$

$$\delta\phi \approx \frac{H}{2\pi}$$

- We can also view this as a typical freezeout problem. Quantum fluctuations behave as a simple harmonic oscillator with frequency or rate $\omega \approx k/a$ and freezeout occurs when $\omega = H$, so $k/a = H$

Perturbation Generation

- Interpretation: universe is expanding quickly enough that various parts of the wave cannot "find" each other to maintain "equilibrium" (continue oscillating)
- Can heuristically understand the freezout value in the same way as Hawking radiation from a black hole – virtual particles become real when separated by the horizon
- Here H defines the horizon area (or in black hole language the Hawking temperature) and dimensional analysis says the field fluctuation must scale with H, the only dimensionful quantity
- Because H remains roughly constant during inflation the result is a scale invariant spectrum of fluctuations
- Let's prove this by analogy to the quantum harmonic oscillator

Perturbation Generation

- For a sufficiently flat potential and assuming slow roll, the field perturbation $\delta\phi$ obeys the free Klein Gordon equation $\Box\delta\phi = 0$

$$\ddot{\delta\phi} + 2\frac{\dot{a}}{a}\dot{\delta\phi} + k^2\delta\phi$$

where overdots are conformal time derivatives with $\tilde{\eta} = 0$ as the end of inflation

- We want to take out the effect of the expansion in the "friction" term so by analogy to comoving coordinates define $u = a\delta\phi$

$$\ddot{u} + \left(k^2 - \frac{\ddot{a}}{a}\right) u = 0$$

and let's further note that in slow roll H is nearly constant so

$$\tilde{\eta} = \int_{a_{\rm end}}^{a} \frac{da}{Ha^2} \approx -\frac{1}{aH}, \qquad \frac{\ddot{a}}{a} \approx \frac{2}{\tilde{\eta}^2}$$

Perturbation Generation

- Under these approximations

$$\ddot{u} + \left(k^2 - \frac{2}{\tilde{\eta}}\right) u = 0$$

- Note that for subhorizon modes $|k\tilde{\eta}| \gg 1$ and u behaves as a simple harmonic oscillator

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator as in ordinary quantum mechanics

$$\hat{u} = u(k, \tilde{\eta})\hat{a} + u^*(k, \tilde{\eta})\hat{a}^\dagger$$

where modefunction $u(k, \tilde{\eta})$ satisfies classical equation of motion

Perturbation Generation

- Creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \qquad a|0\rangle = 0$$

- Field uncertainty principle: $[\hat{x}, \hat{q}] = i \rightarrow [\hat{u}, d\hat{u}/d\tilde{\eta}] = i$

$$u(k, \tilde{\eta}) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

- Equivalently, zero point fluctuations of ground state

$$\begin{aligned}
\langle u^2 \rangle &= \langle 0|u^\dagger u|0\rangle = \langle 0|(u^*\hat{a}^\dagger + u\hat{a})(u\hat{a} + u^*\hat{a}^\dagger)|0\rangle \\
&= \langle 0|\hat{a}\hat{a}^\dagger|0\rangle |u(k,\tilde{\eta})|^2 \\
&= \langle 0|[\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger\hat{a}|0\rangle |u(k,\tilde{\eta})|^2 \\
&= |u(k,\tilde{\eta})|^2 = \frac{1}{2k}
\end{aligned}$$

Perturbation Generation

- From these initial conditions, modefunction then has the exact solution

$$u = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\tilde{\eta}}\right)e^{-ik\tilde{\eta}}$$

- For $|k\tilde{\eta}| \ll 1$ (late times, $\gg$ Hubble length) fluctuation freezes in

$$\lim_{|k\tilde{\eta}|\to 0} u = -\frac{1}{\sqrt{2k}}\frac{i}{k\tilde{\eta}} \approx \frac{iHa}{\sqrt{2k^3}}$$

$$\delta\phi = \frac{iH}{\sqrt{2k^3}}$$

- Power spectrum of field fluctuations

$$\Delta^2_{\delta\phi} = \frac{k^3|\delta\phi|^2}{2\pi^2} = \frac{H^2}{(2\pi)^2}$$

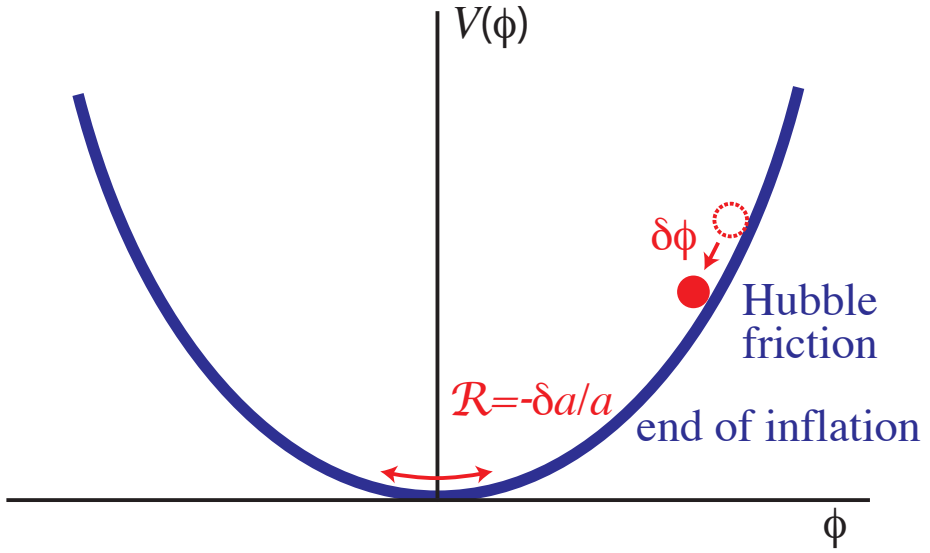
Curvature Fluctuation

- Field fluctuations change the scale factor at which inflation ends

$$\mathcal{R} = -\delta \ln a = -\frac{d \ln a}{dt}\frac{dt}{d\phi}\delta\phi = -\frac{H^2}{2\pi}\frac{dt}{d\phi}$$

- Using the equation of state of ϕ we can convert $d\phi/dt$ to ϵ_H

$$\begin{aligned} w_\phi &= \frac{p_\phi}{\rho_\phi} \\ &= \frac{(d\phi/dt)^2/2 - V}{(d\phi/dt)^2/2 + V} \\ &\approx \frac{(d\phi/dt)^2}{V} - 1 \end{aligned}$$



and $H^2 \approx 8\pi G V/3$ from Friedmann

Curvature Fluctuation

- So

$$\epsilon_H \approx \frac{3}{2}\frac{(d\phi/dt)^2}{V} \approx 4\pi G\frac{(d\phi/dt)^2}{H^2}$$

  and the variance of fluctuations per log wavenumber $d\ln k$

$$\Delta^2_{\mathcal{R}} \equiv \langle \mathcal{R}^2 \rangle \approx \frac{H^4}{4\pi^2}\frac{4\pi G}{H^2\epsilon_H} \approx \frac{G}{\pi}\frac{H^2}{\epsilon_H}$$

- Crucial point: curvature fluctuations are enhanced over field fluctuations by the slowness of roll:

$$\Delta^2_{\mathcal{R}} \propto \frac{\Delta^2_\phi}{\epsilon_H}$$

- We shall see that this means the ratio of gravitational waves to curvature fluctuations are suppressed by ϵ_H

Tilt

- Curvature power spectrum is scale invariant to the extent that H and ϵ_H are constant
- Scalar spectral index

$$\frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln k} \equiv n_S - 1 = 2\frac{d\ln H}{d\ln k} - \frac{d\ln\epsilon_H}{d\ln k}$$

- Evaluate at horizon crossing where fluctuation freezes $k = aH$

$$\frac{d\ln H}{d\ln k} \approx \frac{d\ln H}{d\ln a} = -\epsilon_H$$
$$\frac{d\ln\epsilon}{d\ln k} \approx \frac{d\ln\epsilon}{d\ln a} = 2(\delta_1 + \epsilon_H)$$

- Tilt in the slow-roll approximation

$$n_S - 1 = -4\epsilon_H - 2\delta_1$$

Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation $(K = 0)$, same as scalar field

$$\frac{d^2 h_{+,\times}}{dt^2} + 3H\frac{dh_{+,\times}}{dt} + \frac{k^2}{a^2}h_{+,\times} = 0\,.$$

- Acquires quantum fluctuations in same manner as ϕ. Lagrangian sets the normalization
- Scale-invariant gravitational wave amplitude

$$\Delta^2_{+,\times} = 16\pi G\frac{H^2}{(2\pi)^2}$$

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation

Gravitational Waves

- Tensor-scalar ratio is therefore generally small

$$r \equiv 4\frac{\Delta_+^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon_H$$

- Tensor tilt:

$$\frac{d\ln\Delta_+^2}{d\ln k} \equiv n_T = 2\frac{d\ln H}{d\ln k} = -2\epsilon_H$$

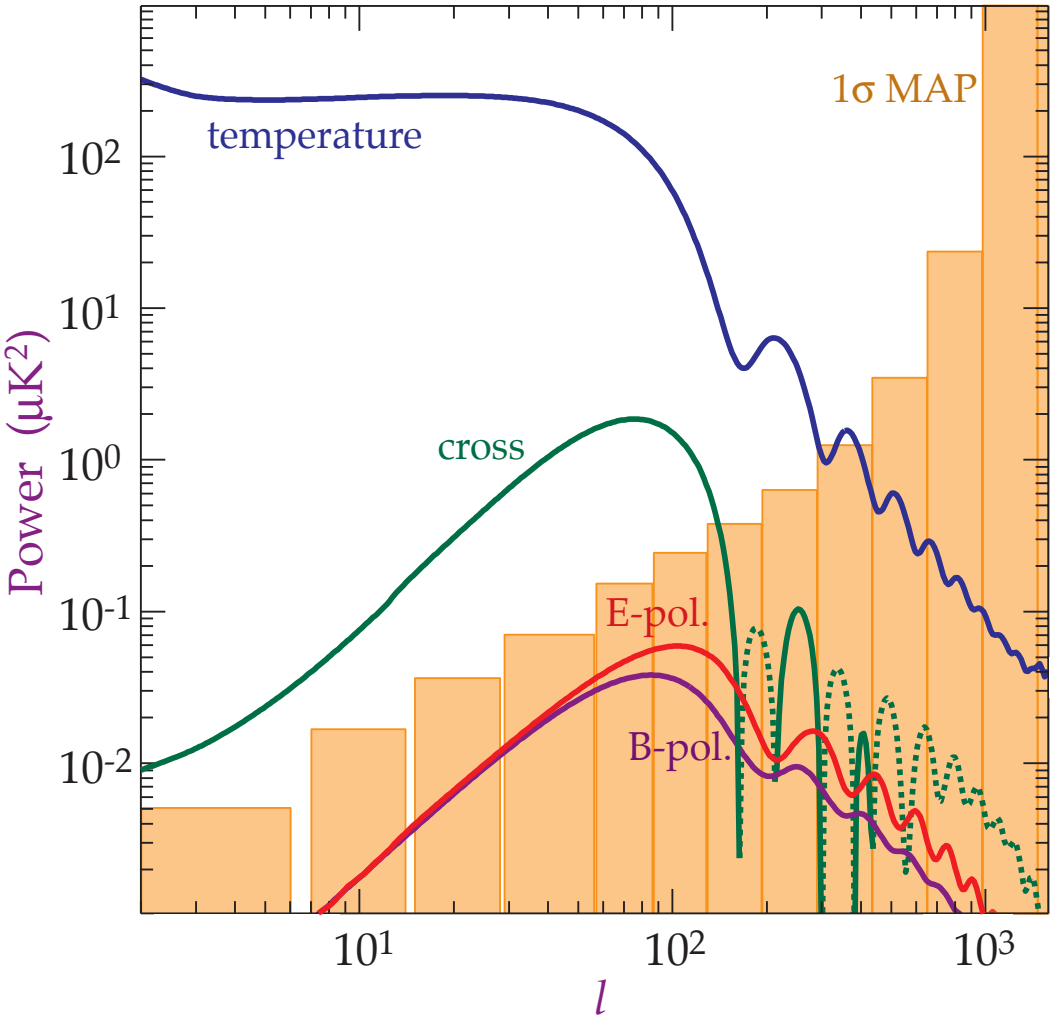
- Consistency relation between tensor-scalar ratio and tensor tilt

$$r = 16\epsilon = -8n_T$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparision of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself

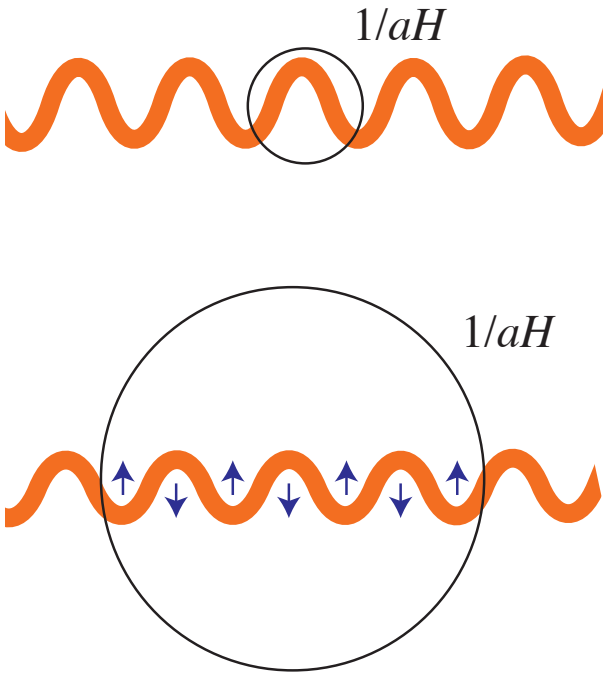
Observability

- Gravitational waves from inflation can be measured via its imprint on the polarization of the CMB



Tensor Power

- Gravitational waves obey a Klein-Gordon like equation
- Like inflation, perturbations generated by quantum fluctuations during inflation
- Freeze out at horizon crossing during inflation an amplitude that reflects the energy scale of inflation

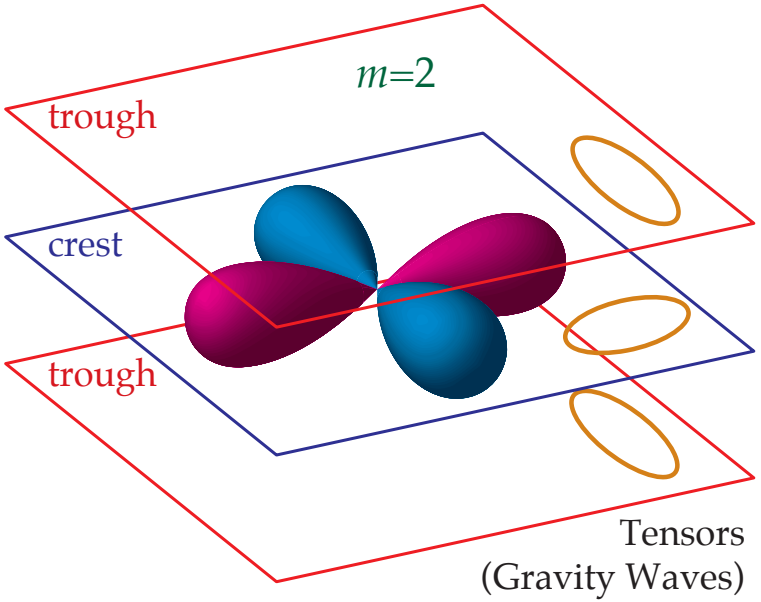


$$\Delta^2_{+,\times} = \frac{H^2}{2\pi^2 M^2_{\rm Pl}} \propto E_i^4$$

- Gravitational waves remain frozen outside the horizon at constant amplitude
- Oscillate inside the horizon and decay or redshift as radiation

Tensor Quadrupoles

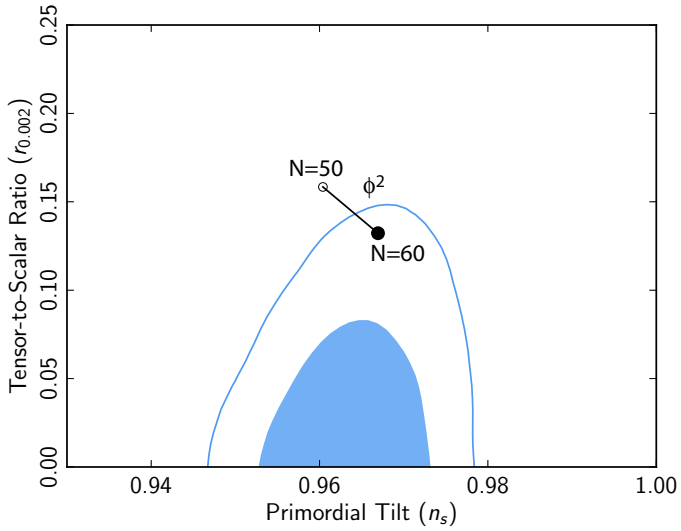
- Changing transverse-traceless distortion of space creates a quadrupole CMB anisotropy much like the distortion of test ring of particles



- As the tensor mode enters the horizon it imprints a quadrupole temperature $\ell = 2, m = \pm 2$ in plane wave coordinates $\mathbf{k} \parallel \mathbf{z}$
- Modes that cross before recombination: effect erased by rescattering $e^{-\tau}$ due to its isotropizing effect
- Modes that cross after recombination: integrate contributions along the line of sight - tensor ISW effect

Tensor Temperature Power Spectrum

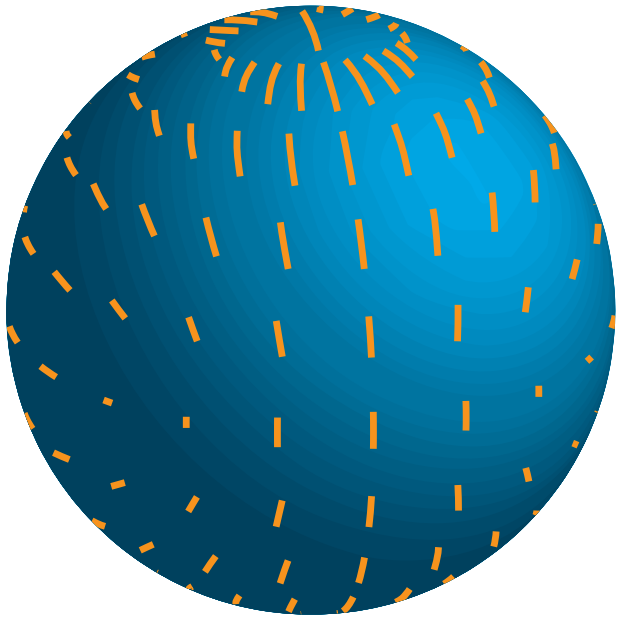
- Resulting spectum,
  near scale invariant out to
  horizon at recombination $\ell < 100$
- Suppressed on smaller scales or
  higher multipoles $\ell > 100$, weakly
  degenerate with tilt



- When added to scalar spectrum, enhances large scale anisotropy over small scale
- Shape of total temperature spectrum can place tight limit $r < 0.1$, for power law curvature spectrum
- Smaller tensor-scalar ratios cannot be constrained by temperature alone due the high cosmic variance of the low multipole specrum

Tensor Polarization Power Spectrum

- Polarization of gravitational wave determines the quadrupole temperature anisotropy
- Scattering of quadrupole temperature anisotropy generates linear polarization aligned with cold lobe



- Direction of CMB polarization is therefore determined by gravitational wave polarization rather than direction of wavevector
- B-mode polarization when the amplitude is modulated by the plane wave
- Requires scattering: two peaks - horizon at recombination and reionization

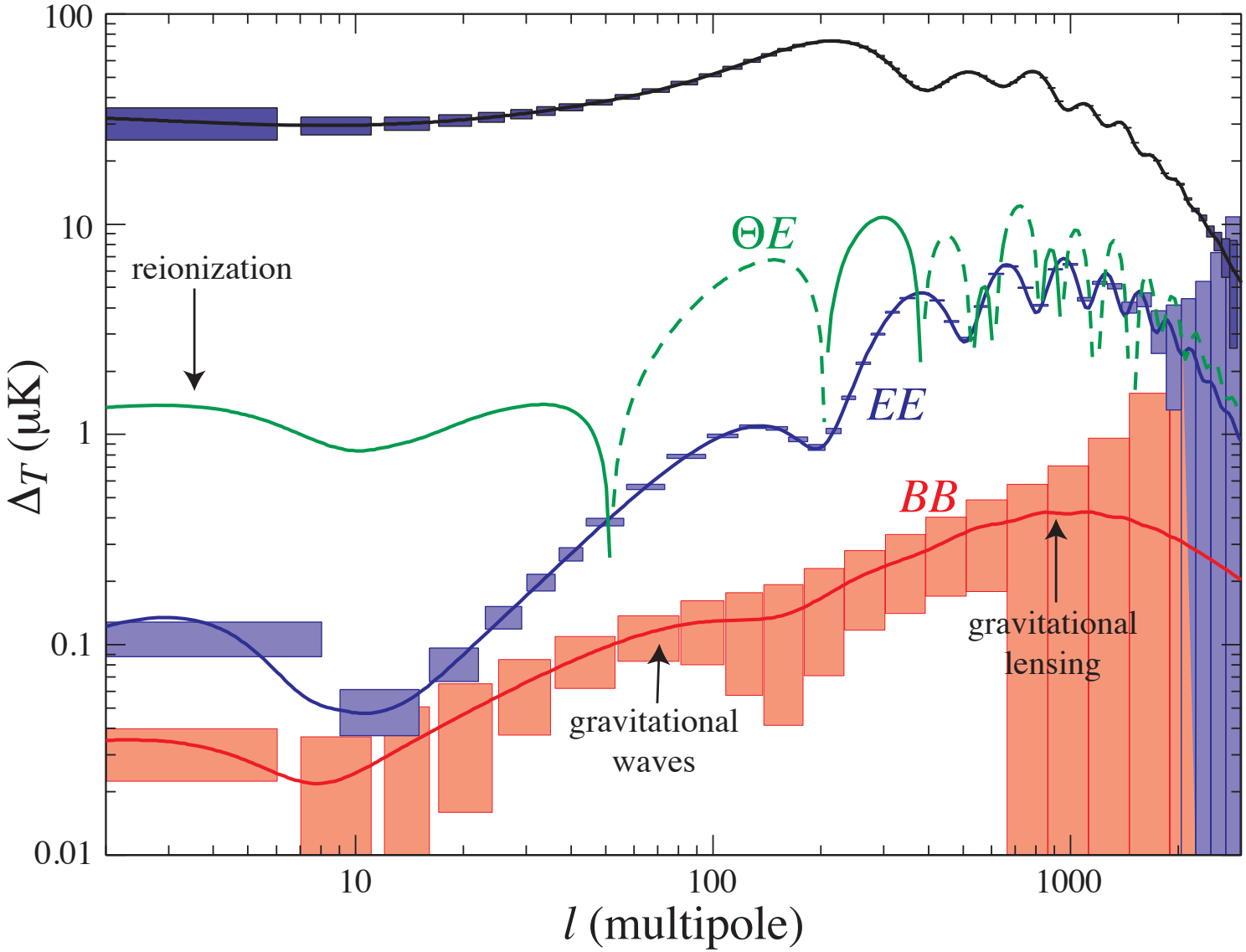
Tensor Polarization Power Spectrum

- Measuring B-modes from gravitational waves determines the energy scale of inflation

$$\Delta B_{\rm peak} \approx 0.024 \left(\frac{E_i}{10^{16}{\rm GeV}} \right)^2 \mu{\rm K}$$

- Also generates E-mode polarization which, like temperature, is a consistency check for $r \sim 0.1$
- Projection is less sharp than for scalar E, so evading temperature bounds by adding features to the curvature spectrum can be tested

Gravitational Lensing: Signal and Noise



Gravitational Lensing

- In general relativity, masses curve space and bend the trajectory of photons - for this discussion lets restore the different units of t and x by restoring c - but note that is now does not represent the coordinate speed of light
- Newtonian approximation to the line element, neglecting the expansion (or in conformal coordinates)

$$ds^2 = -(1 + 2\Psi/c^2)c^2dt^2 + (1 + 2\Phi/c^2)dx^2$$

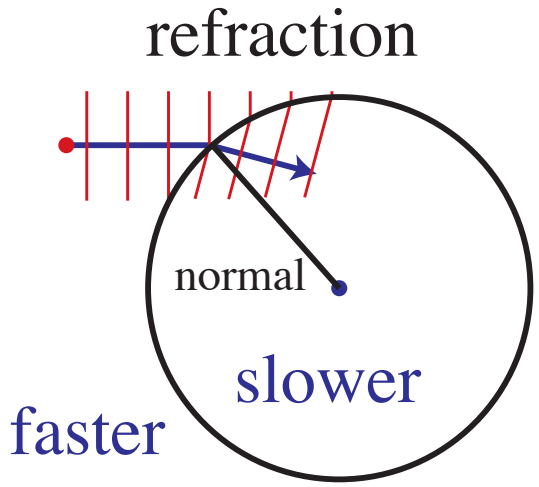
- Photons travel on null geodesics ($ds^2 = 0$) - so the coordinate speed of light is

$$v = \frac{dx}{dt} \approx c\frac{1 + \Psi/c^2}{1 + \Phi/c^2} \approx c(1 - 2\Phi/c^2)$$

Gravitational Lensing

- Coordinate speed of light slows
  in the presence of mass due to
  the warping of spacetime
  as quantified by the gravitational potential

  Can be modelled as an optics problem,
  defines an effective index of refraction



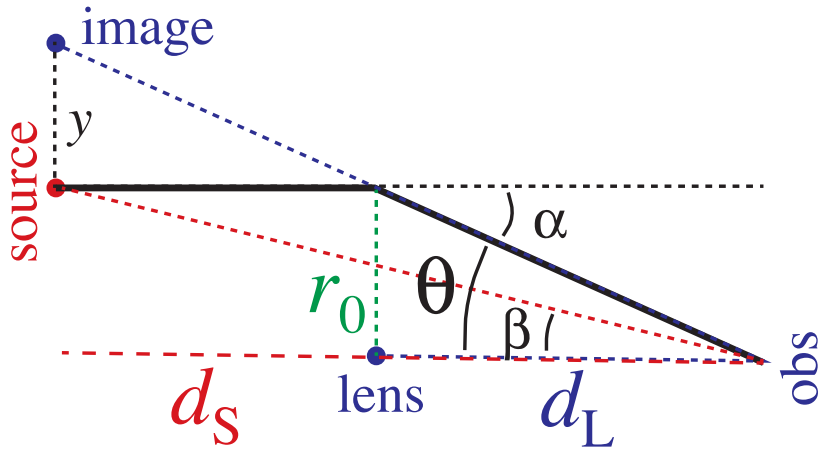
$$n = \frac{c}{v} = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \approx 1 + \frac{2GM}{rc^2}$$

- As light passes by the object, the change in the index of refraction or delay of the propagation of wavefronts bends the trajectory

$$\nabla n = -\frac{2GM}{r^2c^2}\hat{\mathbf{r}}$$

Strong Gravitational Lensing

- Calculation would take the same form if we took a nonrelativistic particle of mass m and used Newtonian mechanics - general relativity just doubles it the deflection for light due to space curvature



- Deflection is small so integrate the transverse ($\perp$) deflection on the unperturbed trajectory

$$\phi = -\int_{-\infty}^{\infty} dx \nabla_\perp n = \int_{-\infty}^{\infty} dx \frac{2GMr_0}{\left(r_0^2 + x^2\right)^{3/2} c^2} = \frac{4GM}{r_0 c^2}$$

Lens Equation

- Given the thin lens deflection formula, the lens equation follows from simple geometry
- Solve for the image position θ with respect to line of sight. Small angle approximation

$$y \approx (d_S - d_L)\phi \approx d_S(\theta - \beta)$$

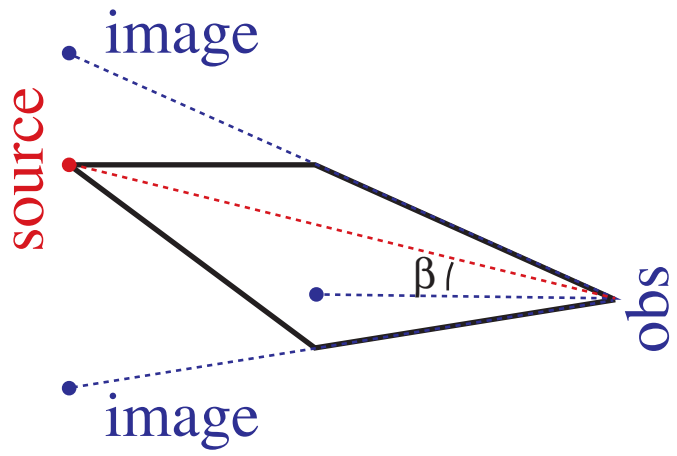
- Substitute in deflection angle

$$(d_S - d_L)\frac{4GM}{r_0 c^2} \approx d_S(\theta - \beta)$$

- Eliminate $r_0 = d_L \sin\theta \approx d_L\theta$
- For cosmological distances replace d's with angular diameter distances D_A

Lens Equation

- Solve for θ to obtain the lens equation



$$\theta^2 - \beta\theta - \frac{4GM}{c^2}\left(\frac{d_S - d_L}{d_S d_L}\right) = 0$$

- A quadratic equation with two solutions for the image position - two images

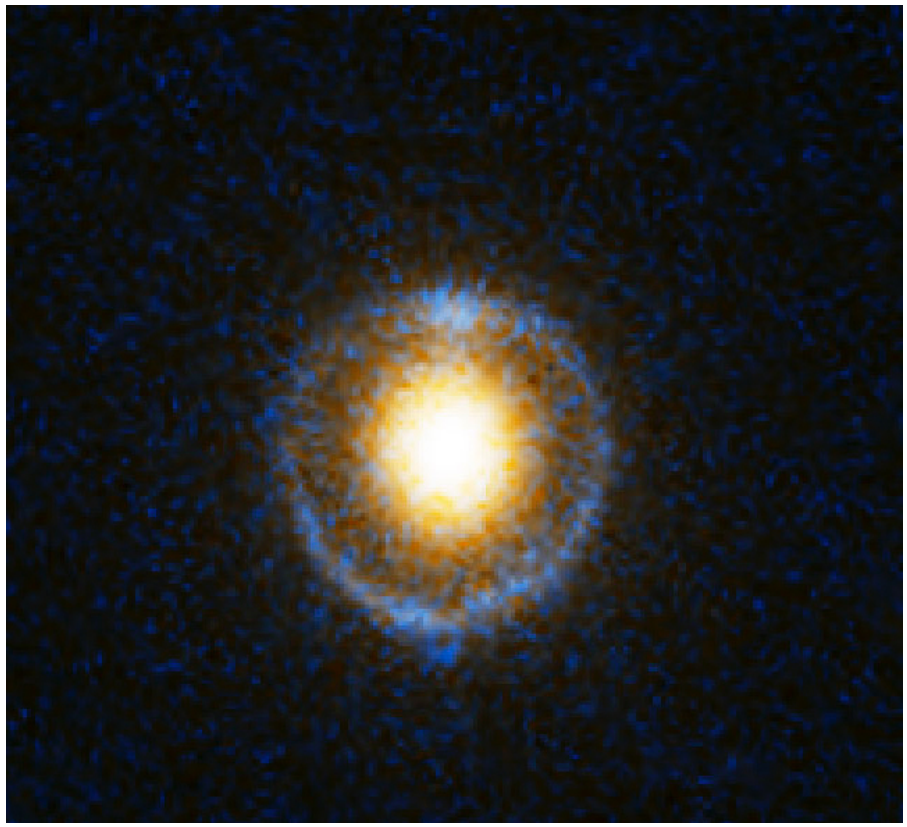
$$\theta_\pm = \frac{\beta}{2} \pm \frac{1}{2}\sqrt{\beta^2 + 16\frac{GM}{c^2}\left(\frac{d_S - d_L}{d_S d_L}\right)}$$

- Sum of angles - second image has negative angle - opposite side of lens

$$\theta_+ + \theta_- = \beta$$

Einstein Ring

- If source is aligned right behind the lens $\beta = 0$ and the two images merge into a ring - Einstein ring - at an angular separation



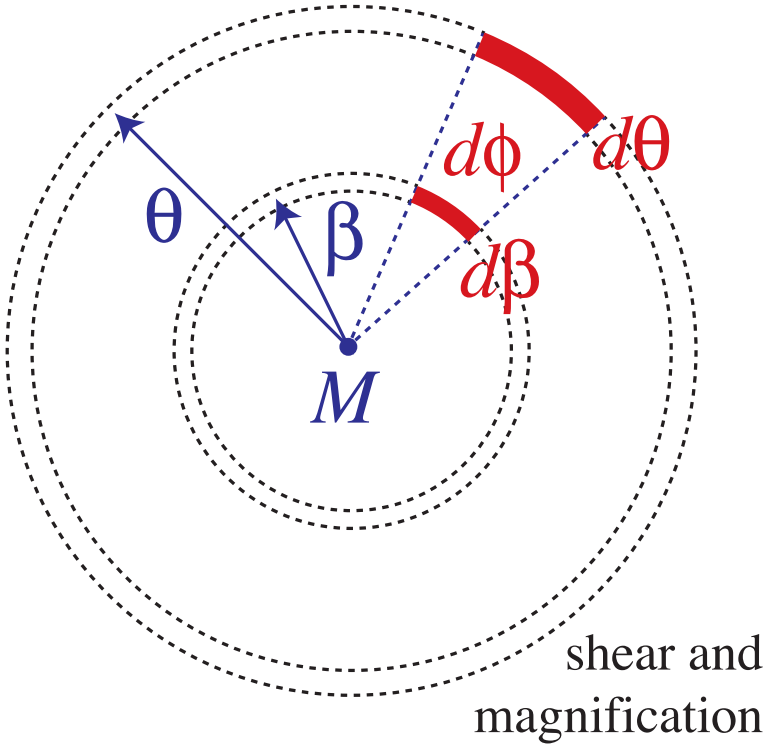
$$\theta_E = \sqrt{\frac{4GM}{c^2}\left(\frac{d_S - d_L}{d_S d_L}\right)}$$

- More generally, quasar is lensed by the extended mass of a galaxy that is not perfectly axially symmetric

Magnification

- Lens equation in terms of Einstein radius

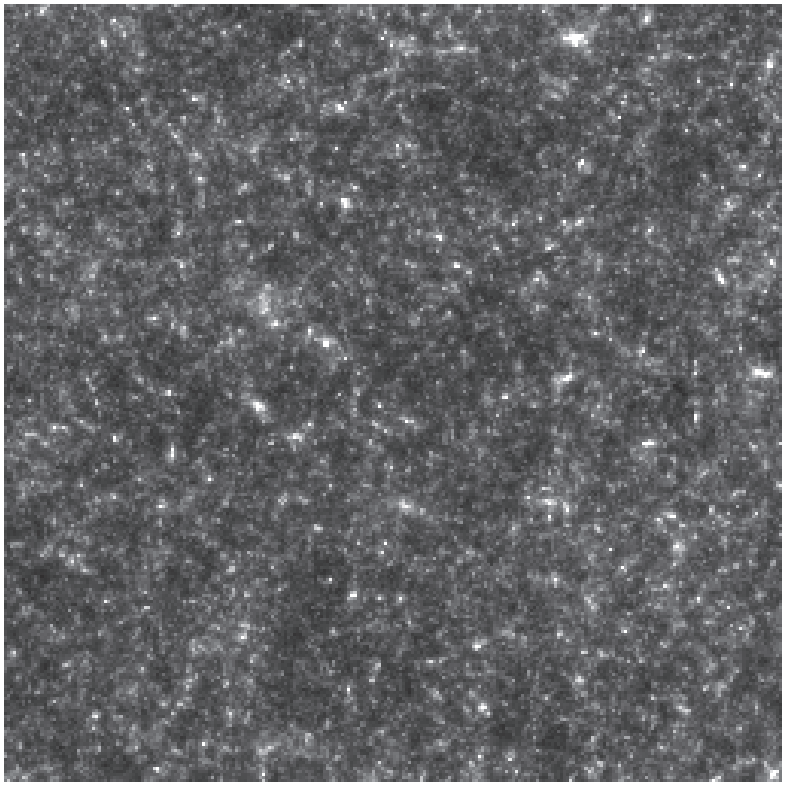
$$\theta^2 - \beta\theta - \theta_E^2 = 0$$

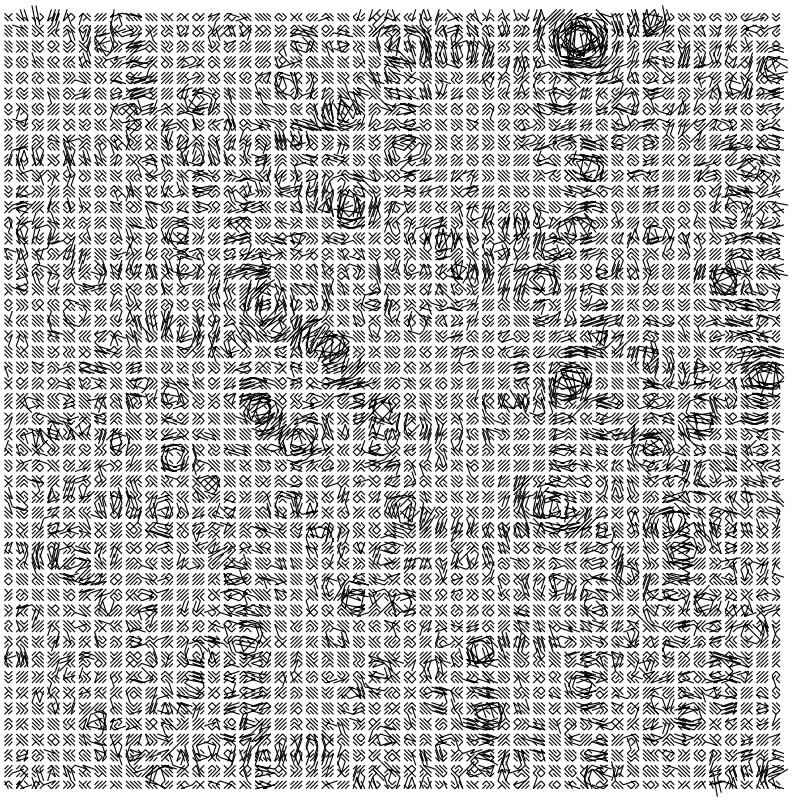


- Given an extended source that covers an angular distance $d\beta$ will have an image cover an angular distance $d\theta_\pm$ related by the derivative $d\theta_\pm/d\beta$
- The displacement in the image is purely radial so that the angular scale of arc $d\phi$ remains unchanged.
- The surface area of the source $\beta d\beta d\phi$ thus becomes $\theta_\pm d\theta_\pm d\phi$.

Cosmic Shear

- On even larger scales, the large-scale structure weakly shears background images: weak lensing





Weak Gravitational Lensing

- Weak lensing is a surface brightness conserving remapping of source to image planes by the gradient of the projected potential

$$\phi(\hat{\mathbf{n}}) = 2 \int_{\eta_*}^{\eta_0} d\eta \, \frac{(D_* - D)}{D \, D_*} \Phi(D\hat{\mathbf{n}}, \eta) \, .$$

  such that the fields are remapped as

$$x(\hat{\mathbf{n}}) \rightarrow x(\hat{\mathbf{n}} + \nabla\phi) \, ,$$

  where $x \in \{\Theta, Q, U\}$ temperature and polarization.

- Taylor expansion leads to product of fields and Fourier mode-coupling

Flat-sky Treatment

- Talyor expand

$$\Theta(\hat{\mathbf{n}}) = \tilde{\Theta}(\hat{\mathbf{n}} + \nabla\phi)$$

$$= \tilde{\Theta}(\hat{\mathbf{n}}) + \nabla_i\phi(\hat{\mathbf{n}})\nabla^i\tilde{\Theta}(\hat{\mathbf{n}}) + \frac{1}{2}\nabla_i\phi(\hat{\mathbf{n}})\nabla_j\phi(\hat{\mathbf{n}})\nabla^i\nabla^j\tilde{\Theta}(\hat{\mathbf{n}}) + \ldots$$

- Fourier decomposition

$$\phi(\hat{\mathbf{n}}) = \int \frac{d^2l}{(2\pi)^2}\phi(\mathbf{l})e^{i\mathbf{l}\cdot\hat{\mathbf{n}}}$$

$$\tilde{\Theta}(\hat{\mathbf{n}}) = \int \frac{d^2l}{(2\pi)^2}\tilde{\Theta}(\mathbf{l})e^{i\mathbf{l}\cdot\hat{\mathbf{n}}}$$

Flat-sky Treatment

- Mode coupling of harmonics

$$\Theta(\mathbf{l}) = \int d\hat{\mathbf{n}}\, \Theta(\hat{\mathbf{n}}) e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}}$$
$$= \tilde{\Theta}(\mathbf{l}) - \int \frac{d^2\mathbf{l}_1}{(2\pi)^2} \tilde{\Theta}(\mathbf{l}_1) L(\mathbf{l}, \mathbf{l}_1)\,,$$

where

$$L(\mathbf{l}, \mathbf{l}_1) = \phi(\mathbf{l} - \mathbf{l}_1)\,(\mathbf{l} - \mathbf{l}_1)\cdot\mathbf{l}_1$$
$$+ \frac{1}{2}\int \frac{d^2\mathbf{l}_2}{(2\pi)^2} \phi(\mathbf{l}_2)\phi^*(\mathbf{l}_2 + \mathbf{l}_1 - \mathbf{l})\,(\mathbf{l}_2\cdot\mathbf{l}_1)(\mathbf{l}_2 + \mathbf{l}_1 - \mathbf{l})\cdot\mathbf{l}_1\,.$$

- Represents a coupling of harmonics separated by $L \approx 60$ peak of deflection power

Power Spectrum

- Power spectra

$$\langle \Theta^*(\mathbf{l})\Theta(\mathbf{l}')\rangle = (2\pi)^2\delta(\mathbf{l}-\mathbf{l}')\, C_l\,,$$
$$\langle \phi^*(\mathbf{l})\phi(\mathbf{l}')\rangle = (2\pi)^2\delta(\mathbf{l}-\mathbf{l}')\, C_l^{\phi\phi}\,,$$

becomes

$$C_l = \left(1 - l^2 R\right)\tilde{C}_l + \int \frac{d^2\mathbf{l}_1}{(2\pi)^2}\tilde{C}_{|\mathbf{l}-\mathbf{l}_1|}C_{l_1}^{\phi\phi}[(\mathbf{l}-\mathbf{l}_1)\cdot\mathbf{l}_1]^2\,,$$

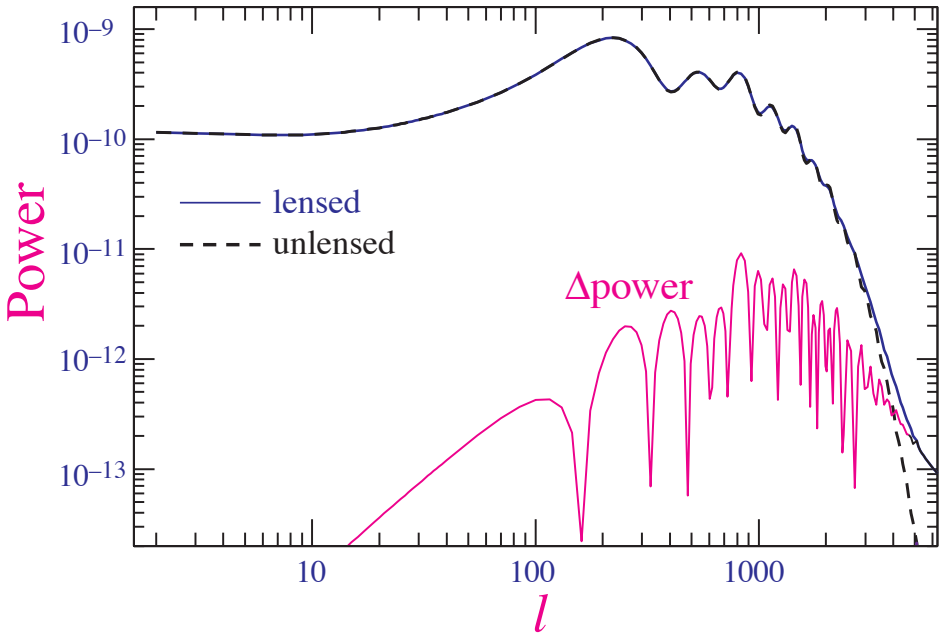
where

$$R = \frac{1}{4\pi}\int \frac{dl}{l}\, l^4 C_l^{\phi\phi}\,.$$

Smoothing Power Spectrum

- If $\tilde{C}_l$ slowly varying then two term cancel

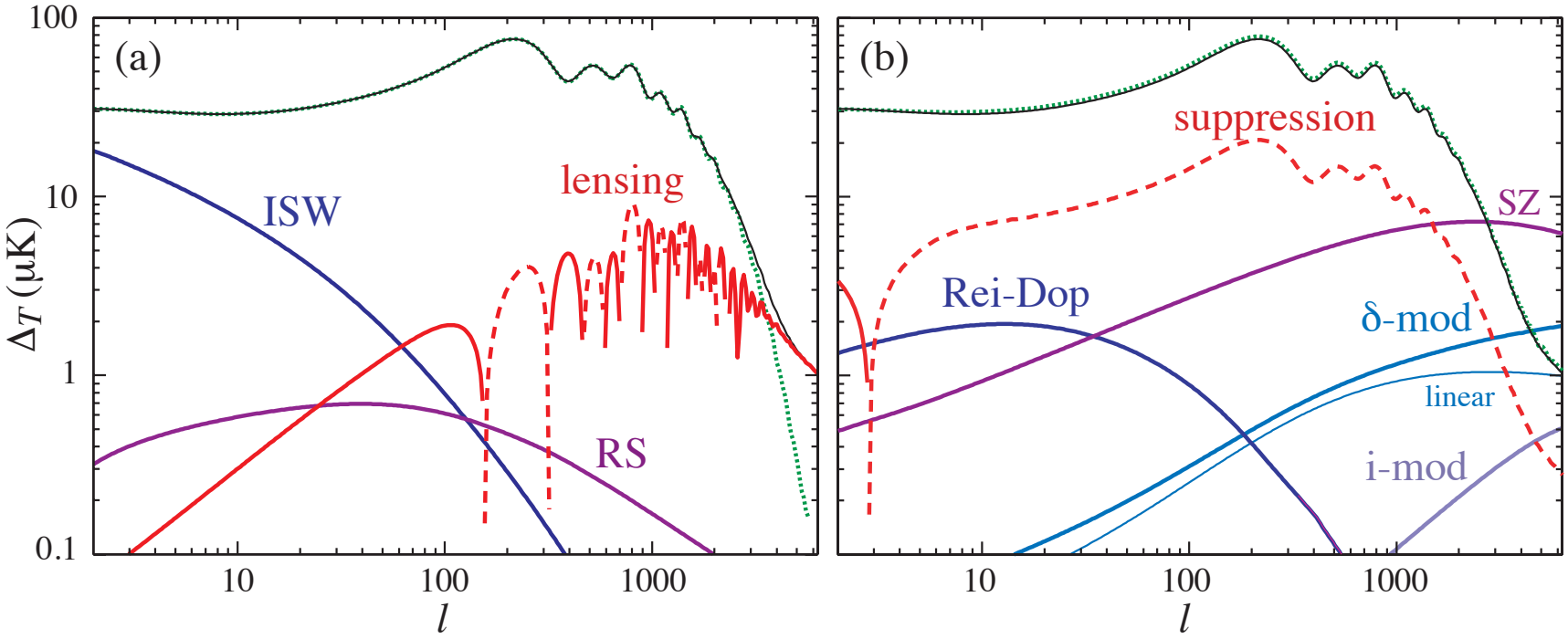
$$\tilde{C}_l \int \frac{d^2\mathbf{l}_1}{(2\pi)^2} C_l^{\phi\phi} (\mathbf{l}\cdot\mathbf{l}_1)^2 \approx l^2 R \tilde{C}_l \,.$$



- So lensing acts to smooth features in the power spectrum. Smoothing kernel is $L \sim 60$ the peak of deflection power spectrum
- Because acoustic feature appear on a scale $l_A \sim 300$, smoothing is a subtle effect in the power spectrum.
- Lensing generates power below the damping scale which directly reflect power in deflections on the same scale

Secondaries: Gravitational

- Gravitational lensing is a gravitational CMB secondary
- Others are the ISW effect from dark energy and its Rees-Sciama (RS) counterpart from nonlinear structure



Polarization Lensing

- Polarization field harmonics lensed similarly

$$[Q \pm iU](\hat{\mathbf{n}}) = -\int \frac{d^2l}{(2\pi)^2}[E \pm iB](\mathbf{l})e^{\pm 2i\phi_{\mathbf{l}}}e^{\mathbf{l}\cdot\hat{\mathbf{n}}}$$

so that

$$\begin{aligned}[Q \pm iU](\hat{\mathbf{n}}) &= [\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}} + \nabla\phi) \\ &\approx [\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}}) + \nabla_i\phi(\hat{\mathbf{n}})\nabla^i[\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}}) \\ &\quad + \frac{1}{2}\nabla_i\phi(\hat{\mathbf{n}})\nabla_j\phi(\hat{\mathbf{n}})\nabla^i\nabla^j[\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}})\end{aligned}$$

Polarization Power Spectra

- Carrying through the algebra

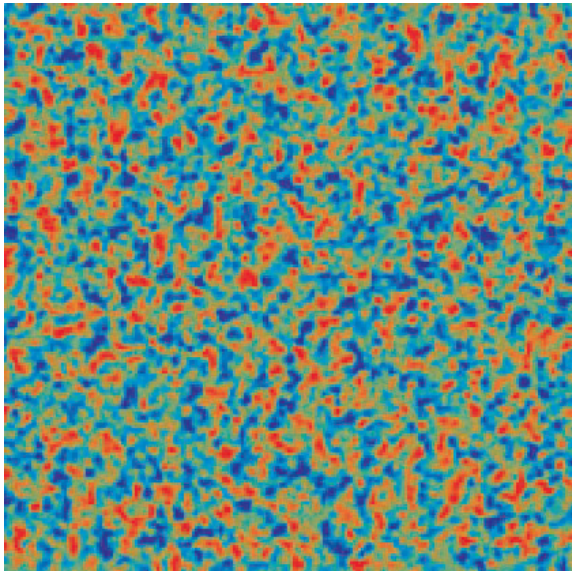
$$C_l^{EE} = \left(1 - l^2 R\right) \tilde{C}_l^{EE} + \frac{1}{2} \int \frac{d^2\mathbf{l}_1}{(2\pi)^2} [(\mathbf{l} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2 C^{\phi\phi}_{|\mathbf{l}-\mathbf{l}_1|} \times \left[(\tilde{C}_{l_1}^{EE} + \tilde{C}_{l_1}^{BB}) + \cos(4\varphi_{l_1}) (\tilde{C}_{l_1}^{EE} - \tilde{C}_{l_1}^{BB}) \right],$$

$$C_l^{BB} = \left(1 - l^2 R\right) \tilde{C}_l^{BB} + \frac{1}{2} \int \frac{d^2\mathbf{l}_1}{(2\pi)^2} [(\mathbf{l} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2 C^{\phi\phi}_{|\mathbf{l}-\mathbf{l}_1|} \times \left[(\tilde{C}_{l_1}^{EE} + \tilde{C}_{l_1}^{BB}) - \cos(4\varphi_{l_1}) (\tilde{C}_{l_1}^{EE} - \tilde{C}_{l_1}^{BB}) \right],$$

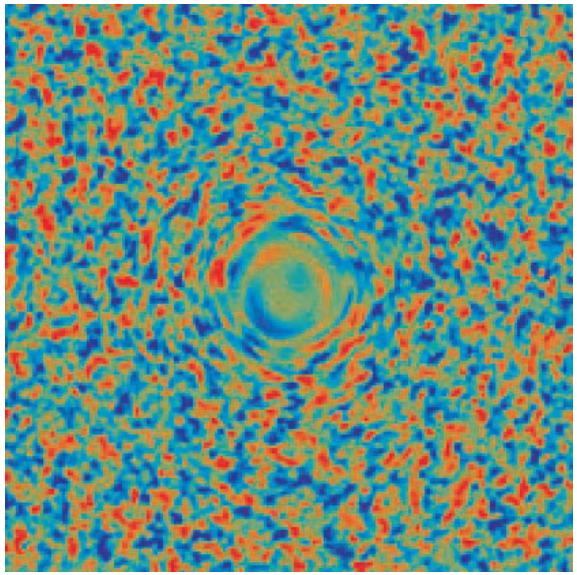
$$C_l^{\Theta E} = \left(1 - l^2 R\right) \tilde{C}_l^{\Theta E} + \int \frac{d^2\mathbf{l}_1}{(2\pi)^2} [(\mathbf{l} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2 C^{\phi\phi}_{|\mathbf{l}-\mathbf{l}_1|} \times \tilde{C}_{l_1}^{\Theta E} \cos(2\varphi_{l_1}),$$

Polarization Lensing

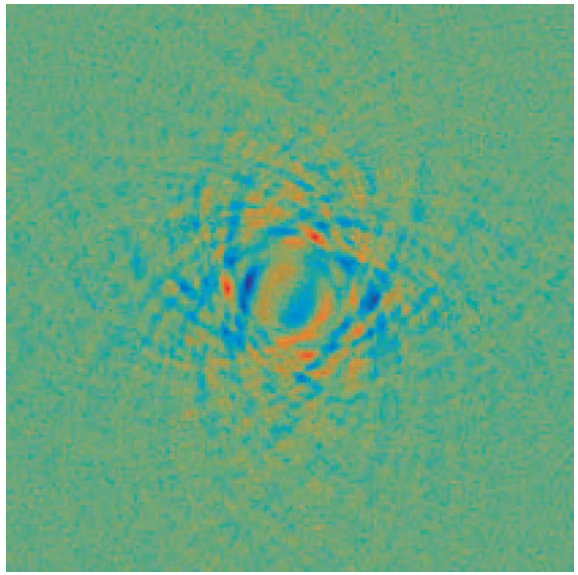
- Lensing generates B-modes out of the acoustic polaraization E-modes contaminates gravitational wave signature if $E_i < 10^{16}$GeV.



Original



Lensed E



Lensed B

Reconstruction from the CMB

- Correlation between Fourier moments for a fixed lens reflect lensing potential

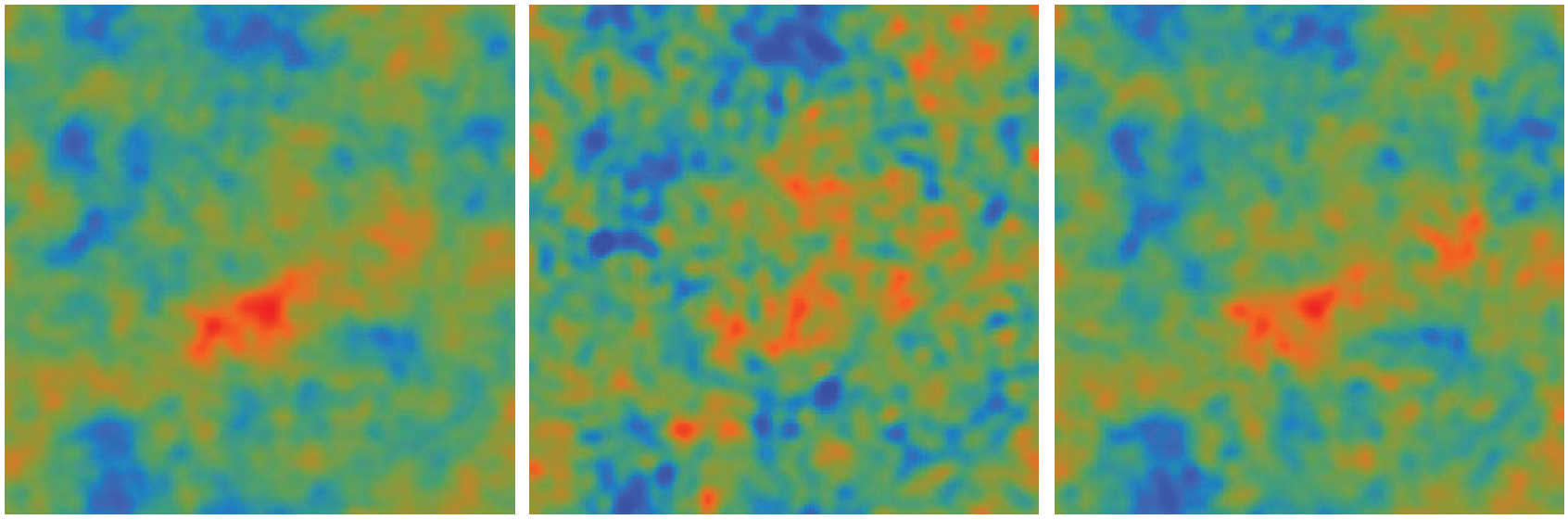
$$\langle x(\mathbf{l})x'(\mathbf{l}')\rangle_{\rm CMB} = f_\alpha(\mathbf{l},\mathbf{l}')\phi(\mathbf{l}+\mathbf{l}')\,,$$

  where $x \in$ temperature, polarization fields and f_α is a fixed weight that reflects geometry

- Each pair forms a noisy estimate of the potential or projected mass - just like a pair of galaxy shears
- Minimum variance weight all pairs to form an estimator of the lensing mass
- Averaging over lenses restore statistical isotropy so that the power spectrum still describes the two point correlation - in this sense lensing produces non-Gaussianity, not statistical anisotropy

Reconstruction from the CMB

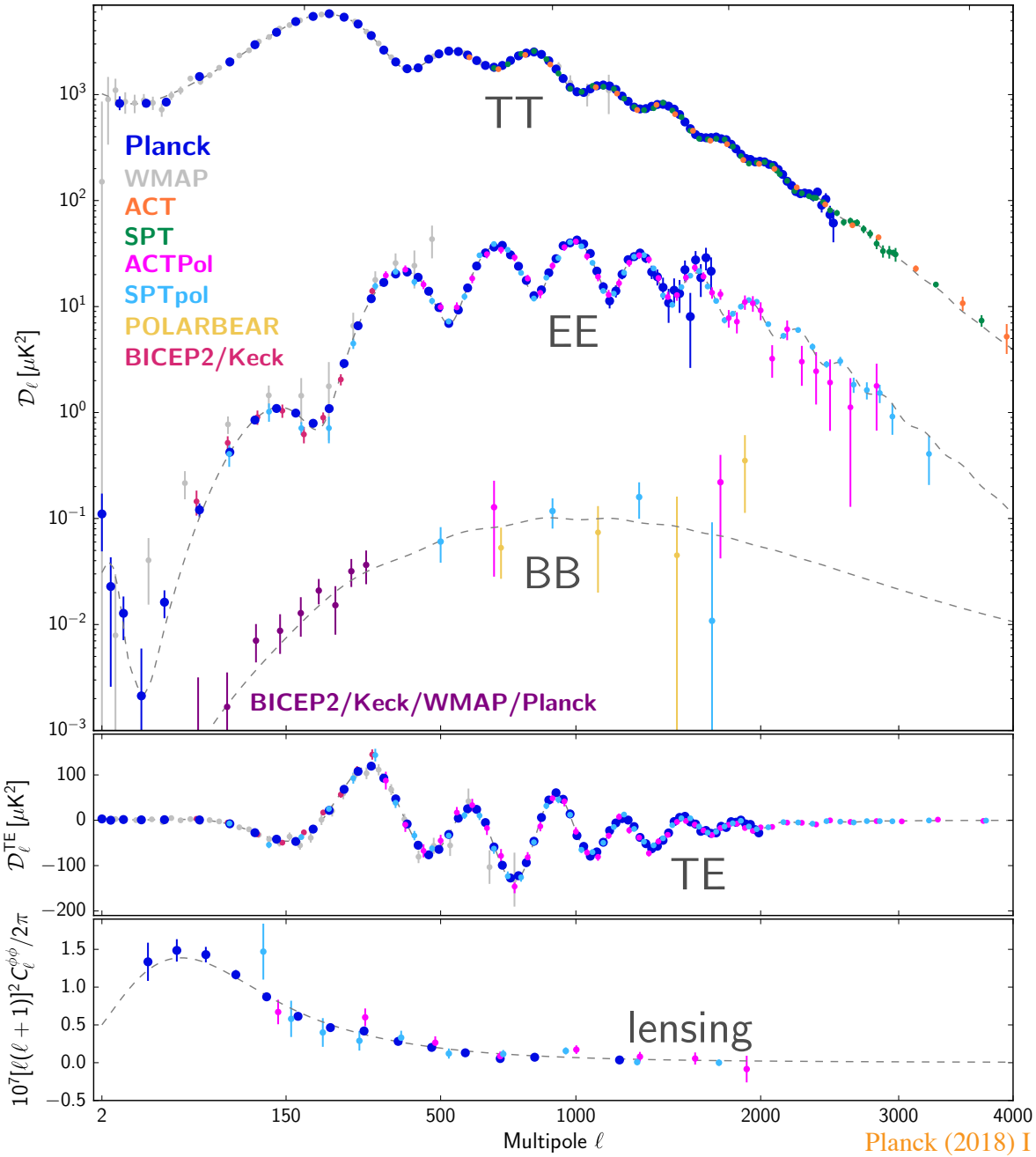
- With high signal-to-noise B-maps, here $1\mu\mathrm{K}'$, quadratic reconstruction signal-to-noise dominated by EB combination
- Use two-point correlations in reconstruction map to measure lensing power spectrum
- Use reconstruction to subtract out lensing contamination of gravitational wave B modes



Lensing Potential TT Reconstruction EB Reconstruction

CMB Power Spectra

- Power spectra of CMB
  - temperature
  - polarization
  - lensing



Planck (2018) I

End of 2021 Lectures

- Supplemental notes follow:

  Boltzmann Equation and Radiative Transfer

  Black Body Formation and Spectral Distortions

  SZ effect and Secondaries

Boltzmann Equation

- Now let's examine some of the formalism behind the previous results with an aim of also understanding

  Thermalization: formation of the blackbody

  Compton y and μ distortions: SZ effect as first step to thermalization with a hot electron plasma

- First let's connect it with other astro and particle courses where you have encountered these issues:

  Radiative transfer

  Particle transport

  and applied it to

  Fluid mechanics

  Thermal relics of big bang

Astro-Particle Dictionary

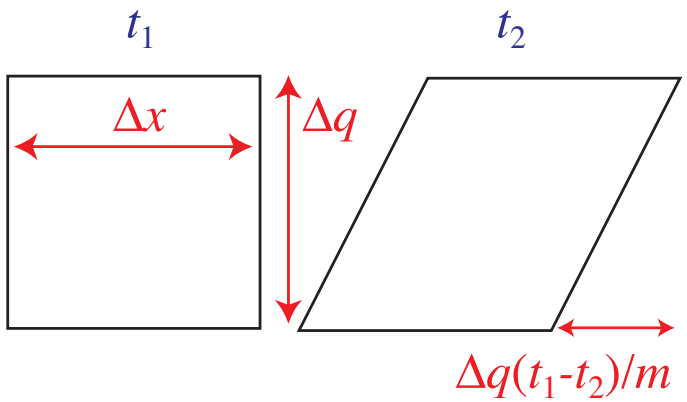
Astrophysicists and physicist use different words to describe same thing:

- Specific intensity $I_\nu = 4\pi\nu^3 f \leftrightarrow$ phase space distribution f

  $I_\nu = \Delta E/\Delta t \Delta\nu \Delta\Omega dA$: "energy per unit everything"
- Surface brightness conservation $\leftrightarrow$ Liouville equation
- Absorption, emission, scattering $\leftrightarrow$ Collision term
- Einstein relations $\leftrightarrow$ Single matrix element
- Radiative transfer equation $\leftrightarrow$ Boltzmann equation
- Eddington approximation $\leftrightarrow$ Fluid approximation
- Moments of $I_\nu \leftrightarrow$ Radiative viscosity
- Rosseland Approximation $\leftrightarrow$ Tight coupling approximation

Liouville Equation

- In absence of interactions, particle conservation implies that the phase space distribution is invariant along the trajectory of the particles



- Follow an element in Δx with spread Δq. For example for non relativistic particles a spread in velocity of $\Delta v = \Delta q/m$.
- After a time δt the low velocity tail will lag the high velocity tail by $\delta x = \Delta v \delta t = \Delta q \delta t/m$
- For ultrarelativistic particles $v = c$ and $\Delta v = 0$, so obviously true

Liouville Equation

- The phase space element can shear but preserves area $\Delta x \Delta q$
- This remains true under Lorentz and even a general coordinate transform
- Therefore $df/dt = 0$ or f is conserved when evaluated along the path of the particles
- Liouville Equation: $f \propto I_\nu/\nu^3$ and $ds = cdt$

$$\frac{df}{dt} = 0 \rightarrow \frac{dI_\nu}{ds} = 0$$

  if frequency is also conserved on the path
- This is the microphysical origin of surface brightness conservation (cf. lensing) – macro it is that flux $\propto r^{-2}$ and angular surface area $\propto r^{-2}$. Now what happens when frequency changes along the path...

Liouville Equation

- In general, expand out the total derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{dx_i}{dt} \frac{\partial f}{\partial x_i} + \frac{dq_i}{dt} \frac{\partial f}{\partial q_i} \right) = 0$$

- The spatial gradient terms are responsible for flow of particles in and out of a fixed volume
- The momentum derivative terms are responsible for redshift effects, including surface brightness diminishing as $(1+z)^4$.

Boltzmann Equation

- Heuristically

$$\frac{df}{dt} = \text{particle sources - sinks}$$

$$\frac{dI_\nu}{ds} = \text{emission - absorption}$$

  the r.h.s. is called the collision term and given as $C[f]$

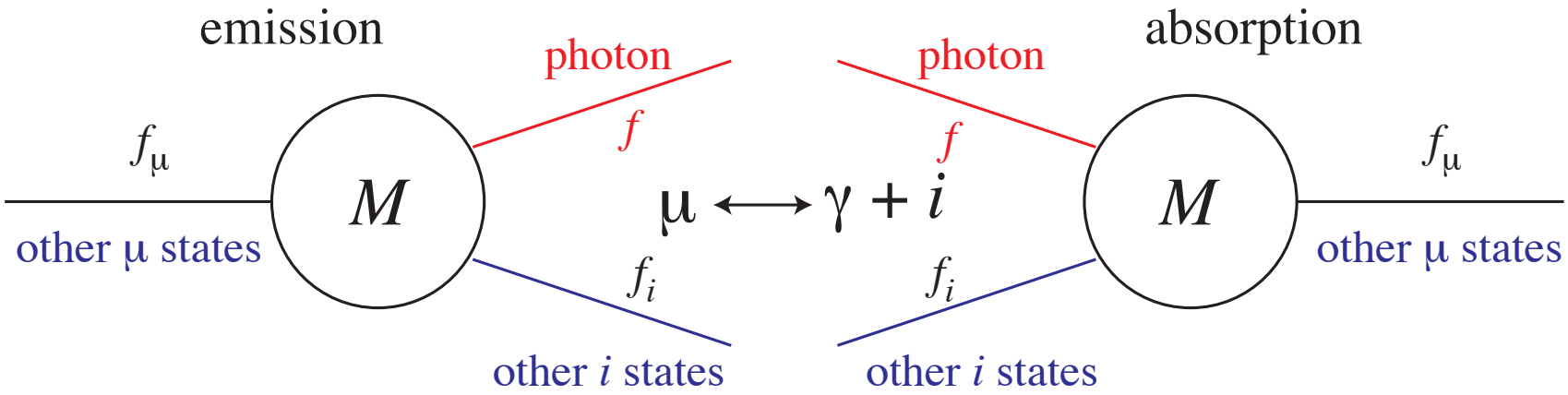
- Collision term: integrate over the phase space of incoming particles, connect to outgoing state with the matrix element of the transition M

- Form:

$$C[f] = \int d(\text{phase space})[\text{ energy-momentum conservation}] \times |M|^2[\text{emission} - \text{absorption}]$$

Boltzmann Equation

- Emission - absorption term involves the particle occupation of the various states
- For concreteness: take f to be the photon distribution function
- Interaction ($\gamma + \sum i \leftrightarrow \sum \mu$); sums are over all incoming and outgoing other particles



- [emission-absorption] $+ =$ boson; $- =$ fermion

$$\Pi_i \Pi_\mu f_\mu (1 \pm f_i)(1 \pm f) - \Pi_i \Pi_\mu (1 \pm f_\mu) f_i f$$

Boltzmann Equation

- Photon Emission: $f_\mu(1 \pm f_i)(1 + f)$

  f_μ: proportional to number of emitters

  $(1 \pm f_i)$: if final state is occupied and a fermion, process blocked; if boson the process enhanced

  $(1 + f)$: final state factor for photons: "1": spontaneous emission (remains if $f = 0$); "$+f$": stimulated and proportional to the occupation of final photon

- Photon Absorption: $-(1 \pm f_\mu) f_i f$

  $(1 \pm f_\mu)$: if final state is occupied and fermion, process blocked; if boson the process enhanced

  f_i: proportional to number of absorbers

  f: proportional to incoming photons

Boltzmann Equation

- If interactions are rapid they will establish an equilibrium distribution where the distribution functions no longer change $C[f_{\rm eq}] = 0$
- Solve by inspection

$$\Pi_i \Pi_\mu f_\mu (1 \pm f_i)(1 \pm f) - \Pi_i \Pi_\mu (1 \pm f_\mu) f_i f = 0$$

- Try $f_a = (e^{-E_a/T} \mp 1)^{-1}$ so that $(1 \pm f_a) = e^{-E_a/T}(e^{-E_a/T} \mp 1)^{-1}$

$$e^{-\sum (E_i + E)/T} - e^{-\sum E_\mu/T} = 0$$

and energy conservation says $E + \sum E_i = \sum E_\mu$, so identity is satisified if the constant T is the same for all species

Boltzmann Equation

- If the interaction does not create or destroy particles of type f (or types i, μ...) then the distribution

$$f_{\rm eq} = (e^{-(E-\mu)/T} \mp 1)^{-1}$$

also solves the equilibrium equation: e.g. a scattering type reaction

$$\gamma_E + i \rightarrow \gamma_{E'} + j$$

$$\sum E_i + (E - \mu) = \sum E_j + (E' - \mu) = 0$$

since the chemical potential μ does not depend on the photon energy, likewise if f is a fermion

- Not surprisingly, this is the Fermi-Dirac distribution for fermions and the Bose-Einstein distribution for bosons

Boltzmann Equation

- Even more generally, for a single reaction, the other species can carry chemical potentials too so long as

$$\sum \mu_i + \mu = \sum \mu_\nu$$

the law of mass action is satisfied

- This general rule applies to interactions that freely create or destroy the particles - e.g. $\gamma + e^- \rightarrow 2\gamma + e^-$

$$\mu_e + \mu = \mu_e + 2\mu \rightarrow \mu = 0$$

so that the chemical potential is driven to zero if particle number is not conserved in interaction

Poor Man's Boltzmann Equation

- Non expanding medium

$$\frac{\partial f}{\partial t} = \Gamma \left(f - f_{\rm eq} \right)$$

where Γ is some rate for collisions

- Add in expansion in a homogeneous medium

$$\frac{\partial f}{\partial t} + \frac{dq}{dt}\frac{\partial f}{\partial q} = \Gamma \left(f - f_{\rm eq} \right)$$

$$\left(q \propto a^{-1} \rightarrow \frac{1}{q}\frac{dq}{dt} = -\frac{1}{a}\frac{da}{dt} = H \right)$$

$$\frac{\partial f}{\partial t} - H\frac{\partial f}{\partial \ln q} = \Gamma \left(f - f_{\rm eq} \right)$$

- So equilibrium will be maintained if collision rate exceeds expansion rate $\Gamma > H$

Boltzmann Equation

- To actually compute the collision term or interaction rates: matrix $|M|^2$ or analogously the cross section for absorption defines all processes (the physical content of the Einstein relations)
- Expect that $\sigma \propto |M|^2$
- Integration over momentum state converts f's to n's

Line Transition

- Example: a line transition from single lower $i = 1$ state to upper $\mu = 2$ state assuming that outgoing states are not occupied
- Absorption: $-(1 \pm f_\mu) f_i f \rightarrow -n_1 f$, $|M|^2 \rightarrow \sigma$, $2h\nu^3 f/c^2 \rightarrow I_\nu$ so that $\alpha_\nu|_{\text{true absorption}} = n_1\sigma$
- Emission: $f_\mu(1 \pm f_i)(1 + f) \rightarrow n_2(1 + f) = n_2 + n_2 f$ so that spontaneous emission $j_\nu \sim n_2\sigma \cdot 2\nu^3 h/c^2$ and stimulated emission is negative absorption with $\alpha_\nu|_{\text{stim emiss}} \sim -n_2\sigma$
- Implies a source function

$$S_\nu = j_\nu/\alpha_\nu \sim \frac{1}{n_1/n_2 - 1}\frac{2h\nu^3}{c^2}$$

Line Transition

- More generally, the full Einstein relationship is

$$S_\nu = j_\nu/\alpha_\nu = \frac{1}{(n_1 g_2/n_2 g_1 - 1)} \frac{2h\nu^3}{c^2}$$

  where degeneracy factors appear for levels that have multiple states
- Interactions drive I_ν to S_ν which nulls the rhs radiative trans. eqn.
- Likewise collisions drive f to some equilibrium distribution and then remains constant thereafter in spite of further collisions $\rightarrow$ black body distribution

- Now let's turn to the Boltzmann equation relevant for CMB

Compton Collision Term

- Collision term for compton scattering (set $\hbar = c = k = 1$ and neglect Pauli blocking and polarization)

$$C[f] = \frac{1}{2E(p_f)} \int \frac{d^3p_i}{(2\pi)^3} \frac{1}{2E(p_i)} \int \frac{d^3q_f}{(2\pi)^3} \frac{1}{2E(q_f)} \int \frac{d^3q_i}{(2\pi)^3} \frac{1}{2E(q_i)}$$
$$\times (2\pi)^4 \delta(p_f + q_f - p_i - q_i) |M|^2$$
$$\times \{f_e(q_i) f(p_i)[1 + f(p_f)] - f_e(q_f) f(p_f)[1 + f(p_i)]\}$$

where the matrix element is calculated in field theory and is Lorentz invariant. In terms of the rest frame $\alpha = e^2/\hbar c$ (cf. Klein Nishina Cross Section)

$$|M|^2 = 2(4\pi)^2 \alpha^2 \left[\frac{E(p_i)}{E(p_f)} + \frac{E(p_f)}{E(p_i)} - \sin^2 \beta \right]$$

with β as the rest frame scattering angle

Liouville Equation

- In absence of scattering, the phase space distribution of photons in each polarization state a is conserved along the propagation path
- Rewrite variables in terms of the photon propagation direction $\mathbf{q} = q\hat{\mathbf{n}}$, so $f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta)$ and

$$\frac{D}{D\eta} f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta) = 0 = \left(\frac{\partial}{\partial \eta} + \frac{d\mathbf{x}}{d\eta} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{d\hat{\mathbf{n}}}{d\eta} \cdot \frac{\partial}{\partial \hat{\mathbf{n}}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a$$

- For simplicity, assume spatially flat universe $K = 0$ then $d\hat{\mathbf{n}}/d\eta = 0$ and $d\mathbf{x} = \hat{\mathbf{n}} d\eta$

$$\dot{f}_a + \hat{\mathbf{n}} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0$$

- The spatial gradient describes the conversion from inhomogeneity to anisotropy and the $\dot{q}$ term the gravitational sources.

Photon Moments

- The photon stress-energy tensor is given by moments of distribution function

$$T^{\mu}{}_{\nu} = g \int \frac{d^3q}{(2\pi)^3} \frac{q^\mu q_\nu}{E(q)} f$$

- $\ell = 0$ Boltzmann moment is continuity equation: $\Theta_0^{(0)} = \delta\rho_\gamma/4\rho_\gamma$
- $\ell = 1$ moment is Navier-Stokes equation with $\Theta_1^{(m)} = v_\gamma^{(m)}$ and

$$\Theta_2^{(0)} = \frac{5}{12}(1 - 3K/k^2)^{1/2}\Pi_\gamma^{(0)}$$

and similarly up to normalization for vector and tensor cases

Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies
- Spatial gradient term hits plane wave:

$$\hat{\mathbf{n}} \cdot \nabla e^{i\mathbf{k}\cdot\mathbf{x}} = i\hat{\mathbf{n}} \cdot \mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} = i\sqrt{\frac{4\pi}{3}} k Y_1^0(\hat{\mathbf{n}}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

- Dipole term adds to angular dependence through the addition of angular momentum

$$\sqrt{\frac{4\pi}{3}} Y_1^0 Y_\ell^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell+1)(2\ell-1)}} Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell+1)(2\ell+3)}} Y_{\ell+1}^m$$

where $\kappa_\ell^m = \sqrt{\ell^2 - m^2}$ is given by Clebsch-Gordon coefficients.

Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

$$\dot{\Theta}_\ell^{(m)} = k\left[\frac{\kappa_\ell^m}{2\ell+1}\Theta_{\ell-1}^{(m)} - \frac{\kappa_{\ell+1}^m}{2\ell+3}\Theta_{\ell+1}^{(m)}\right] - \dot{\tau}\Theta_\ell^{(m)} + S_\ell^{(m)}$$

  where $S_\ell^{(m)}$ are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic $\ell = 0$ temperature perturbation will eventually become a high order anisotropy by "free streaming" or simple projection

- Original CMB codes solved the full hierarchy equations out to the ℓ of interest.

Gravitational Source

- Either extract the source $S_\ell^{(m)}$ from these associations or by noting that the geodesic equation gives the redshifting term,
  $g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu})$

$$\frac{\dot{q}}{q} = -\frac{\dot{a}}{a} - \frac{1}{2}n^i n^j \dot{h}_{ij} - n^i \dot{h}_{0i} - \frac{1}{2}n^i \nabla_i h_{00}$$

- We can further separate out the pieces of the metric fluctuation h_{ij} in a 3+1 ADM split

$$h_{00} = -2A, \quad h_{0i} = -B_i, \quad h_{ij} = 2H_L\delta_{ij} + 2H_{T\,ij}$$

  and Fourier decompose these metric perturbation fields in their scalar (0), vector (1), and tensor (2) components

Source Terms

- Temperature source terms $S_l^{(m)}$ (rows $\pm|m|$; flat assumption

$$\begin{pmatrix} \dot{\tau}\Theta_0^{(0)} - \dot{H}_L^{(0)} & \dot{\tau}v_b^{(0)} + \dot{B}^{(0)} & \dot{\tau}P^{(0)} - \frac{2}{3}\dot{H}_T^{(0)} \\ 0 & \dot{\tau}v_b^{(\pm 1)} + \dot{B}^{(\pm 1)} & \dot{\tau}P^{(\pm 1)} - \frac{\sqrt{3}}{3}\dot{H}_T^{(\pm 1)} \\ 0 & 0 & \dot{\tau}P^{(\pm 2)} - \dot{H}_T^{(\pm 2)} \end{pmatrix}$$

  where $\dot{\tau} \equiv n_e \sigma_T a$ terms are Thomson scattering sources with

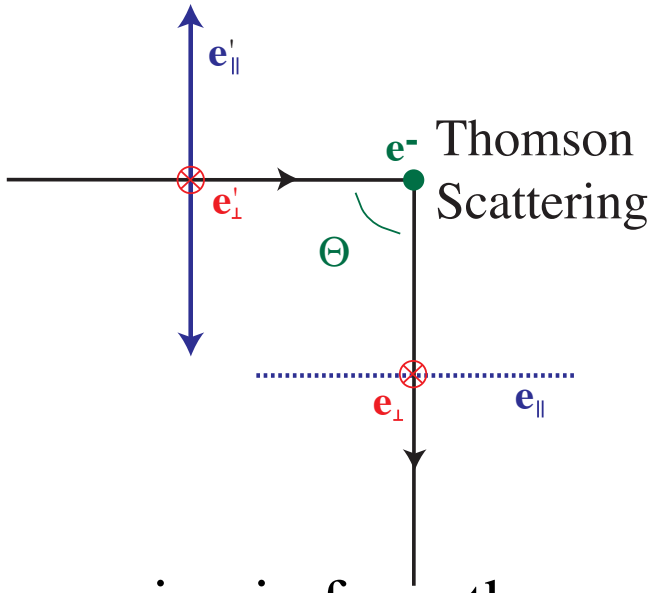
$$P^{(m)} \equiv \frac{1}{10}(\Theta_2^{(m)} - \sqrt{6}E_2^{(m)})$$

- Polarization source terms are generated through Thomson scattering from temperature quadrupoles

$$\frac{d\sigma_T}{d\Omega} = \frac{3}{8\pi}|\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \,,$$

Polarized Source Term

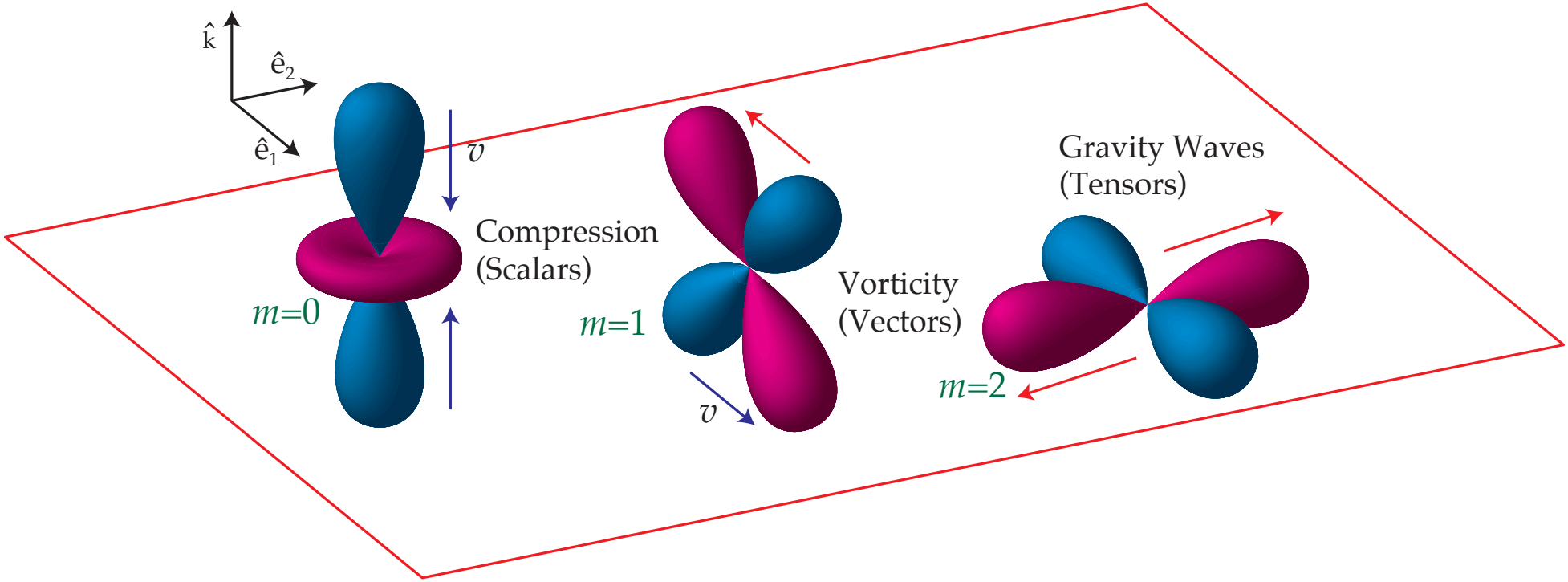
- Heuristically, incoming photon electric field accelerates electron in the same direction and radiates out a photon whose polarization is given by projection of this direction in transverse plane



- Consider scattering by 90 degrees: photons coming in from the left/right supply one polarization state, in/out of page the other
- A quadrupole temperature anisotropy in left/right vs top/bottom leads to net linear polarization
- Polarization source term

$$\mathcal{E}_\ell^{(m)} = -\dot{\tau}\sqrt{6}P^{(m)}\delta_{\ell,2} \qquad \mathcal{B}_\ell^{(m)} = 0$$

Quadrupole Source Term



- Each type leads to quadrupoles with different azimuthal symmetry, polarization aligned with cold lobe
- For the vector and tensor cases, the breaking of azimuthal symmetry leads to B-mode polarization

Gravitational Wave Observability

- A gravitational wave makes a quadrupolar (transverse-traceless) distortion to metric
- Just like the scale factor or spatial curvature, a temporal variation in its amplitude leaves a residual temperature variation in CMB photons – here anisotropic
- Before recombination, anisotropic variation is eliminated by scattering
- Gravitational wave temperature effect drops sharply at the horizon scale at recombination - distorts the spectrum
- Source to polarization goes as $\dot{\tau}/\dot{H}_T$ and peaks at the horizon not damping scale
- B modes since symmetry of plane wave broken by the transverse nature of gravity wave polarization

Truncated Hierarchy

- CMBFast introduced the hybrid truncated hierarchy, integral solution technique
- Formal integral solution contains sources that are not external to system but defined through the Boltzmann hierarchy itself
- Solution: recall the fluid approximation where interactions suppress all but the $\ell = 0$ (density) and $\ell = 1$ (velocity) terms
- CMBFast extends this idea by solving a truncated hierarchy of equations, e.g. out to $\ell = 25$ with non-reflecting boundary conditions

- For completeness, we explicitly derive the scattering source term via polarized radiative transfer in the last part of the notes

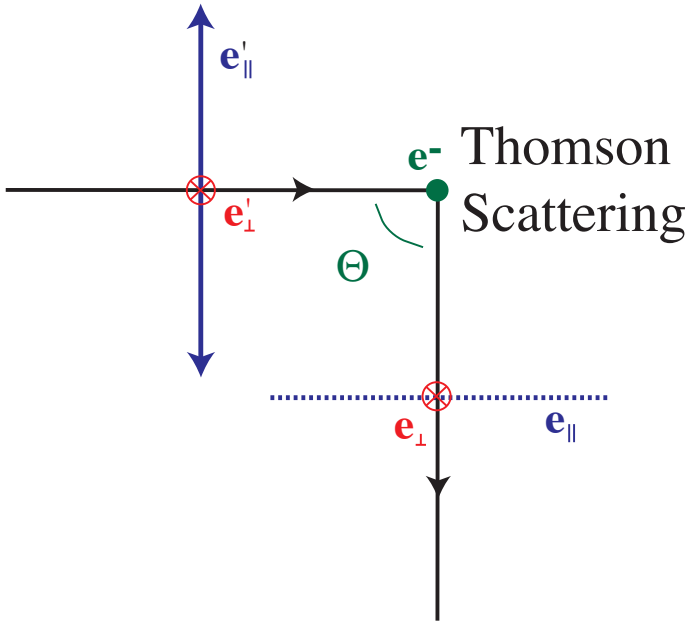
Polarized Radiative Transfer

- Define a specific intensity "vector": $\mathbf{I}_\nu = (\Theta_\parallel, \Theta_\perp, U, V)$ where $\Theta = \Theta_\parallel + \Theta_\perp$, $Q = \Theta_\parallel - \Theta_\perp$

$$\frac{d\mathbf{I}_\nu}{d\eta} = \dot{\tau}(\mathbf{S}_\nu - \mathbf{I}_\nu)$$

- Thomson collision
  based on differential cross section

$$\frac{d\sigma_T}{d\Omega} = \frac{3}{8\pi}|\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \,,$$



Polarized Radiative Transfer

- $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.
- Thomson scattering by 90 deg: $\Theta_\perp \rightarrow \Theta_\perp$ but $\Theta_\parallel$ does not scatter
- More generally if β is the scattering angle

$$\mathbf{S}_\nu = \frac{3}{8\pi} \int d\Omega' \begin{pmatrix} \cos^2\beta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & \cos\beta \end{pmatrix} \mathbf{I}'_\nu$$

- But to calculate Stokes parameters in a fixed coordinate system must rotate into the scattering basis, scatter and rotate back out to the fixed coordinate system

Thomson Collision Term

- The $U \rightarrow U'$ transfer follows by writing down the polarization vectors in the 45° rotated basis

$$\hat{\mathbf{E}}_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_\parallel + \hat{\mathbf{E}}_\perp), \qquad \hat{\mathbf{E}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_\parallel - \hat{\mathbf{E}}_\perp)$$

- Define the temperature in this basis

$$\begin{aligned}
\Theta_1 &\propto |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 + |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta'_1 + \frac{1}{4}(\cos\beta - 1)^2 \Theta'_2 \\
\Theta_2 &\propto |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 + |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta'_2 + \frac{1}{4}(\cos\beta - 1)^2 \Theta'_1
\end{aligned}$$

or $\Theta_1 - \Theta_2 \propto \cos\beta(\Theta'_1 - \Theta'_2)$

Scattering Matrix

- Transfer matrix of Stokes state $\mathbf{T} \equiv (\Theta, Q + iU, Q - iU)$

$$\mathbf{T} \propto \mathbf{S}(\beta)\mathbf{T}'$$

$$\mathbf{S}(\beta) = \frac{3}{4}\begin{pmatrix} \cos^2\beta + 1 & -\frac{1}{2}\sin^2\beta & -\frac{1}{2}\sin^2\beta \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta + 1)^2 & \frac{1}{2}(\cos\beta - 1)^2 \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta - 1)^2 & \frac{1}{2}(\cos\beta + 1)^2 \end{pmatrix}$$

normalization factor of 3 is set by photon conservation in scattering

Scattering Matrix

- Transform to a fixed basis, by a rotation of the incoming and outgoing states $\mathbf{T} = \mathbf{R}(\psi)\mathbf{T}$ where

$$\mathbf{R}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & e^{2i\psi} \end{pmatrix}$$

giving the scattering matrix

$$\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha) = \tag{1}$$

$$\frac{1}{2}\sqrt{\frac{4\pi}{5}} \begin{pmatrix} Y_2^0(\beta,\alpha) + 2\sqrt{5}Y_0^0(\beta,\alpha) & -\sqrt{\tfrac{3}{2}}Y_2^{-2}(\beta,\alpha) & -\sqrt{\tfrac{3}{2}}Y_2^{2}(\beta,\alpha) \\ -\sqrt{6}\,{}_2Y_2^0(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^{-2}(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^{2}(\beta,\alpha)e^{2i\gamma} \\ -\sqrt{6}\,{}_{-2}Y_2^0(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^{-2}(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^{2}(\beta,\alpha)e^{-2i\gamma} \end{pmatrix} \tag{2}$$

Addition Theorem for Spin Harmonics

- Spin harmonics are related to rotation matrices as

$$ {}_sY_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\mathcal{D}^\ell_{-ms}(\phi,\theta,0) $$

Note: for explicit evaluation sign convention differs from usual (e.g. Jackson) by $(-1)^m$

- Multiplication of rotations

$$ \sum_{m''} \mathcal{D}^\ell_{mm''}(\alpha_2,\beta_2,\gamma_2)\mathcal{D}^\ell_{m''m}(\alpha_1,\beta_1,\gamma_1) = \mathcal{D}^\ell_{mm'}(\alpha,\beta,\gamma) $$

- Implies

$$ \sum_m {}_{s_1}Y_\ell^{m*}(\theta',\phi') \; {}_{s_2}Y_\ell^m(\theta,\phi) = (-1)^{s_1-s_2}\sqrt{\frac{2\ell+1}{4\pi}} \; {}_{s_2}Y_\ell^{-s_1}(\beta,\alpha)e^{is_2\gamma} $$

Sky Basis

- Scattering into the state (rest frame)

$$C_{\rm in}[\mathbf{T}] = \dot{\tau}\int \frac{d\hat{\mathbf{n}}'}{4\pi}\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha)\mathbf{T}(\hat{\mathbf{n}}')\,,$$

$$= \dot{\tau}\int \frac{d\hat{\mathbf{n}}'}{4\pi}(\Theta', 0, 0) + \frac{1}{10}\dot{\tau}\int d\hat{\mathbf{n}}' \sum_{m=-2}^{2} \mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}')\mathbf{T}(\hat{\mathbf{n}}')\,.$$

where the quadrupole coupling term is $\mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') =$

$$\begin{pmatrix} Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\,{}_2Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\,{}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_2Y_2^m(\hat{\mathbf{n}}) & 3\,{}_2Y_2^{m*}(\hat{\mathbf{n}}')\,{}_2Y_2^m(\hat{\mathbf{n}}) & 3\,{}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_2Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\,{}_2Y_2^{m*}(\hat{\mathbf{n}}')\,{}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\,{}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_{-2}Y_2^m(\hat{\mathbf{n}}) \end{pmatrix},$$

expression uses angle addition relation above. We call this term C_Q.

Scattering Matrix

- Full scattering matrix involves difference of scattering into and out of state

$$C[\mathbf{T}] = C_{\rm in}[\mathbf{T}] - C_{\rm out}[\mathbf{T}]$$

- In the electron rest frame

$$C[\mathbf{T}] = \dot{\tau} \int \frac{d\hat{\mathbf{n}}'}{4\pi}(\Theta', 0, 0) - \dot{\tau}\mathbf{T} + C_Q[\mathbf{T}]$$

which describes isotropization in the rest frame. All moments have $e^{-\tau}$ suppression except for isotropic temperature Θ_0.
Transformation into the background frame simply induces a dipole term

$$C[\mathbf{T}] = \dot{\tau} \left(\hat{\mathbf{n}} \cdot \mathbf{v}_b + \int \frac{d\hat{\mathbf{n}}'}{4\pi}\Theta', 0, 0 \right) - \dot{\tau}\mathbf{T} + C_Q[\mathbf{T}]$$

Integral Solution

- Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface
- In general, the solution describes the decomposition of the source $S_\ell^{(m)}$ with its local angular dependence as seen at a distance D.
- Proceed by decomposing the angular dependence of the plane wave

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} j_\ell(kD) Y_\ell^0(\hat{\mathbf{n}})$$

- Recouple to the local angular dependence of G_ℓ^m

$$G_{\ell_s}^m = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} \alpha_{\ell_s \ell}^{(m)}(kD) Y_\ell^m(\hat{\mathbf{n}})$$

Integral Solution

- Projection kernels:

$$\alpha^{(m=0)}_{\ell_s=0\ell} \equiv j_\ell \qquad \alpha^{(m=0)}_{\ell_s=1\ell} \equiv j'_\ell$$

- Integral solution:

$$\frac{\Theta^{(m)}_\ell(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S^{(m)}_{\ell_s} \alpha^{(m)}_{\ell_s\ell}(k(\eta_0-\eta))$$

- Power spectrum:

$$C_\ell = 4\pi \int \frac{dk}{k} \frac{k^3}{2\pi^2} \sum_m \frac{\langle \Theta^{(m)*}_\ell \Theta^{(m)}_\ell \rangle}{(2\ell+1)^2}$$

- Integration over an oscillatory radial source with finite width - suppression of wavelengths that are shorter than width leads to reduction in power by $k\Delta\eta/\ell$ in the "Limber approximation"

Polarization Hierarchy

- In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hierarchy:

$$\dot{E}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell+1)}B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}E_{\ell+1}^{(m)}\right] - \dot{\tau}E_\ell^{(m)} + \mathcal{E}_\ell^{(m)}$$

$$\dot{B}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell+1)}E_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}B_{\ell+1}^{(m)}\right] - \dot{\tau}B_\ell^{(m)} + \mathcal{B}_\ell^{(m)}$$

where ${}_2\kappa_\ell^m = \sqrt{(\ell^2-m^2)(\ell^2-4)/\ell^2}$ is given by the Clebsch-Gordon coefficients and $\mathcal{E}$, $\mathcal{B}$ are the sources (scattering only).

- Note that for vectors and tensors $|m| > 0$ and B modes may be generated from E modes by projection. Cosmologically $\mathcal{B}_\ell^{(m)} = 0$

Polarization Integral Solution

- Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

$$\frac{E_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \epsilon_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

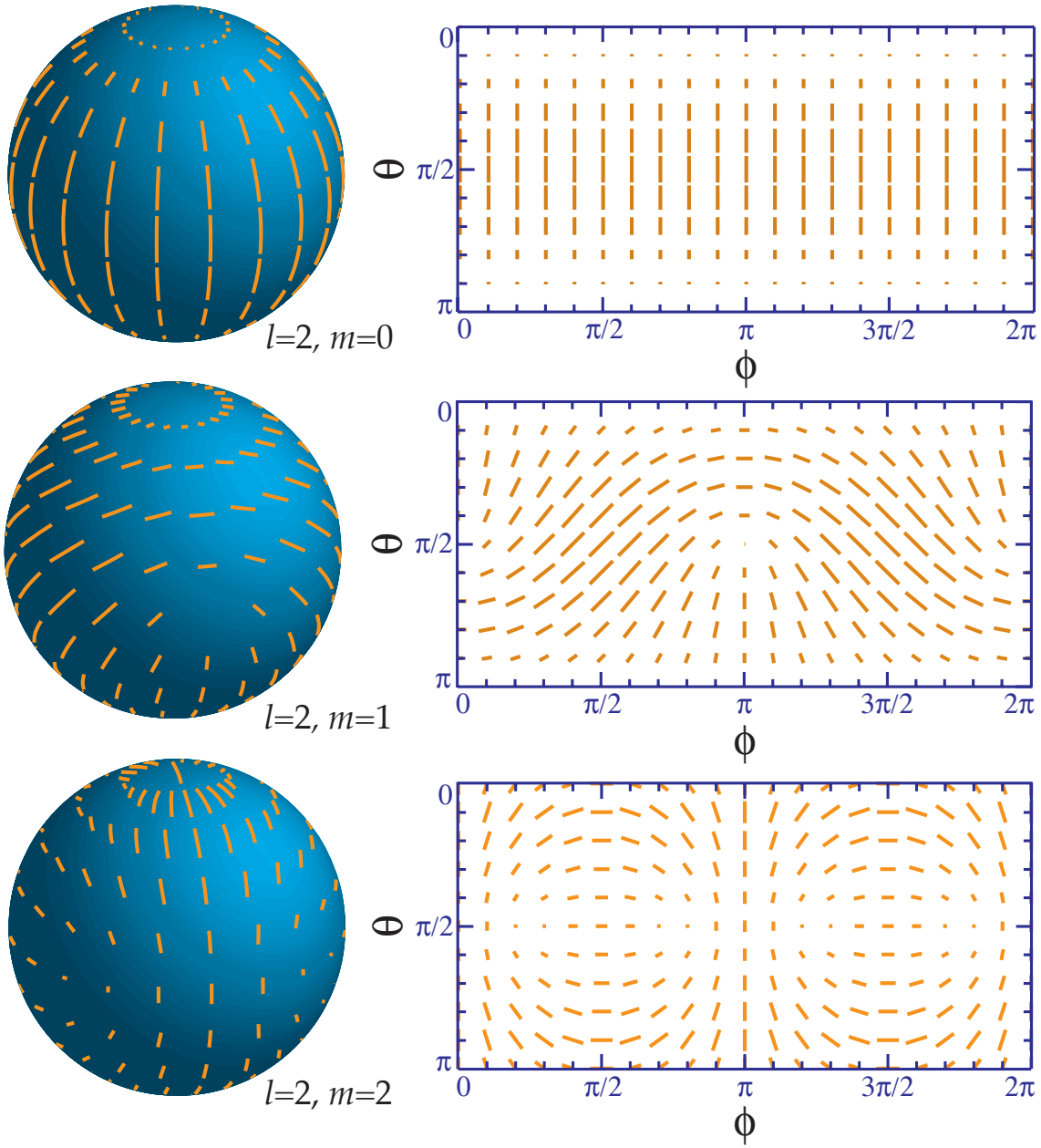
$$\frac{B_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \beta_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

- Power spectrum $XY = \Theta\Theta, \Theta E, EE, BB$:

$$C_\ell^{XY} = 4\pi \int \frac{dk}{k} \frac{k^3}{2\pi^2} \sum_m \frac{\langle X_\ell^{(m)*} Y_\ell^{(m)} \rangle}{(2\ell+1)^2}$$

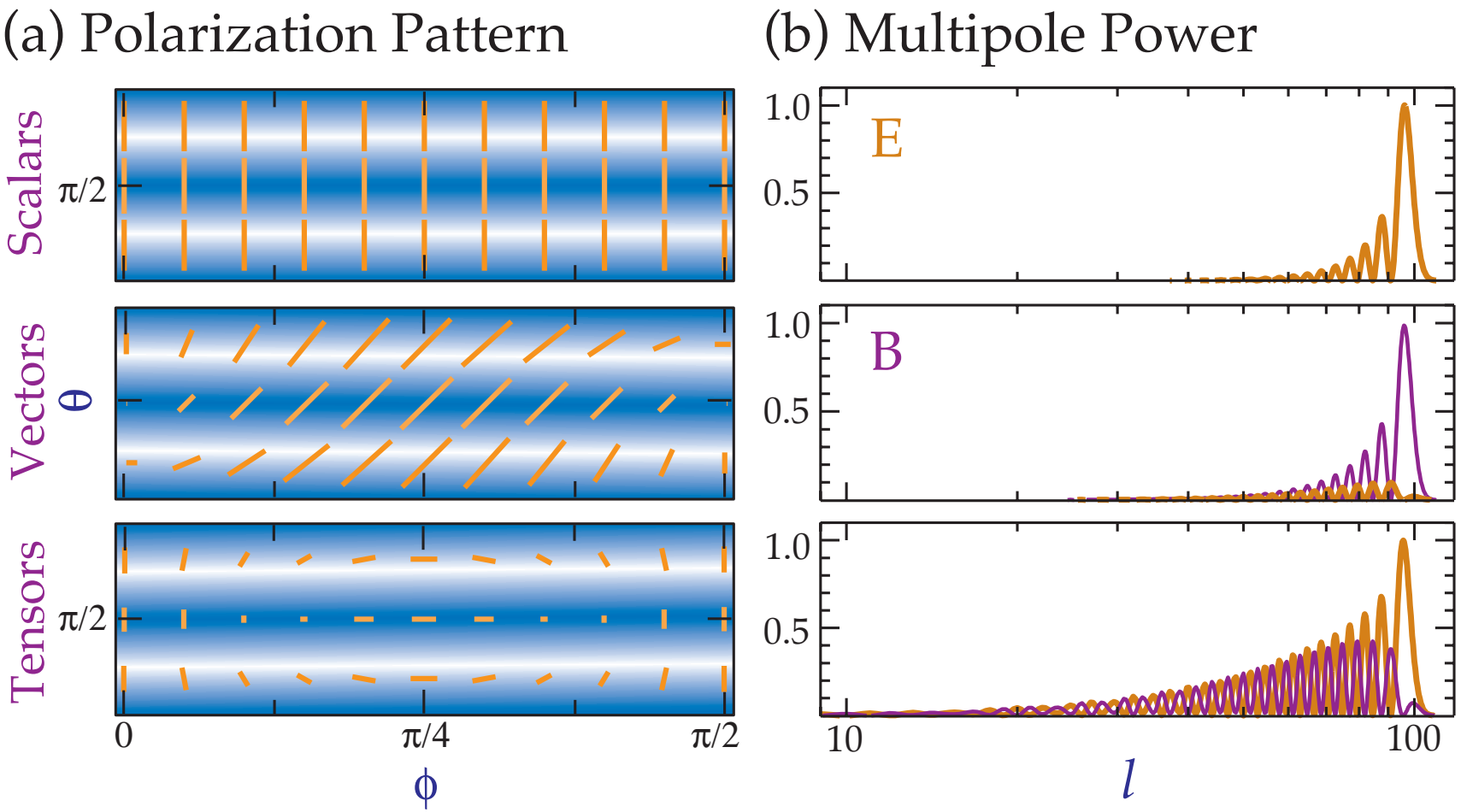
- We shall see that the only sources of temperature anisotropy are $\ell = 0, 1, 2$ and polarization anisotropy $\ell = 2$
- In the basis of $\hat{\mathbf{z}} = \hat{\mathbf{k}}$ there are only $m = 0, \pm 1, \pm 2$ or scalar, vector and tensor components

Polarization Sources



Polarization Transfer

- A polarization source function with $\ell = 2$, modulated with plane wave orbital angular momentum
- Scalars have no B mode contribution, vectors mostly B and tensor comparable B and E



Thermalization and Spectral Distortions

- Full Boltzmann equation with Compton scattering (set $\hbar = c = k = 1$ and neglect Pauli blocking and polarization)

$$\frac{\partial f}{\partial t} = \frac{1}{2E(p_f)} \int \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E(p_i)} \int \frac{d^3 q_f}{(2\pi)^3} \frac{1}{2E(q_f)} \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E(q_i)}$$
$$\times (2\pi)^4 \delta(p_f + q_f - p_i - q_i) |M|^2$$
$$\times \{ f_e(q_i) f(p_i)[1 + f(p_f)] - f_e(q_f) f(p_f)[1 + f(p_i)] \}$$

where the matrix element is calculated in field theory and is Lorentz invariant. In terms of the rest frame $\alpha = e^2/\hbar c$ (c.f. Klein Nishina Cross Section)

$$|M|^2 = 2(4\pi)^2 \alpha^2 \left[\frac{E(p_i)}{E(p_f)} + \frac{E(p_f)}{E(p_i)} - \sin^2 \beta \right]$$

with β as the rest frame scattering angle

Kompaneets Equation

- The Kompaneets equation is the radiative transfer equation in the limit that electrons are thermal

$$f_e = e^{-(m-\mu)/T_e} e^{-q^2/2mT_e} \qquad \left[n_e = e^{-(m-\mu)/T_e} \left(\frac{mT_e}{2\pi} \right)^{3/2} \right]$$

$$= \left(\frac{2\pi}{mT_e} \right)^{3/2} n_e e^{-q^2/2mT_e}$$

and assume that the energy transfer is small (non-relativistic electrons, $E_i \ll m$)

$$\frac{E_f - E_i}{E_i} \ll 1 \qquad [\mathcal{O}(T_e/m, E_i/m)]$$

second order Doppler effect $T_e \propto \langle v^2 \rangle$ and electron recoil

Kompaneets Equation

- Kompaneets equation has derivatives of f because of this expansion

$$\frac{\partial f}{\partial t} = n_e \sigma_T \left(\frac{T_e}{m}\right) \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial f}{\partial x} + f(1+f) \right) \right] \qquad x = E/T_e$$

- Equilibrium solution must still be a Bose-Einstein distribution $\partial f / \partial t = 0$

$$\left[x^4 \left(\frac{\partial f}{\partial x} + f(1+f) \right) \right] = K$$

$$\frac{\partial f}{\partial x} + f(1+f) = \frac{K}{x^4}$$

Kompaneets Equation

- Assume that as $x \to 0$, $f \to 0$ then $K = 0$ and

$$\frac{df}{dx} = -f(1+f) \qquad \to \frac{df}{f(1+f)} = dx$$

$$\ln \frac{f}{1+f} = -x + c \qquad \to \frac{f}{1+f} = e^{-x+c}$$

$$f = \frac{e^{-x+c}}{1 - e^{-x+c}} = \frac{1}{e^{x-c} - 1}$$

Kompaneets Equation

- More generally, no evolution in the number density

$$n_\gamma \propto \int d^3p f \propto \int dx x^2 f$$

$$\frac{\partial n_\gamma}{\partial t} \propto \int dx x^2 \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial f}{\partial x} + f(1+f)\right)\right]$$

$$\propto x^4\left[\frac{\partial f}{\partial x} + f(1+f)\right]_0^\infty = 0$$

- Energy evolution $R \equiv n_e \sigma_T (T_e/m)$

$$\rho = 2\int \frac{d^3p}{(2\pi)^3} Ef = 2\int \frac{p^3 dp}{2\pi^2} f = \frac{T_e^4}{\pi^2}\int x^3 dx f$$

$$\frac{\partial \rho}{\partial t} = \frac{T_e^4}{\pi^2} R \int dx x \frac{\partial}{\partial x}\left[x^4\left(\frac{\partial f}{\partial x} + f(1+f)\right)\right]$$

Kompaneets Equation

- Integrate by parts

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\frac{T_e^4}{\pi^2} R \int dx x^4 \left(\frac{\partial f}{\partial x} + f(1+f) \right) \\ &= \frac{T_e^4}{\pi^2} R \int dx 4x^3 f - \frac{T_e^4}{\pi^2} R \int dx x^4 f(1+f) \\ &= 4R\rho - \frac{T_e^4}{\pi^2} R \int dx x^4 f(1+f)\end{aligned}$$

Change in energy is difference between Doppler and recoil

- If f is a Bose-Einstein distribution at temperature T_γ

$$\frac{\partial f}{\partial x_\gamma} = -f(1+f) \qquad x_\gamma = \frac{E}{T_\gamma}$$

$$\int dx x^4 f(1+f) = -\int dx x^4 \frac{\partial f}{\partial x_\gamma} = \int dx 4x^3 \frac{dx}{dx_\gamma} f$$

Kompaneets Equation

- Radiative transfer equation for energy density

$$\frac{\partial \rho}{\partial t} = 4 n_e \sigma_T \frac{T_e}{m} \left[1 - \frac{T_\gamma}{T_e} \right] \rho$$

$$\frac{1}{\rho} \frac{\partial \rho}{\partial t} = 4 n_e \sigma_T \frac{(T_e - T_\gamma)}{m}$$

- The analogue to the optical depth for energy transfer is the Compton y parameter

$$d\tau = n_e \sigma_t dt = n_e \sigma_T ds$$

$$dy = \frac{(T_e - T_\gamma)}{m} d\tau$$

- Notice $dy \propto n_e T_e ds$, i.e. y is the line of sight integral of the electron gas pressure $p_e = n_e T_e$

Kompaneets Equation

- Radiative transfer equation for spectral distortion
- Rewrite Kompaneets equation with y as the time variable
- Assume that initial distribution is a blackbody at temperature $T_\gamma \neq T_e$ on the RHS
- Integrate in the $y \ll 1$ limit

$$\frac{\Delta f}{f} = -y x_\gamma e^{x_\gamma} \left(4 - x_\gamma \coth \frac{x_\gamma}{2}\right)$$

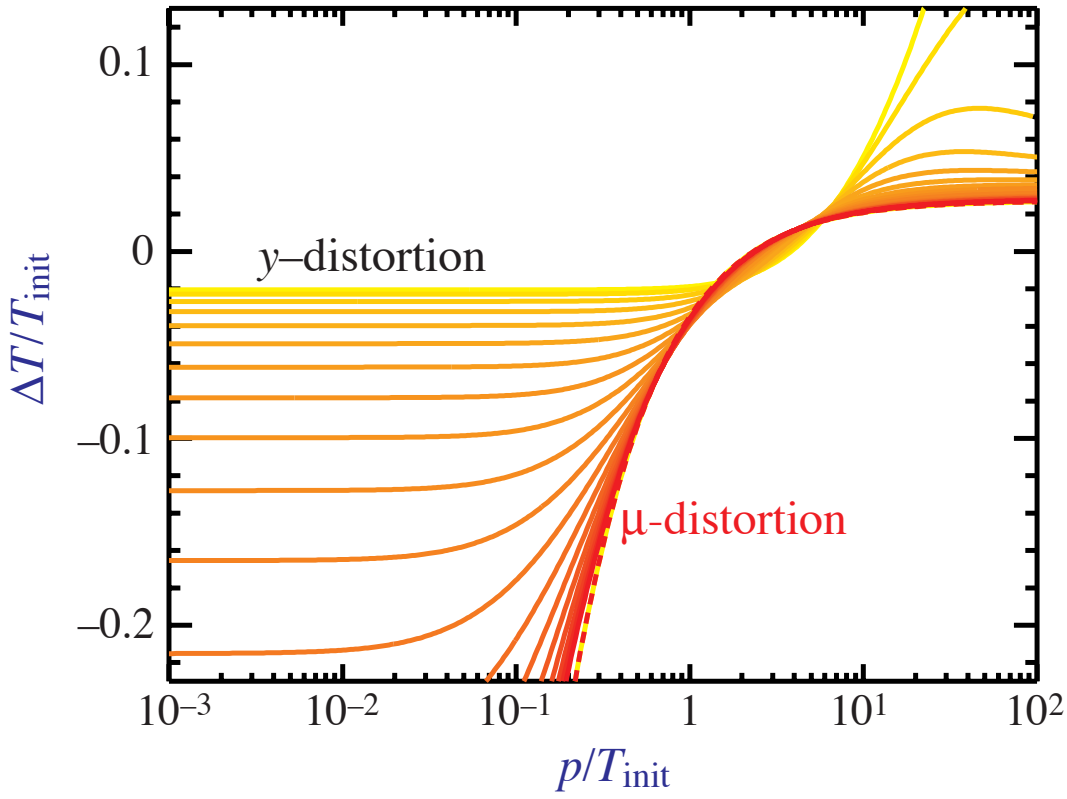
- Deficit in Rayleigh-Jeans ($= -2y$), excess in Wien, null at $x_\gamma = 3.83$ or 217GHz
- "Compton-y" spectral distortion

SZ effect

- Example: hot X-ray cluster with $kT \sim$ keV and the CMB: $T_e \gg T_\gamma$
- Inverse Compton scattering transfers energy to the photons while conserving the photon number
- Optically thin conditions: low energy photons boosted to high energy leaving a deficit in the number density in the RJ tail and an enhancement in the Wien tail called a Compton-y distortion — see problem set
- Compton scattering off high energy electrons can give low energy photons a large boost in energy but cannot create the photons in the first place

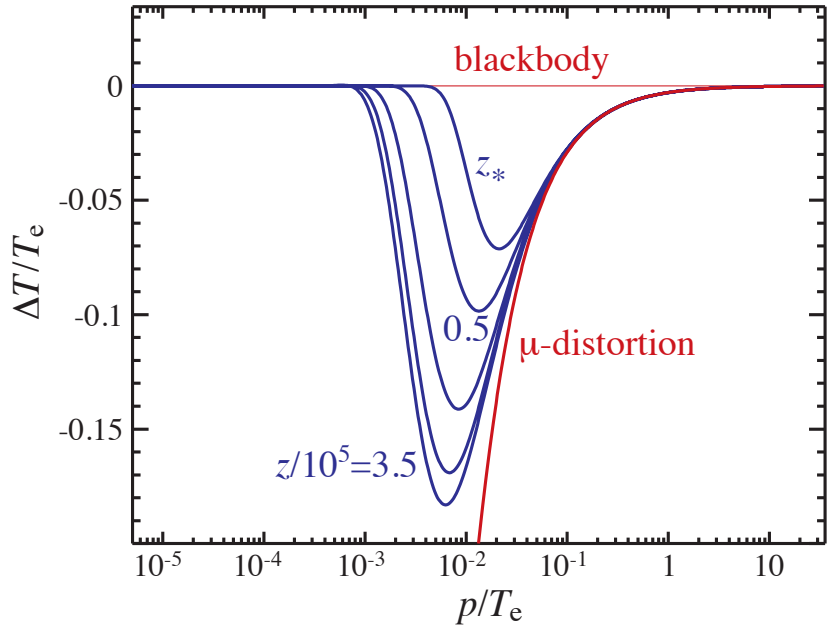
Kompaneets Equation

- Numerical solution of the Kompaneets equation going from a Compton-y distortion to a chemical potential distortion of a blackbody



Black Body Formation

- After $z \sim 10^6$, photon creating processes $\gamma + e^- \leftrightarrow 2\gamma + e^-$ and bremmstrahlung $e^- + p \leftrightarrow e^- + p + \gamma$ drop out of equilibrium for photon energies $E \sim T$.



- Compton scattering remains effective in redistributing energy via exchange with electrons
- Out of equilibrium processes like decays leave residual photon chemical potential imprint
- Observed black body spectrum places tight constraints on any that might dump energy into the CMB

Bremmstrahlung

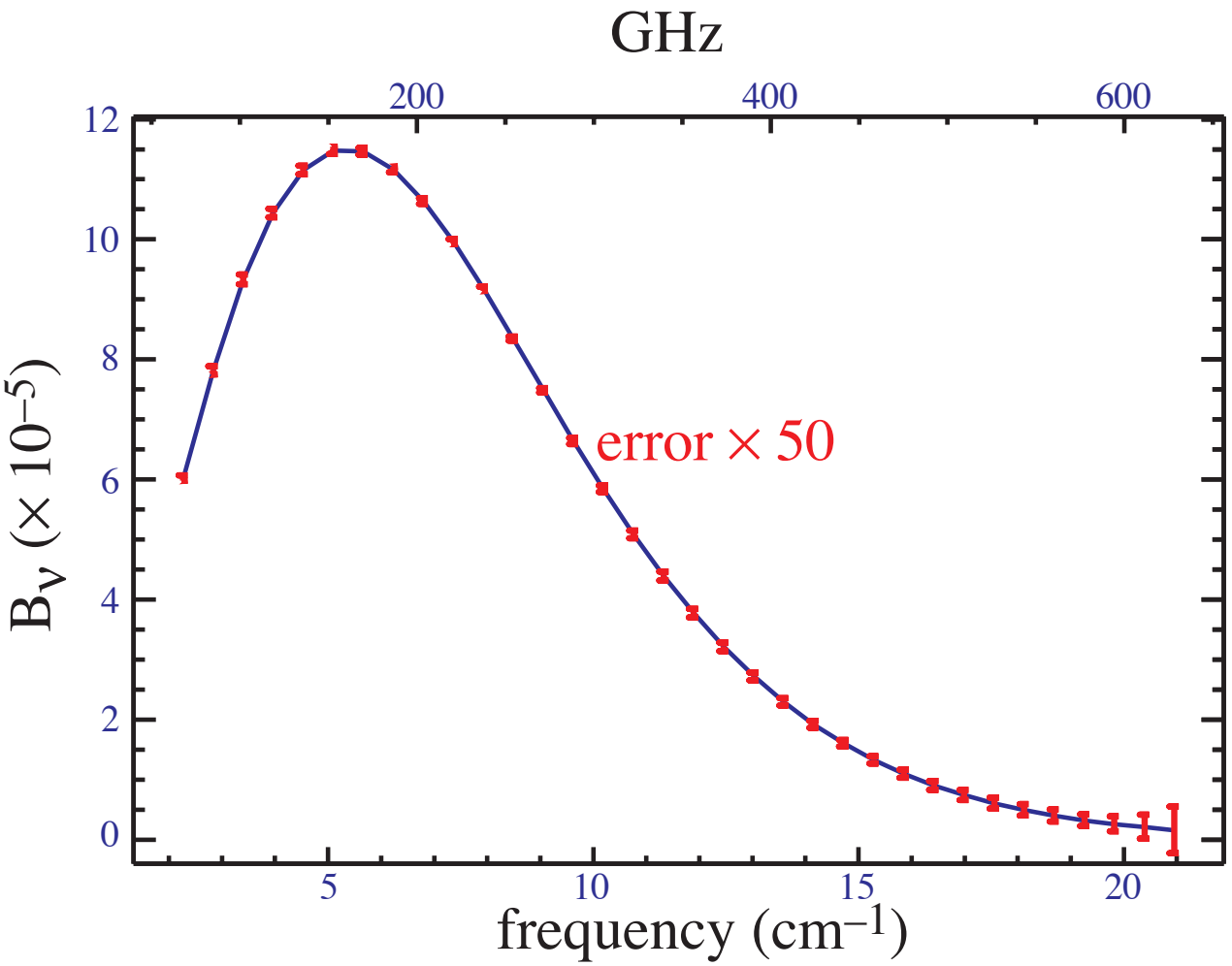
- Bemsstrahlung can be characterized by a collision term like the Kompaneets equation ($k = \hbar = c = 1$, $x = h\nu/kT_e$)

$$C_{\rm ff}[f] = \sqrt{\frac{2}{\pi}} \left(\frac{T_e}{m}\right)^{-1/2} Z^2 \alpha T_e^{-3} n_i n_e \sigma_T g_{\rm ff} \frac{e^{-x}}{x^3} [1 - (e^x - 1) f]$$

note that emission and absorption is balanced only if $f = 1/(e^x - 1)$, a true blackbody (no chemical potential)

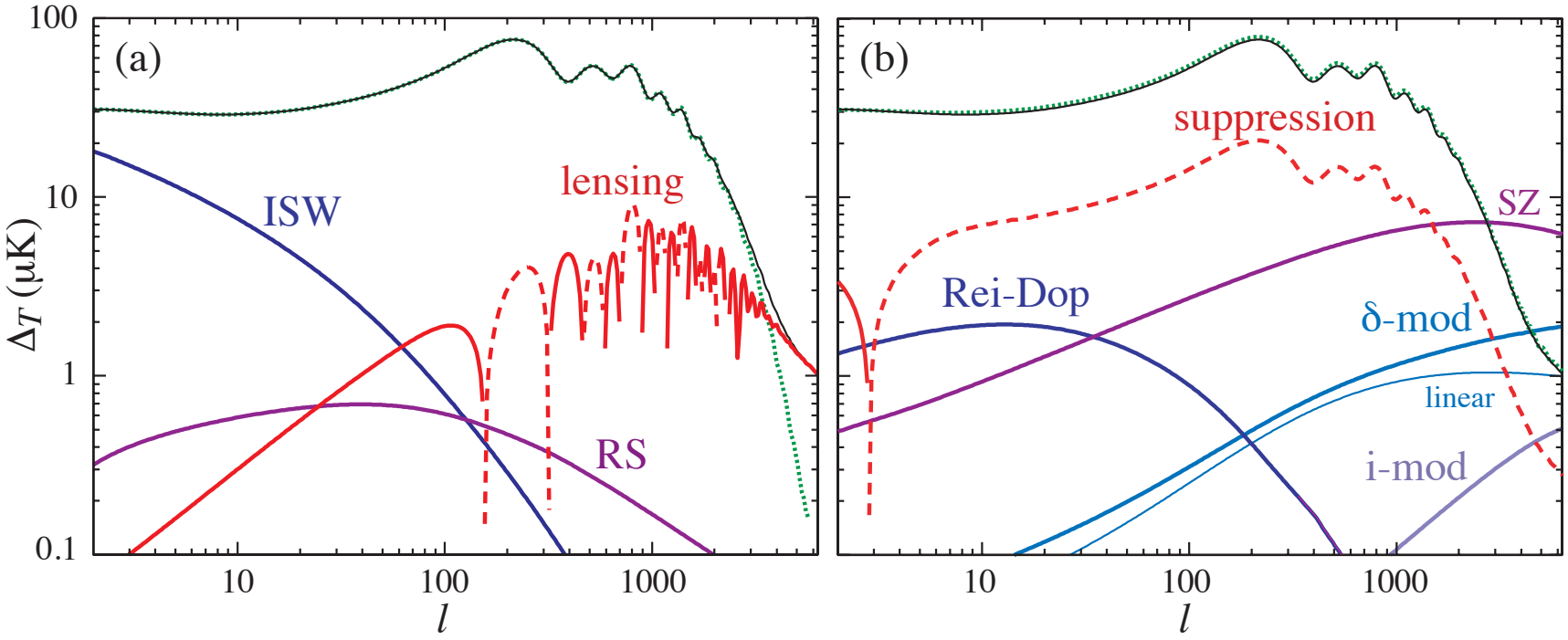
CMB Blackbody

- COBE FIRAS revealed a blackbody spectrum at $T = 2.725\text{K}$ (or cosmological density $\Omega_\gamma h^2 = 2.471 \times 10^{-5}$)



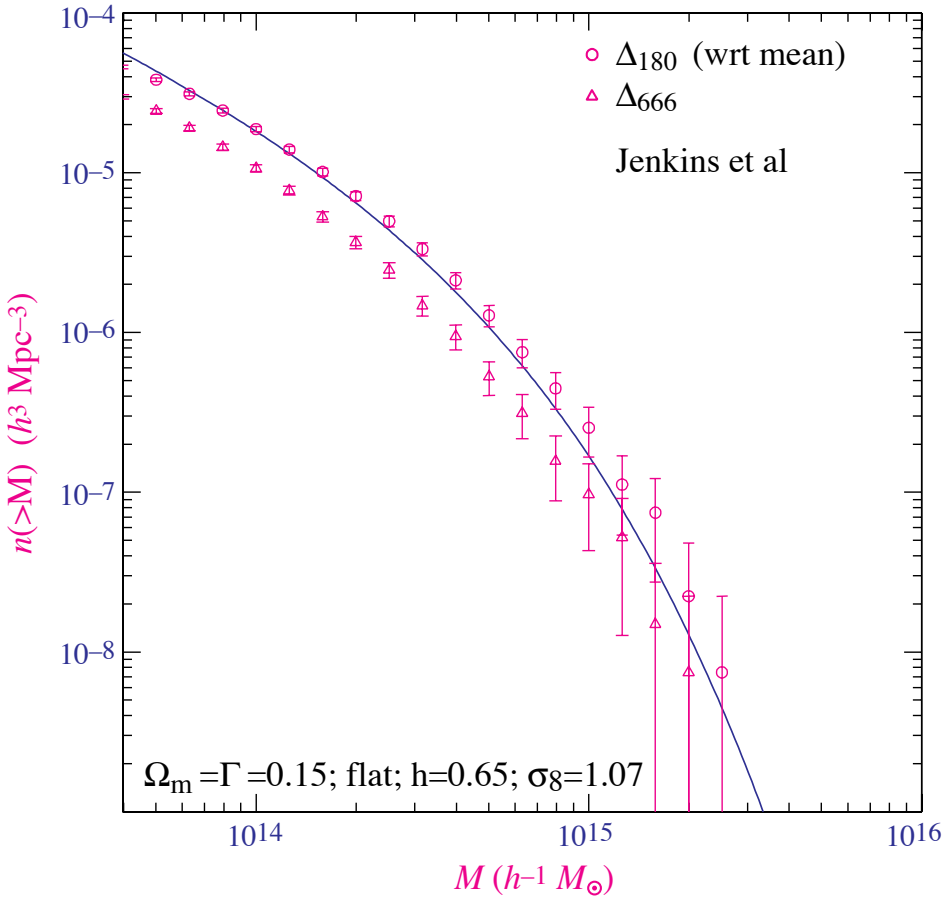
Secondaries: Scattering

- Scattering secondaries: modulated Doppler effect and Sunyaev-Zeldovich effect
- SZ effect from hot gas in halos: associate a temperature (or better, a pressure profile) to each halo in simulations or semi-analytically in the halo model



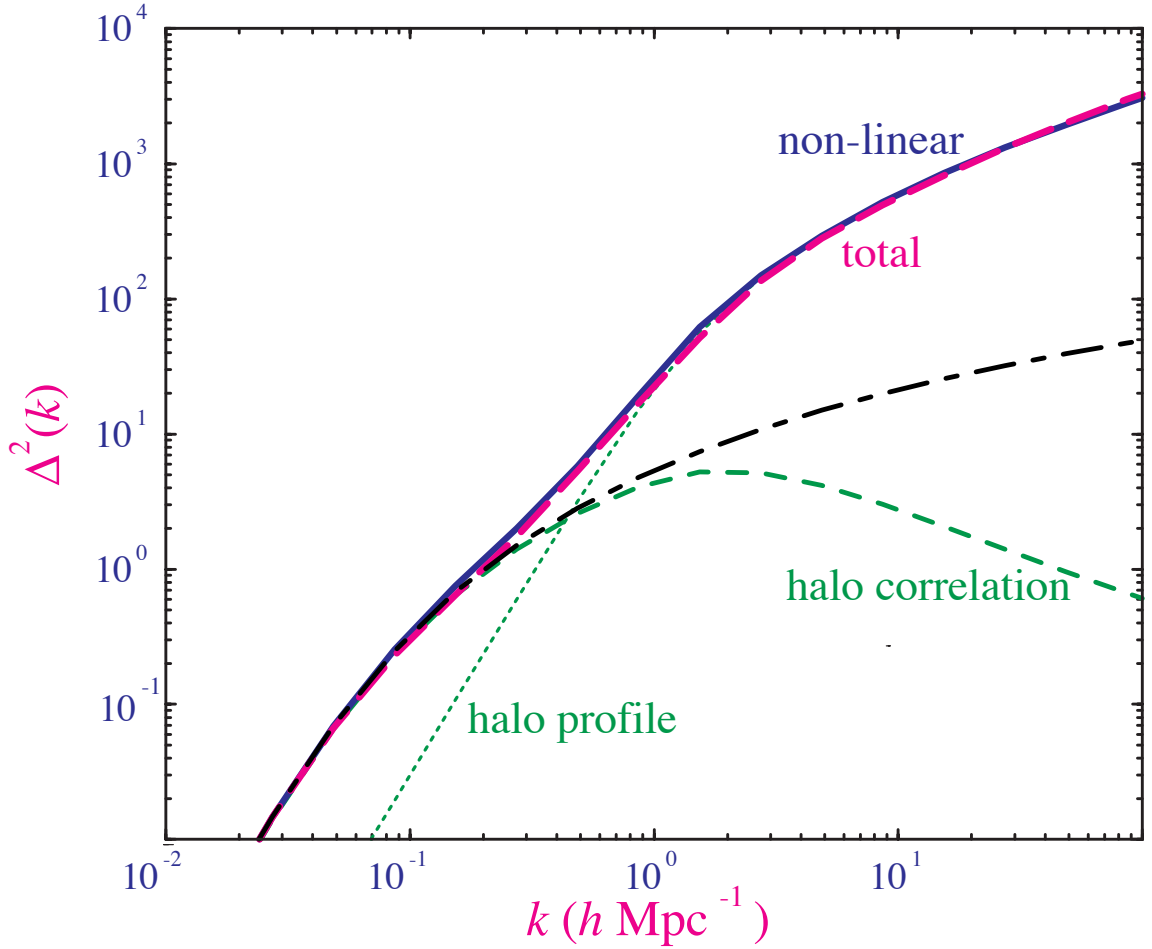
Numerical Mass Function

- Cumulative halo abundance as function of mass: exponential suppression at high mass, exponential sensitivity to amplitude of linear structure σ_8.



The Halo Model

- NFW halos, of abundance n_M given by mass function, clustered according to the halo bias $b(M)$ and the linear theory $P(k)$
- Power spectrum example:



SZ Halo Model

- Instead of the mass profile of NFW use the pressure profile (e.g. from hydrostatic equilibrium)
- Simple example to get scaling of assigning a virial temperature to a halo of mass M
- Solve for velocity dispersion for a self gravitating system

$$\sigma = \left(\frac{3}{5}\frac{GM}{R}\right)^{1/2}$$

- Associate the average kinetic energy with a temperature, called the virial temperature

$$\frac{1}{2}\mu m_H \sigma^2 = \frac{3}{2} k T_{\rm virial}$$

where μ is the mean molecular weight.

SZ Halo Model

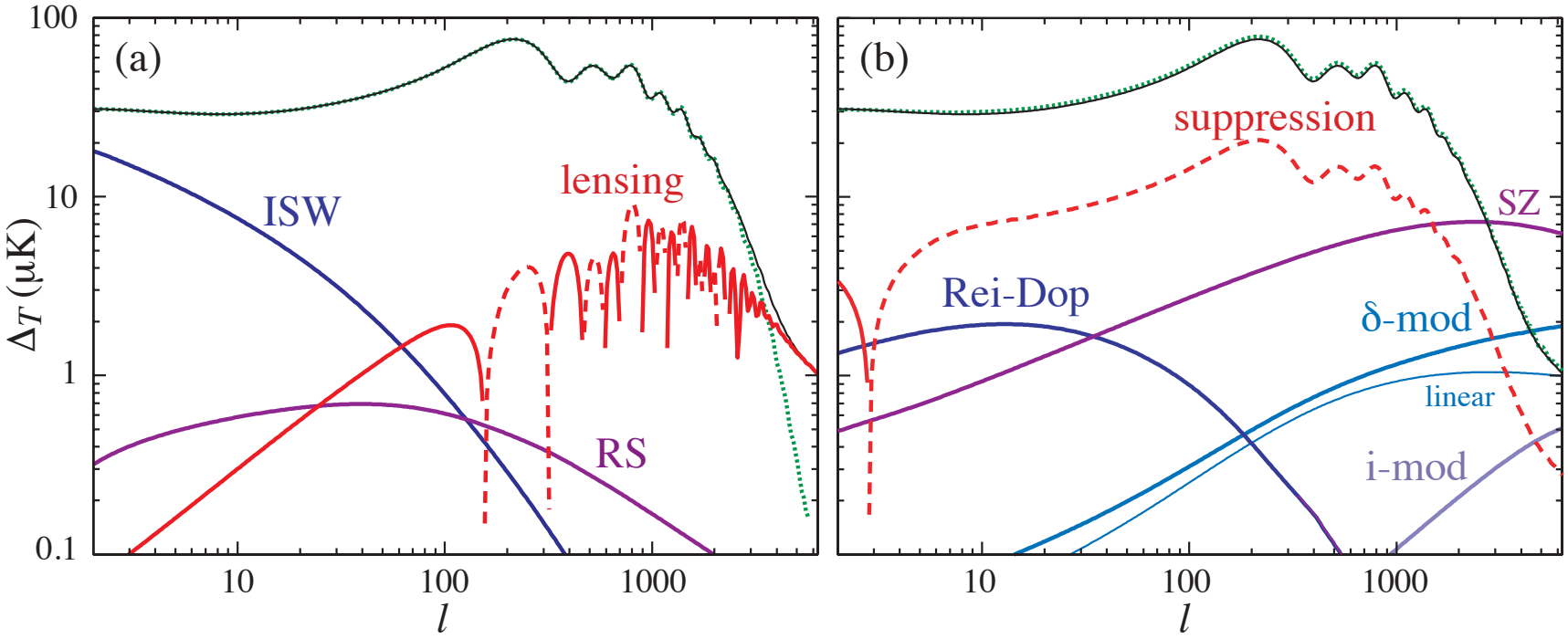
- Solve for virial temperature

$$T_{\rm virial} = \frac{\mu m_H \sigma^2}{3k} = \frac{\mu m_H}{5k}\frac{GM}{R} \approx \frac{\mu m_H}{5k} G M^{2/3} \left(\frac{4\pi\rho}{3}\right)^{1/3}$$

- Mass dependence $T_{\rm virial} \propto M^{2/3}$ further weights the SZ contribution to the high mass end
- Mass function says that the abundance of high mass haloes is exponentially sensitive to the linear power spectrum
- SZ power spectrum extremely sensitive to amplitude of linear power spectrum, i.e. σ_8
- Unfortunately, also highly sensitive to astrophysical assumptions in obtaining the gas pressure

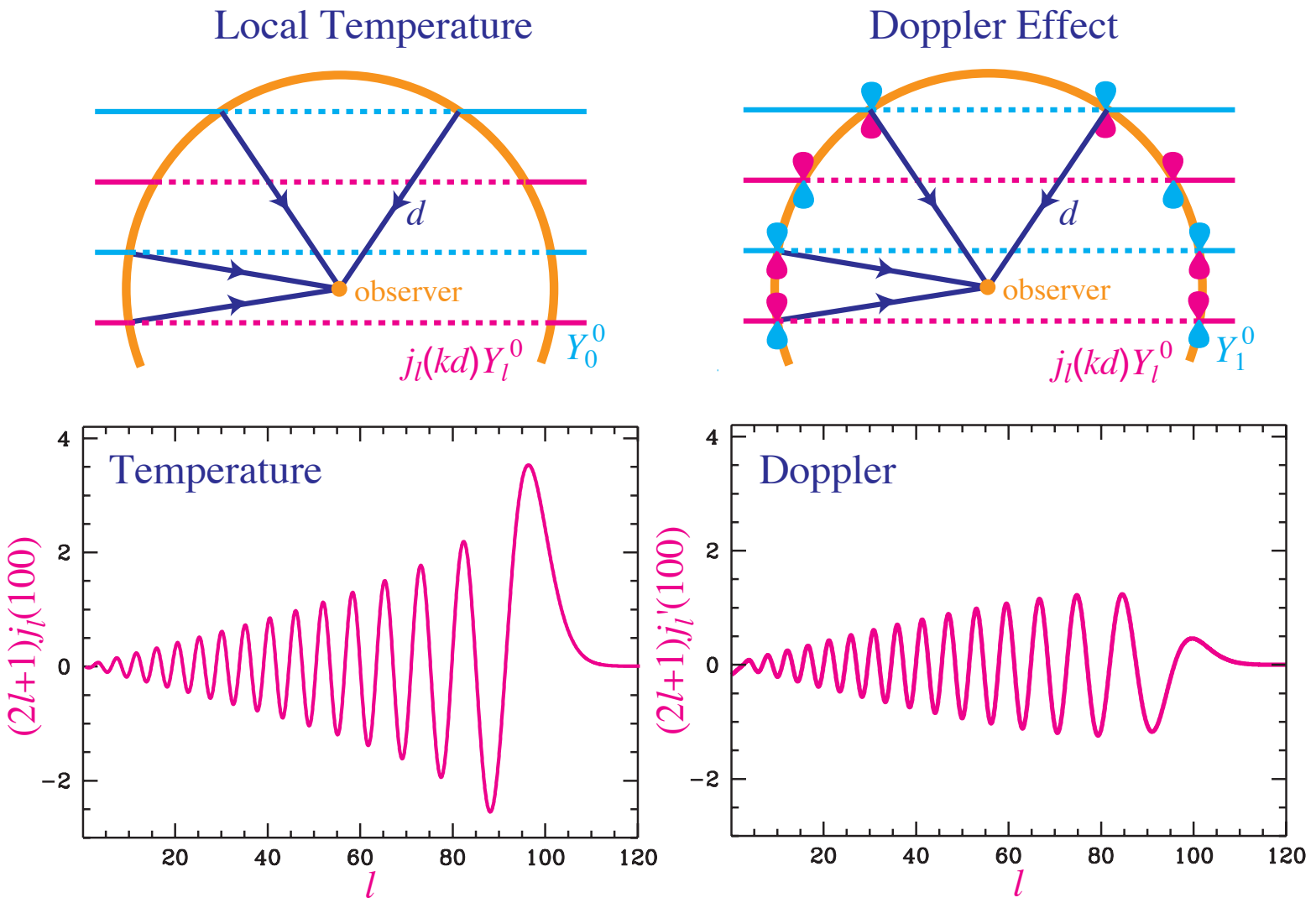
Secondaries: SZ effect

- Halo model + simple virial temperature scaling



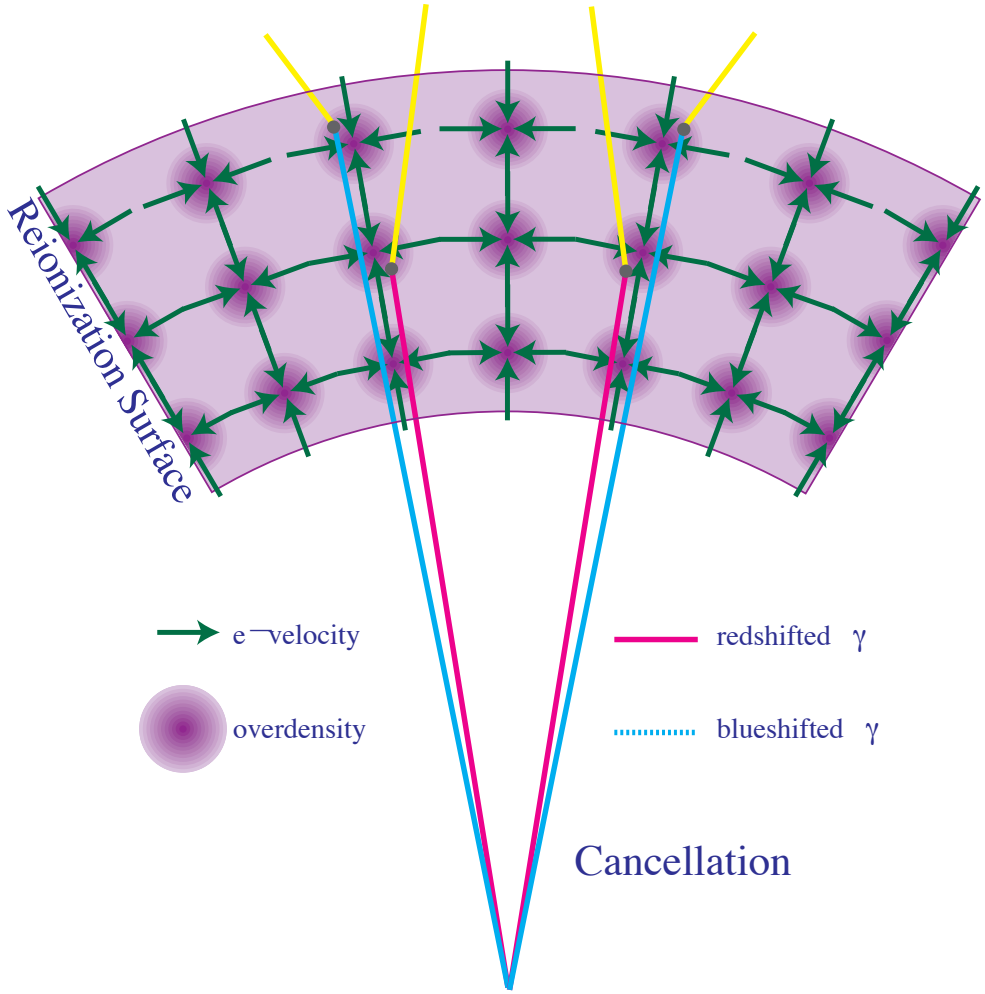
Linear Doppler Effect

- Linear Doppler effect does not contribute when wave is transverse to line of sight



Cancellation

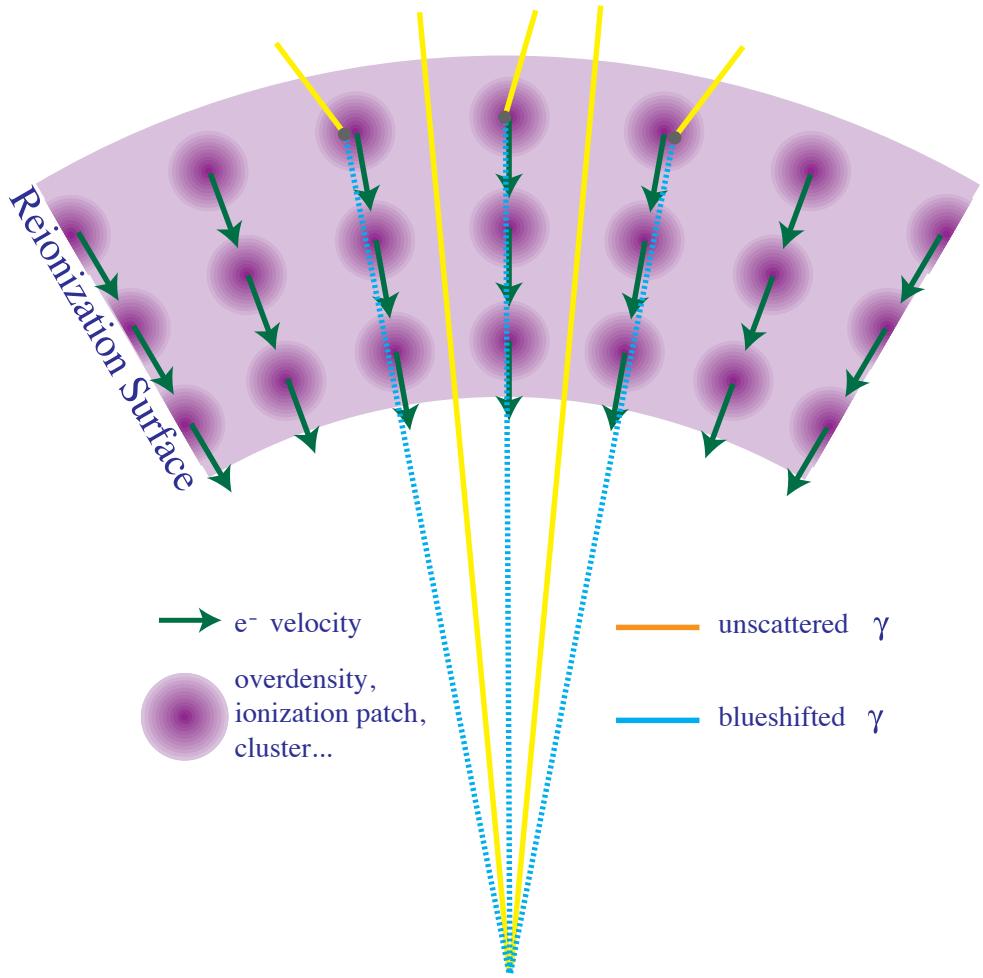
- During reionization, velocities reach $v \sim 10^{-3}$ but Doppler $\Delta T/T \ll v\tau$.
- In linear theory each plane wave contributes both positive and negative line of sight contributions which cancel over the extended duration of reionization



- This behavior is typical for secondaries: Limber approximation of the radiative transfer integral solution - suppression of small scale contributions

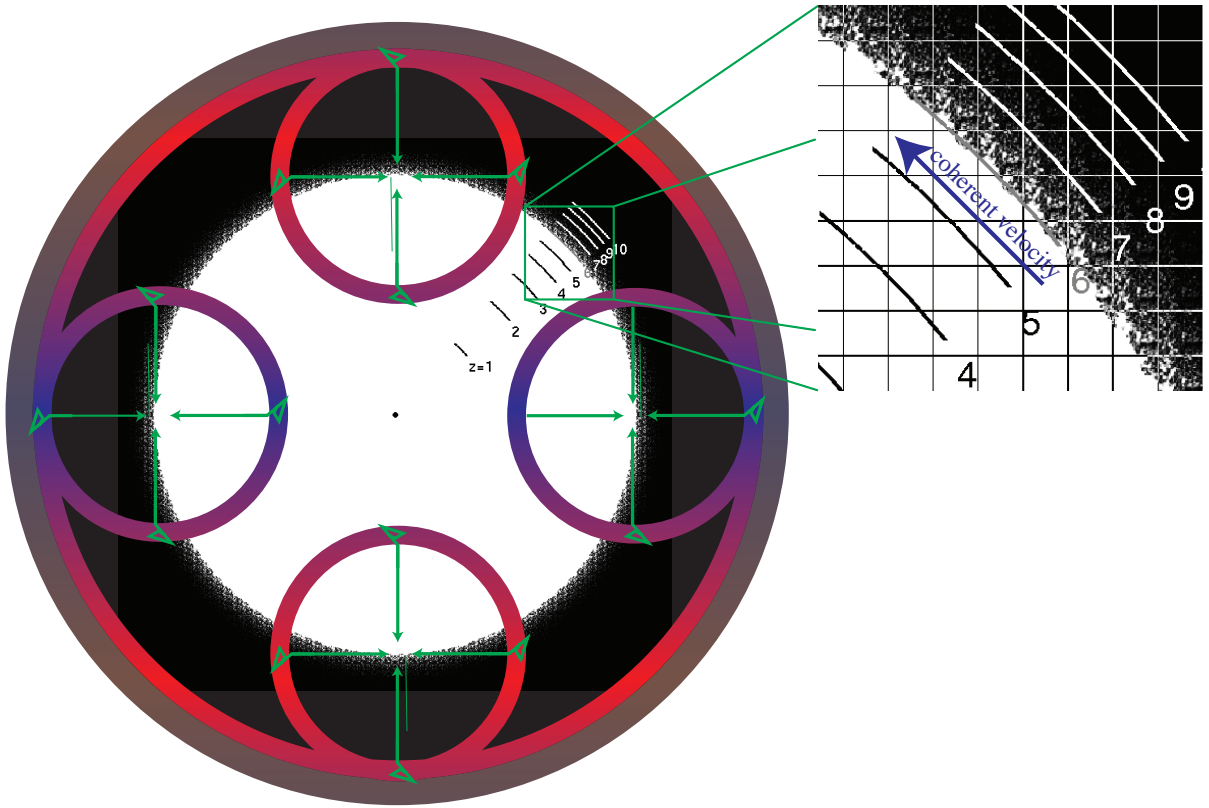
Doppler Modulation

- For large scale velocity fields, the probability of scattering can be modulated on small scales $\Delta T/T \sim v\,\delta\tau$.
- If modulation is from small scale density fluctuations in the (quasi)linear regime: Ostriker-Vishniac effect
- If modulation is from collapsed objects: kinetic SZ effect
- If modulation is from ionization fluctuations: patchy or inhomogeneous ionization effect



Patchy Reionization

- Models of reionization predict size and correlation of ionization bubbles
- In extended Press-Schechter model, predict bubbles in same way as predict halos
- Power spectrum of ionization fluctuation and large scale velocity fields predict modulated temperature (and polarization) secondary power spectra



Secondaries: Modulated Doppler

- Modulated Doppler effect contributes blackbody fluctuations beyond the damping tail