Astro 448

Set 4: C_ℓ Basics

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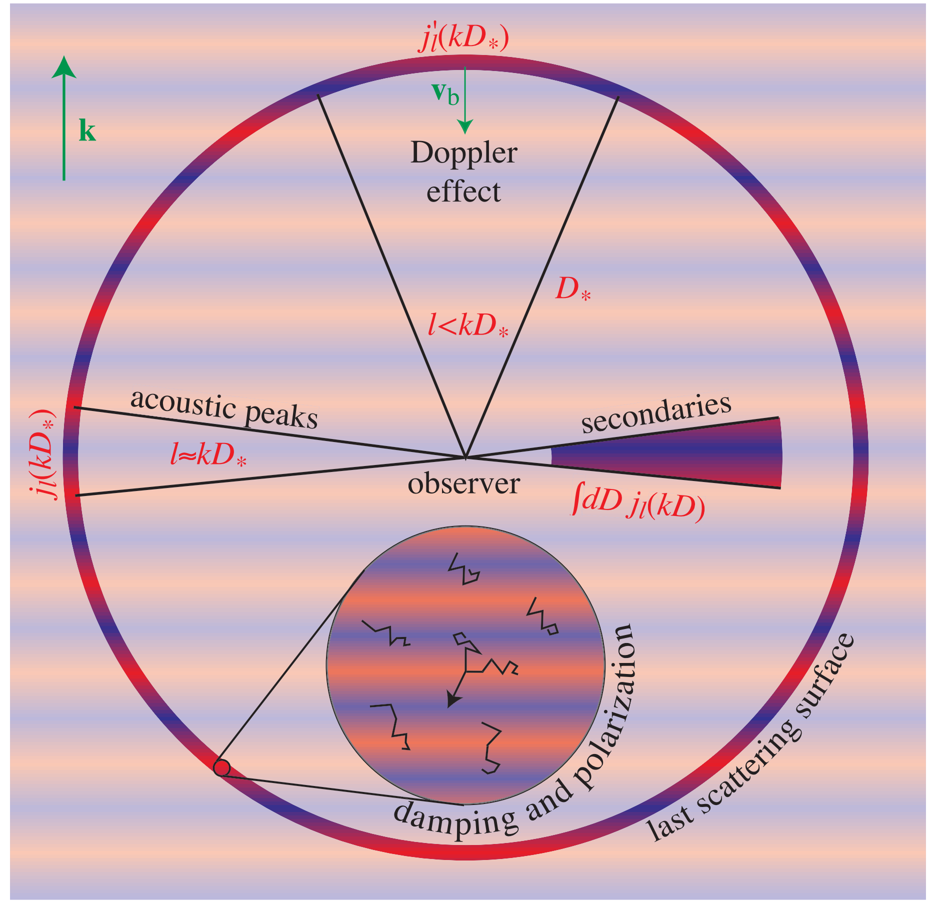
Inhomogeneity vs Anisotropy

- Θ is a function of position as well as direction but we only have access to our position
- Light travels at the speed of light so the radiation we receive in direction $\hat{\mathbf{n}}$ was $(\eta_0 - \eta)\hat{\mathbf{n}}$ at conformal time η
- Inhomogeneity at a distance appears as an anisotopy to the observer
- We need to transport the radiation from the initial conditions to the observer
- This is done with the Boltzmann or radiative transfer equation
- In the absence of scattering, emission or absorption the Boltzmann equation is simply

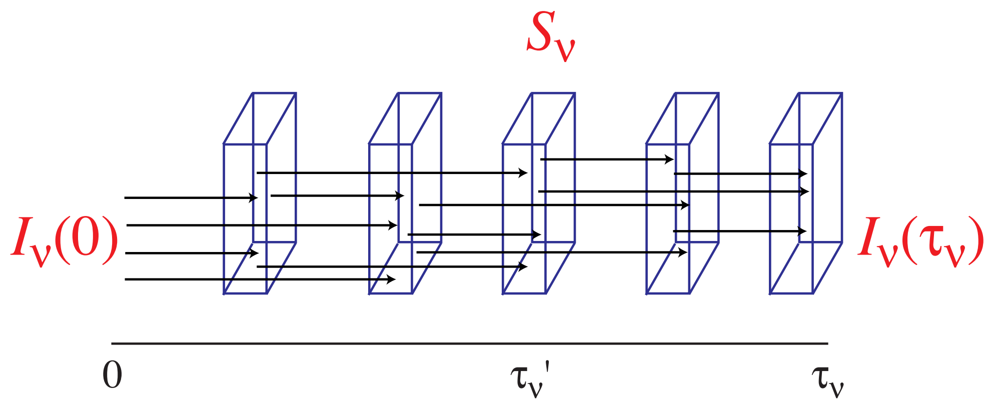
$$\frac{Df}{Dt} = 0$$

Last Scattering

- Angular distribution of radiation is the 3D temperature field projected onto a shell - surface of last scattering
- Shell radius is distance from the observer to recombination: called the last scattering surface
- Take the radiation distribution at last scattering to also be described by an isotropic temperature fluctuation field $\Theta(\mathbf{x})$



Integral Solution to Radiative Transfer



- Formal solution for specific intensity $I_\nu = 2h\nu^3 f/c^2$

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} d\tau'_\nu S_\nu(\tau'_\nu)e^{-(\tau_\nu - \tau'_\nu)}$$

- Specific intensity I_ν attenuated by absorption and replaced by source function, attenuated by absorption from foreground matter
- Here Θ plays the role of specific intensity and $\tau_\nu - \tau'_\nu = \tau$ is optical depth to Compton scattering from $\mathbf{x} = \mathbf{0}$ to $D\hat{\mathbf{n}}$

Angular Power Spectrum

- Take recombination to be instantaneous: $d\tau e^{-\tau} = dD\delta(D - D_*)$ and the source to be the local temperature inhomogeneity

$$\Theta(\hat{\mathbf{n}}) = \int dD \, \Theta(\mathbf{x})\delta(D - D_*)$$

  where D is the comoving distance and D_* denotes recombination.

- Describe the temperature field by its Fourier moments

$$\Theta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

- Note that Fourier moments $\Theta(\mathbf{k})$ have units of volume k^{-3}
- 2 point statistics of the real-space field are translationally and rotationally invariant
- Described by power spectrum

Spatial Power Spectrum

- Translational invariance

$$\langle \Theta(\mathbf{x}')\Theta(\mathbf{x})\rangle = \langle \Theta(\mathbf{x}'+\mathbf{d})\Theta(\mathbf{x}+\mathbf{d})\rangle$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\langle \Theta^*(\mathbf{k}')\Theta(\mathbf{k})\rangle e^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{x}'}$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\langle \Theta^*(\mathbf{k}')\Theta(\mathbf{k})\rangle e^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{x}'+i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{d}}$$

So two point function requires $\delta(\mathbf{k}-\mathbf{k}')$; rotational invariance says coefficient depends only on magnitude of k not it's direction

$$\langle \Theta(\mathbf{k})^*\Theta(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P_T(k)$$

Note that $\delta(\mathbf{k}-\mathbf{k}')$ has units of volume and so P_T must have units of volume

Dimensionless Power Spectrum

- Variance

$$\sigma_\Theta^2 \equiv \langle \Theta(\mathbf{x})\Theta(\mathbf{x})\rangle = \int \frac{d^3k}{(2\pi)^3} P_T(k)$$
$$= \int \frac{k^2 dk}{2\pi^2} \int \frac{d\Omega}{4\pi} P_T(k)$$
$$= \int d\ln k \, \frac{k^3}{2\pi^2} P_T(k)$$

- Define power per logarithmic interval

$$\Delta_T^2(k) \equiv \frac{k^3 P_T(k)}{2\pi^2}$$

- This quantity is dimensionless.

Angular Power Spectrum

- Temperature field

$$\Theta(\hat{\mathbf{n}}) = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) e^{i\mathbf{k}\cdot D_* \hat{\mathbf{n}}}$$

- Multipole moments $\Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m}$
- Expand out plane wave in spherical coordinates

$$e^{i\mathbf{k}D_*\cdot\hat{\mathbf{n}}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kD_*) Y^*_{\ell m}(\mathbf{k}) Y_{\ell m}(\hat{\mathbf{n}})$$

- Angular moment

$$\Theta_{\ell m} = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) 4\pi i^\ell j_\ell(kD_*) Y^*_{\ell m}(\mathbf{k})$$

Angular Power Spectrum

- Power spectrum

$$\langle \Theta_{\ell m}^* \Theta_{\ell' m'} \rangle = \int \frac{d^3k}{(2\pi)^3} (4\pi)^2 i^{\ell-\ell'} j_\ell(kD_*) j_{\ell'}(kD_*) Y_{\ell m}(\mathbf{k}) Y_{\ell' m'}^*(\mathbf{k}) P_T(k)$$

$$= \delta_{\ell\ell'} \delta_{mm'} 4\pi \int d\ln k \, j_\ell^2(kD_*) \Delta_T^2(k)$$

with $\int_0^\infty j_\ell^2(x) d\ln x = 1/(2\ell(\ell+1))$, slowly varying Δ_T^2

- Angular power spectrum:

$$C_\ell = \frac{4\pi \Delta_T^2(\ell/D_*)}{2\ell(\ell+1)} = \frac{2\pi}{\ell(\ell+1)} \Delta_T^2(\ell/D_*)$$

- Not surprisingly, a relationship between $\ell^2 C_\ell / 2\pi$ and Δ_T^2 at $\ell \gg 1$. By convention use $\ell(\ell+1)$ to make relationship exact

Generalized Source

- More generally, we know the Y_ℓ^m's are a complete angular basis and plane waves are complete spatial basis
- General distribution can be decomposed into

$$Y_\ell^m(\hat{\mathbf{n}})\exp(i\mathbf{k}\cdot\mathbf{x})$$

- The observer at the origin sees this distribution in projection

$$Y_\ell^m(\hat{\mathbf{n}})e^{i\mathbf{k}D_*\cdot\hat{\mathbf{n}}} = 4\pi\sum_{\ell' m'} i^{\ell'} j_{\ell'}(kD_*)Y_{\ell'}^{m'*}(\mathbf{k})Y_{\ell'}^{m'}(\hat{\mathbf{n}})Y_\ell^m(\hat{\mathbf{n}})$$

- We extract the observed multipoles by the addition of angular momentum $Y_{\ell'}^{m'}(\hat{\mathbf{n}})Y_\ell^m(\hat{\mathbf{n}}) \to Y_L^M(\hat{\mathbf{n}})$
- Radial functions become linear sums over j_ℓ with the recoupling (Clebsch-Gordan) coefficients
- Formal integral solution to the radiative transfer equation

Boltzmann Equation

- General integral solution for radiative transfer as long as the angular distribution at emission is known
- Formalize further the evolution of angular moments in the cosmological context:

$$\frac{Df}{Dt} = \dot{f} + \dot{\mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{q}} + \dot{\mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0$$

- Momentum $\mathbf{q} = q\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a directional unit vector and in a flat universe $\dot{\mathbf{q}} = \dot{q}\hat{\mathbf{n}}$
- Particle velocity $\dot{\mathbf{x}} = \mathbf{q}/E$

$$\dot{f} + \dot{q}\frac{\partial f}{\partial q} + \frac{\mathbf{q}}{E} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0$$

Angular Moments

- Define the angularly dependent Stokes perturbation

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta), \quad Q(\mathbf{x}, \hat{\mathbf{n}}, \eta), \quad U(\mathbf{x}, \hat{\mathbf{n}}, \eta)$$

- Decompose into normal modes: plane waves for spatial part and spherical harmonics for angular part

$$G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$

$${}_{\pm 2}G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} {}_{\pm 2}Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$

- In a spatially curved universe generalize the plane wave part

Normal Modes

- Temperature and polarization fields

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} \Theta_\ell^{(m)} G_\ell^m$$

$$[Q \pm iU](\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} [E_\ell^{(m)} \pm iB_\ell^{(m)}]_{\pm 2} G_\ell^m$$

- For each $\mathbf{k}$ mode, work in coordinates where $\mathbf{k} \parallel \mathbf{z}$ and so $m = 0$ represents scalar modes, $m = \pm 1$ vector modes, $m = \pm 2$ tensor modes, $|m| > 2$ vanishes. Since modes add incoherently and $Q \pm iU$ is invariant up to a phase, rotation back to a fixed coordinate system is trivial.

Liouville Equation

- In absence of scattering, the phase space distribution of photons in each polarization state a is conserved along the propagation path
- Rewrite variables in terms of the photon propagation direction $\mathbf{q} = q\hat{\mathbf{n}}$, so $f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta)$ and

$$\frac{d}{d\eta} f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta) = 0$$

$$= \left(\frac{\partial}{\partial \eta} + \frac{d\mathbf{x}}{d\eta} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{d\hat{\mathbf{n}}}{d\eta} \cdot \frac{\partial}{\partial \hat{\mathbf{n}}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a$$

- For simplicity, assume spatially flat universe $K = 0$ then $d\hat{\mathbf{n}}/d\eta = 0$ and $d\mathbf{x} = \hat{\mathbf{n}} d\eta$

$$\dot{f}_a + \hat{\mathbf{n}} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0$$

Scalar, Vector, Tensor

- Normalization of modes is chosen so that the lowest angular mode for scalars, vectors and tensors are normalized in the same way as the mode function

$$G_0^0 = Q^{(0)} \quad G_1^0 = n^i Q_i^{(0)} \quad G_2^0 \propto n^i n^j Q_{ij}^{(0)}$$
$$G_1^{\pm 1} = n^i Q_i^{(\pm 1)} \quad G_2^{\pm 1} \propto n^i n^j Q_{ij}^{(\pm 1)}$$
$$G_2^{\pm 2} = n^i n^j Q_{ij}^{(\pm 2)}$$

where recall

$$\begin{aligned}
Q^{(0)} &= \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k}\cdot\mathbf{x})
\end{aligned}$$

Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies
- Spatial gradient term hits plane wave:

$$\hat{\mathbf{n}} \cdot \nabla e^{i\mathbf{k}\cdot\mathbf{x}} = i\hat{\mathbf{n}} \cdot \mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} = i\sqrt{\frac{4\pi}{3}} k Y_1^0(\hat{\mathbf{n}}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

- Dipole term adds to angular dependence through the addition of angular momentum

$$\sqrt{\frac{4\pi}{3}} Y_1^0 Y_\ell^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell+1)(2\ell-1)}} Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell+1)(2\ell+3)}} Y_{\ell+1}^m$$

where $\kappa_\ell^m = \sqrt{\ell^2 - m^2}$ is given by Clebsch-Gordon coefficients.

Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

$$\dot{\Theta}_\ell^{(m)} = k\left[\frac{\kappa_\ell^m}{2\ell+1}\Theta_{\ell-1}^{(m)} - \frac{\kappa_{\ell+1}^m}{2\ell+3}\Theta_{\ell+1}^{(m)}\right] - \dot{\tau}\Theta_\ell^{(m)} + S_\ell^{(m)}$$

  where $S_\ell^{(m)}$ are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic $\ell = 0$ temperature perturbation will eventually become a high order anisotropy by “free streaming” or simple projection
- Original CMB codes solved the full hierarchy equations out to the ℓ of interest.

Integral Solution

- Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface
- In general, the solution describes the decomposition of the source $S_\ell^{(m)}$ with its local angular dependence as seen at a distance $\mathbf{x} = D\hat{\mathbf{n}}$.
- Proceed by decomposing the angular dependence of the plane wave

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} j_\ell(kD) Y_\ell^0(\hat{\mathbf{n}})$$

- Recouple to the local angular dependence of G_ℓ^m

$$G_{\ell_s}^m = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} \alpha_{\ell_s \ell}^{(m)}(kD) Y_\ell^m(\hat{\mathbf{n}})$$

Integral Solution

- Projection kernels:

$$\ell_s = 0, \quad m = 0 \qquad\qquad \alpha^{(0)}_{0\ell} \equiv j_\ell$$

$$\ell_s = 1, \quad m = 0 \qquad\qquad \alpha^{(0)}_{1\ell} \equiv j'_\ell$$

- Integral solution:

$$\frac{\Theta^{(m)}_\ell(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S^{(m)}_{\ell_s} \alpha^{(m)}_{\ell_s \ell}(k(\eta_0 - \eta))$$

- Power spectrum:

$$C_\ell = \frac{2}{\pi} \int \frac{dk}{k} \sum_m \frac{k^3 \langle \Theta^{(m)*}_\ell \Theta^{(m)}_\ell \rangle}{(2\ell+1)^2}$$

- Solving for C_ℓ reduces to solving for the behavior of a handful of sources

Polarization Hierarchy

- In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hierarchy:

$$\dot{E}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell+1)}B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}E_{\ell+1}^{(m)}\right] - \dot{\tau}E_\ell^{(m)} + \mathcal{E}_\ell^{(m)}$$

$$\dot{B}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell+1)}E_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}B_{\ell+1}^{(m)}\right] - \dot{\tau}B_\ell^{(m)} + \mathcal{B}_\ell^{(m)}$$

  where ${}_2\kappa_\ell^m = \sqrt{(\ell^2 - m^2)(\ell^2 - 4)/\ell^2}$ is given by the Clebsch-Gordon coefficients and $\mathcal{E}$, $\mathcal{B}$ are the sources (scattering only).

- Note that for vectors and tensors $|m| > 0$ and B modes may be generated from E modes by projection. Cosmologically $\mathcal{B}_\ell^{(m)} = 0$

Polarization Integral Solution

- Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

$$\frac{E_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \epsilon_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$
$$\frac{B_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \beta_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

- The only source to the polarization is from the quadrupole anisotropy so we only need $\ell_s = 2$, e.g. for scalars

$$\epsilon_{2\ell}^{(0)}(x) = \sqrt{\frac{3}{8}\frac{(\ell+2)!}{(\ell-2)!}}\frac{j_\ell(x)}{x^2} \qquad \beta_{2\ell}^{(0)} = 0$$

Gravitational Terms

- As in our Newtonian gauge calculation, gravitational terms - now including vectors and tensors in an arbitrary gauge, come from the geodesic equation
- First define the slicing (lapse function A, shift function B^i)

$$
\begin{aligned}
g^{00} &= -a^{-2}(1-2A)\,, \\
g^{0i} &= -a^{-2}B^i\,,
\end{aligned}
$$

A defines the lapse of proper time between 3-surfaces whereas B^i defines the threading or relationship between the 3-coordinates of the surfaces

Gravitational Terms

- This absorbs 1+3=4 degrees of freedom in the metric, remaining 6 is in the spatial surfaces which we parameterize as

$$g^{ij} = a^{-2}(\gamma^{ij} - 2H_L\gamma^{ij} - 2H_T^{ij}) .$$

here (1) H_L a perturbation to the spatial curvature; (5) H_T^{ij} a trace-free distortion to spatial metric (which also can perturb the curvature)

- Geodesic equation gives the redshifting term

$$\frac{\dot{q}}{q} = -\frac{\dot{a}}{a} - \frac{1}{2}n^i n^j \dot{H}_{Tij} - \dot{H}_L + n^i \dot{B}_i - \hat{\mathbf{n}} \cdot \nabla A$$

which is incorporated in the conservation and gauge transformation equations

Source Terms

- Temperature source terms $S_l^{(m)}$ (rows $\pm|m|$; flat assumption

$$\begin{pmatrix} \dot\tau\Theta_0^{(0)} - \dot H_L^{(0)} & \dot\tau v_b^{(0)} + \dot B^{(0)} & \dot\tau P^{(0)} - \frac{2}{3}\dot H_T^{(0)} \\ 0 & \dot\tau v_b^{(\pm1)} + \dot B^{(\pm1)} & \dot\tau P^{(\pm1)} - \frac{\sqrt{3}}{3}\dot H_T^{(\pm1)} \\ 0 & 0 & \dot\tau P^{(\pm2)} - \dot H_T^{(\pm2)} \end{pmatrix}$$

where

$$P^{(m)} \equiv \frac{1}{10}(\Theta_2^{(m)} - \sqrt{6}E_2^{(m)})$$

- Polarization source term

$$\mathcal{E}_\ell^{(m)} = -\dot\tau\sqrt{6}P^{(m)}\delta_{\ell,2}$$

$$\mathcal{B}_\ell^{(m)} = 0$$

Truncated Hierarchy

- CMBFast introduced the hybrid truncated hierarchy, integral solution technique
- Formal integral solution contains sources that are not external to system but defined through the Boltzmann hierarchy itself
- Solution: recall that we used this technique in the tight coupling regime by applying a closure condition from tight coupling
- CMBFast extends this idea by solving a truncated hierarchy of equations, e.g. out to $\ell = 25$ with non-reflecting boundary conditions
- For completeness, we explicitly derive the scattering source term via polarized radiative transfer in the last part of the notes

Polarized Radiative Transfer

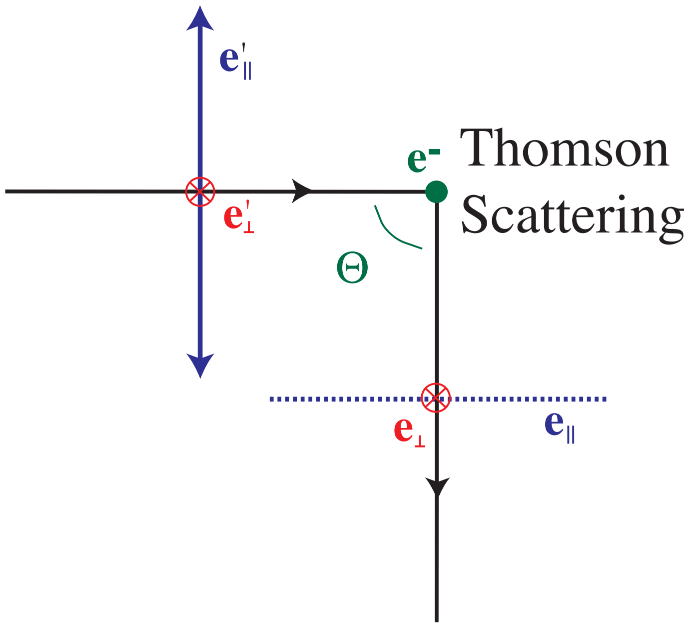
- Define a specific intensity "vector": $\mathbf{I}_\nu = (\Theta_\parallel, \Theta_\perp, U, V)$ where $\Theta = \Theta_\parallel + \Theta_\perp$, $Q = \Theta_\parallel - \Theta_\perp$

$$\frac{d\mathbf{I}_\nu}{d\eta} = \dot{\tau}(\mathbf{S}_\nu - \mathbf{I}_\nu)$$

- Thomson collision

  based on differential cross section

$$\frac{d\sigma_T}{d\Omega} = \frac{3}{8\pi}|\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \ ,$$



Polarized Radiative Transfer

- $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.
- Thomson scattering by 90 deg: $\Theta_\perp \to \Theta_\perp$ but $\Theta_\parallel$ does not scatter
- More generally if Θ is the scattering angle

$$\mathbf{S}_\nu = \frac{3}{8\pi}\int d\Omega' \begin{pmatrix} \cos^2\Theta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\Theta & 0 \\ 0 & 0 & 0 & \cos\Theta \end{pmatrix} \mathbf{I}'_\nu$$

- But to calculate Stokes parameters in a fixed coordinate system must rotate into the scattering basis, scatter and rotate back out to the fixed coordinate system

Thomson Collision Term

- The $U \to U'$ transfer follows by writing down the polarization vectors in the $45°$ rotated basis

$$\hat{\mathbf{E}}_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_\parallel + \hat{\mathbf{E}}_\perp), \qquad \hat{\mathbf{E}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_\parallel - \hat{\mathbf{E}}_\perp)$$

- Define the temperature in this basis

$$\begin{aligned}
\Theta_1 &\propto |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_1|^2 \Theta_1' + |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2|^2 \Theta_2' \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta_1' + \frac{1}{4}(\cos\beta - 1)^2 \Theta_2' \\
\Theta_2 &\propto |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_2|^2 \Theta_2' + |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_1|^2 \Theta_1' \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta_2' + \frac{1}{4}(\cos\beta - 1)^2 \Theta_1'
\end{aligned}$$

or $\Theta_1 - \Theta_2 \propto \cos\beta(\Theta_1' - \Theta_2')$

Scattering Matrix

- Transfer matrix of Stokes state $\mathbf{T} \equiv (\Theta, Q + iU, Q - iU)$

$$\mathbf{T} \propto \mathbf{S}(\beta)\mathbf{T}'$$

$$\mathbf{S}(\beta) = \frac{3}{4} \begin{pmatrix} \cos^2\beta + 1 & -\frac{1}{2}\sin^2\beta & -\frac{1}{2}\sin^2\beta \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta + 1)^2 & \frac{1}{2}(\cos\beta - 1)^2 \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta - 1)^2 & \frac{1}{2}(\cos\beta + 1)^2 \end{pmatrix}$$

normalization factor of 3 is set by photon conservation in scattering

Scattering Matrix

- Transform to a fixed basis, by a rotation of the incoming and outgoing states $\mathbf{T} = \mathbf{R}(\psi)\mathbf{T}$ where

$$\mathbf{R}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & e^{2i\psi} \end{pmatrix}$$

giving the scattering matrix

$$\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha) =$$

$$\frac{1}{2}\sqrt{\frac{4\pi}{5}} \begin{pmatrix} Y_2^0(\beta,\alpha) + 2\sqrt{5}Y_0^0(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^{-2}(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^2(\beta,\alpha) \\ -\sqrt{6}\,{}_2Y_2^0(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^{-2}(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^2(\beta,\alpha)e^{2i\gamma} \\ -\sqrt{6}\,{}_{-2}Y_2^0(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^{-2}(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^2(\beta,\alpha)e^{-2i\gamma} \end{pmatrix}$$

Addition Theorem for Spin Harmonics

- Spin harmonics are related to rotation matrices as

$$ {}_sY_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\mathcal{D}^\ell_{-ms}(\phi,\theta,0) $$

  Note: for explicit evaluation sign convention differs from usual (e.g. Jackson) by $(-1)^m$

- Multiplication of rotations

$$ \sum_{m''}\mathcal{D}^\ell_{mm''}(\alpha_2,\beta_2,\gamma_2)\mathcal{D}^\ell_{m''m}(\alpha_1,\beta_1,\gamma_1) = \mathcal{D}^\ell_{mm'}(\alpha,\beta,\gamma) $$

- Implies

$$ \sum_m {}_{s_1}Y_\ell^{m*}(\theta',\phi')\ {}_{s_2}Y_\ell^m(\theta,\phi) = (-1)^{s_1-s_2}\sqrt{\frac{2\ell+1}{4\pi}}\ {}_{s_2}Y_\ell^{-s_1}(\beta,\alpha)e^{is_2\gamma} $$

Sky Basis

- Scattering into the state (rest frame)

$$C_{\rm in}[\mathbf{T}] = \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi}\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha)\mathbf{T}(\hat{\mathbf{n}}')\,,$$

$$= \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi}(\Theta', 0, 0) + \frac{1}{10}\dot\tau \int d\hat{\mathbf{n}}' \sum_{m=-2}^{2} \mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}')\mathbf{T}(\hat{\mathbf{n}}')\,.$$

where the quadrupole coupling term is $\mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') =$

$$\begin{pmatrix} Y_2^{m*}(\hat{\mathbf{n}}')\,Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\,{}_2Y_2^{m*}(\hat{\mathbf{n}}')\,Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\,{}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\,Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_2Y_2^m(\hat{\mathbf{n}}) & 3\,{}_2Y_2^{m*}(\hat{\mathbf{n}}')\,{}_2Y_2^m(\hat{\mathbf{n}}) & 3\,{}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_2Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\,{}_2Y_2^{m*}(\hat{\mathbf{n}}')\,{}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\,{}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_{-2}Y_2^m(\hat{\mathbf{n}}) \end{pmatrix},$$

expression uses angle addition relation above. We call this term C_Q.

Scattering Matrix

- Full scattering matrix involves difference of scattering into and out of state

$$C[\mathbf{T}] = C_{\rm in}[\mathbf{T}] - C_{\rm out}[\mathbf{T}]$$

- In the electron rest frame

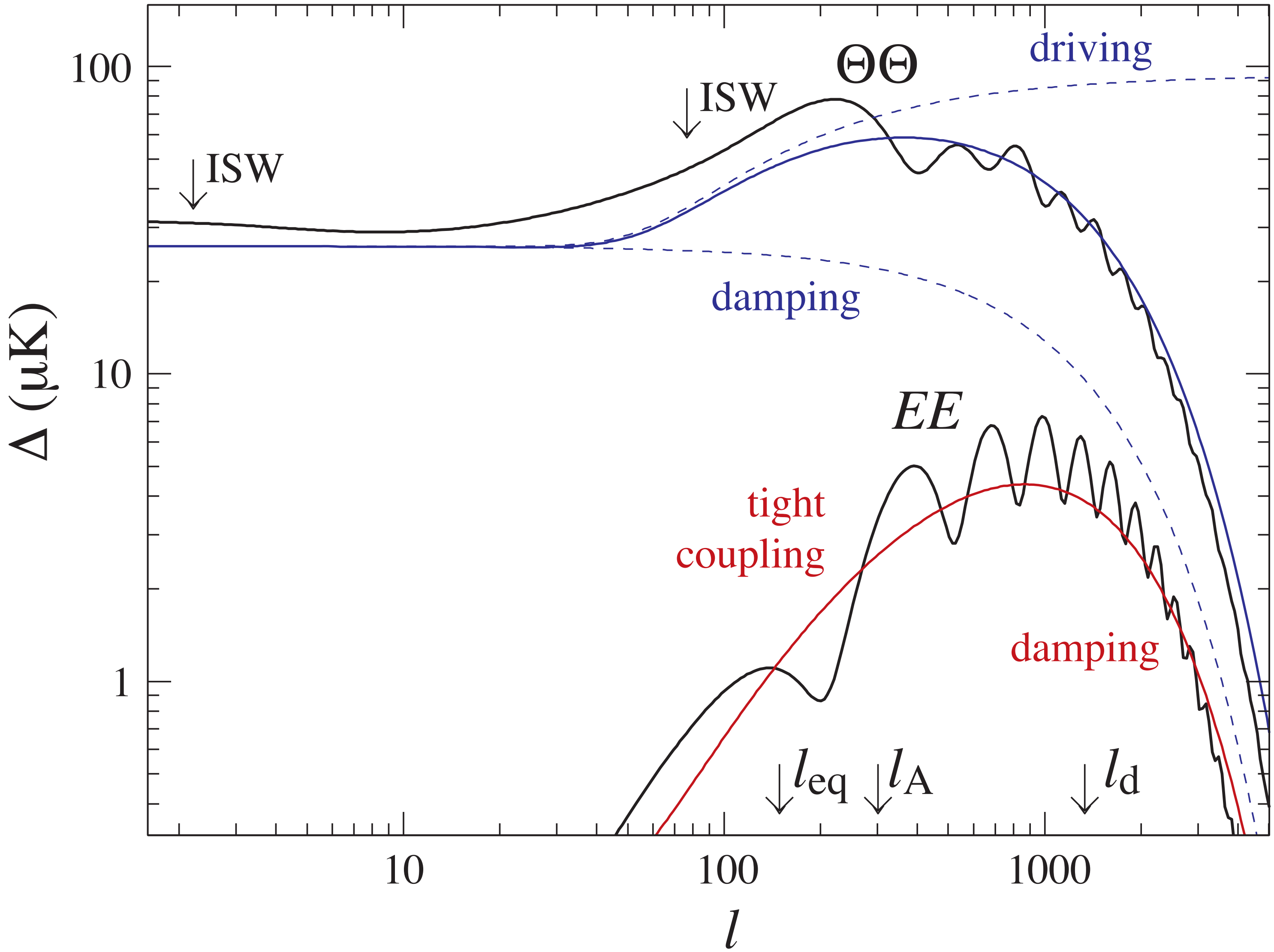
$$C[\mathbf{T}] = \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi}(\Theta', 0, 0) - \dot\tau \mathbf{T} + C_Q[\mathbf{T}]$$

which describes isotropization in the rest frame. All moments have $e^{-\tau}$ suppression except for isotropic temperature Θ_0. Transformation into the background frame simply induces a dipole term

$$C[\mathbf{T}] = \dot\tau \left(\hat{\mathbf{n}} \cdot \mathbf{v}_b + \int \frac{d\hat{\mathbf{n}}'}{4\pi}\Theta', 0, 0 \right) - \dot\tau \mathbf{T} + C_Q[\mathbf{T}]$$

Schematic Outline

- Take apart features in the power spectrum



Thomson Scattering

- Thomson scattering of photons off of free electrons is the most important CMB process with a cross section (averaged over polarization states) of

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25}\text{cm}^2$$

- Density of free electrons in a fully ionized $x_e = 1$ universe

$$n_e = (1 - Y_p/2)x_e n_b \approx 10^{-5}\Omega_b h^2 (1+z)^3 \text{cm}^{-3}\,,$$

where $Y_p \approx 0.24$ is the Helium mass fraction, creates a high (comoving) Thomson opacity

$$\dot{\tau} \equiv n_e \sigma_T a$$

where dots are conformal time $\eta \equiv \int dt/a$ derivatives and τ is the optical depth.

Tight Coupling Approximation

- Near recombination $z \approx 10^3$ and $\Omega_b h^2 \approx 0.02$, the (comoving) mean free path of a photon

$$\lambda_C \equiv \frac{1}{\dot{\tau}} \sim 2.5\text{Mpc}$$

small by cosmological standards!

- On scales $\lambda \gg \lambda_C$ photons are tightly coupled to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions
- Specifically, their bulk velocities are defined by a single fluid velocity $v_\gamma = v_b$ and the photons carry no anisotropy in the rest frame of the baryons
- $\rightarrow$ No heat conduction or viscosity (anisotropic stress) in fluid

Zeroth Order Approximation

- Momentum density of a fluid is $(\rho + p)v$, where p is the pressure
- Neglect the momentum density of the baryons

$$R \equiv \frac{(\rho_b + p_b)v_b}{(\rho_\gamma + p_\gamma)v_\gamma} = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}$$

$$\approx 0.6 \left(\frac{\Omega_b h^2}{0.02}\right) \left(\frac{a}{10^{-3}}\right)$$

since $\rho_\gamma \propto T^4$ is fixed by the CMB temperature $T = 2.73(1 + z)$K – OK substantially before recombination

- Neglect radiation in the expansion

$$\frac{\rho_m}{\rho_r} = 3.6 \left(\frac{\Omega_m h^2}{0.15}\right) \left(\frac{a}{10^{-3}}\right)$$

- Neglect gravity

Fluid Equations

- Density $\rho_\gamma \propto T^4$ so define temperature fluctuation Θ

$$\delta_\gamma = 4\frac{\delta T}{T} \equiv 4\Theta$$

- Real space continuity equation

$$\dot{\delta}_\gamma = -(1 + w_\gamma)kv_\gamma$$

$$\dot{\Theta} = -\frac{1}{3}kv_\gamma$$

- Euler equation (neglecting gravity)

$$\dot{v}_\gamma = -(1 - 3w_\gamma)\frac{\dot{a}}{a}v_\gamma + \frac{kc_s^2}{1 + w_\gamma}\delta_\gamma$$

$$\dot{v}_\gamma = kc_s^2\frac{3}{4}\delta_\gamma = 3c_s^2 k\Theta$$

Oscillator: Take One

- Combine these to form the simple harmonic oscillator equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = 0$$

where the sound speed is adiabatic

$$c_s^2 = \frac{\delta p_\gamma}{\delta \rho_\gamma} = \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma}$$

here $c_s^2 = 1/3$ since we are photon-dominated

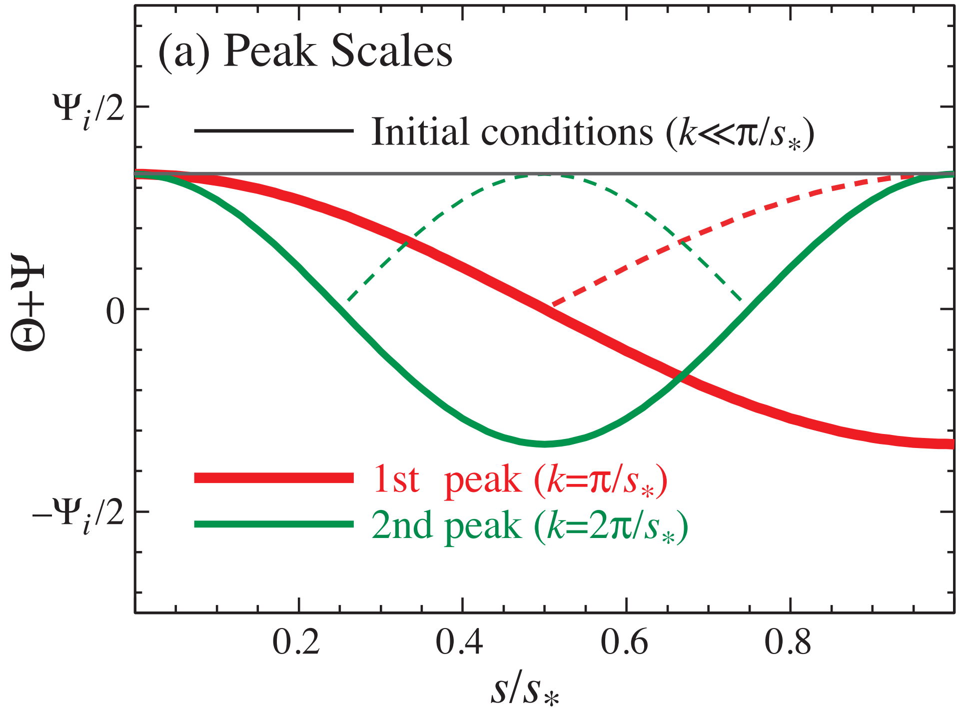
- General solution:

$$\Theta(\eta) = \Theta(0)\cos(ks) + \frac{\dot{\Theta}(0)}{kc_s}\sin(ks)$$

where the sound horizon is defined as $s \equiv \int c_s d\eta$

Harmonic Extrema

- All modes are frozen in at recombination (denoted with a subscript $*$)
- Temperature perturbations of different amplitude for different modes.
- For the adiabatic (curvature mode) initial conditions



$$\dot{\Theta}(0) = 0$$

- So solution

$$\Theta(\eta_*) = \Theta(0)\cos(ks_*)$$

Harmonic Extrema

- Modes caught in the extrema of their oscillation will have enhanced fluctuations

$$k_n s_* = n\pi$$

yielding a fundamental scale or frequency, related to the inverse sound horizon

$$k_A = \pi / s_*$$

and a harmonic relationship to the other extrema as $1 : 2 : 3...$

Peak Location

- The fundmental physical scale is translated into a fundamental angular scale by simple projection according to the angular diameter distance D_A

$$\begin{aligned} \theta_A &= \lambda_A / D_A \\ \ell_A &= k_A D_A \end{aligned}$$

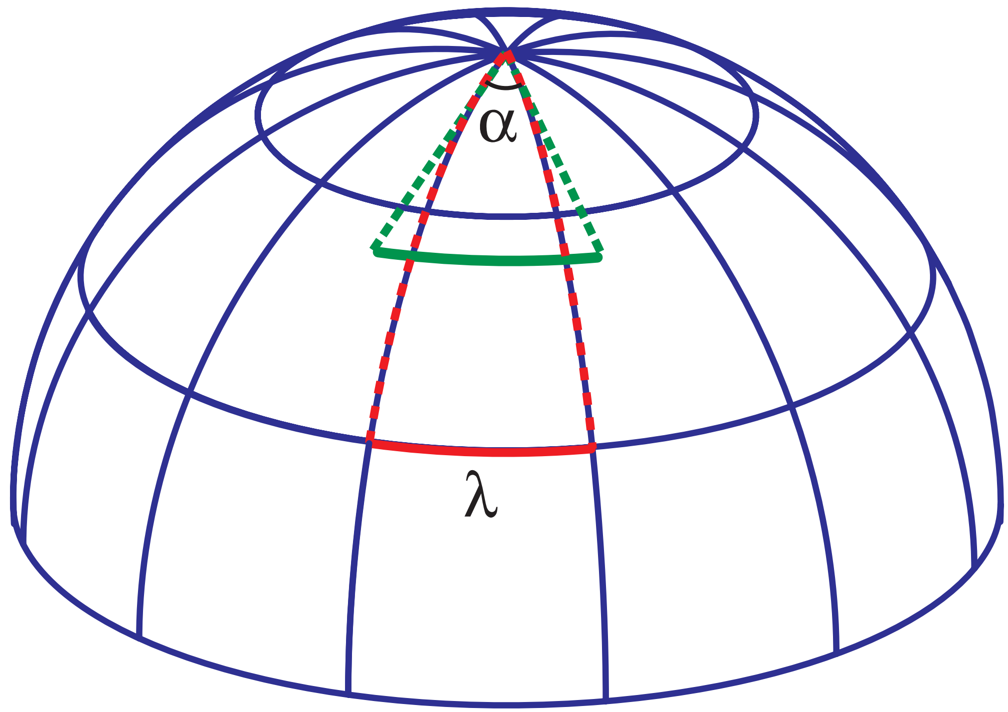
- In a flat universe, the distance is simply $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$, the horizon distance, and $k_A = \pi / s_* = \sqrt{3}\pi / \eta_*$ so

$$\theta_A \approx \frac{\eta_*}{\eta_0}$$

- In a matter-dominated universe $\eta \propto a^{1/2}$ so $\theta_A \approx 1/30 \approx 2^\circ$ or

$$\ell_A \approx 200$$

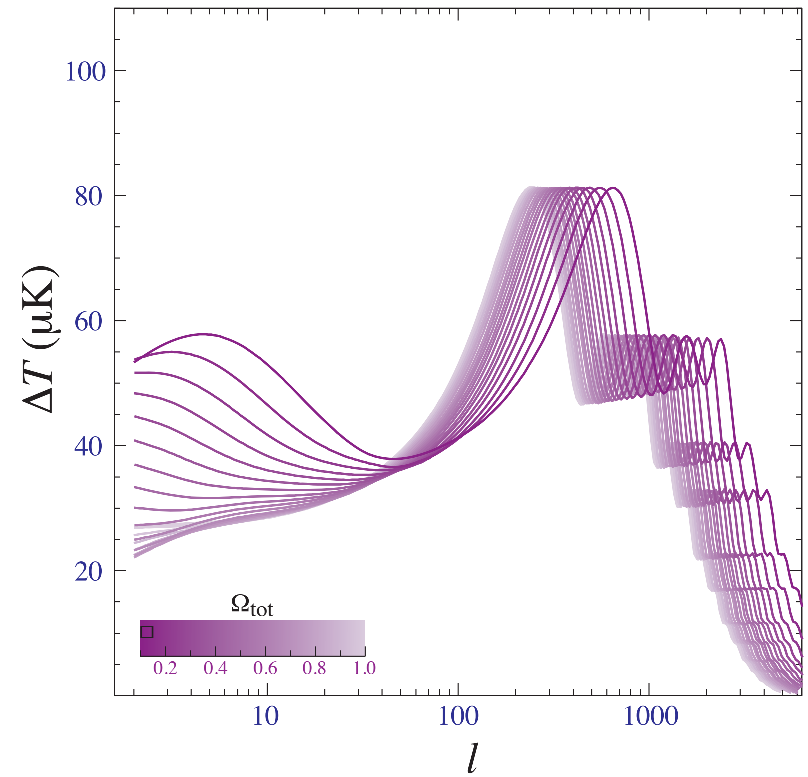
Curvature



- In a curved universe, the apparent or angular diameter distance is no longer the conformal distance $D_A = R\sin(D/R) \neq D$
- Objects in a closed universe are further than they appear! gravitational lensing of the background...
- Curvature scale of the universe must be substantially larger than current horizon

Curvature

- Flat universe indicates critical density and implies missing energy given local measures of the matter density "dark energy"
- D also depends on dark energy density $\Omega_{\rm DE}$ and equation of state $w = p_{\rm DE}/\rho_{\rm DE}$.
- Expansion rate at recombination or matter-radiation ratio enters into calculation of k_A.



Fixed Deceleration Epoch

- CMB determination of matter density controls all determinations in the deceleration (matter dominated) epoch
- WMAP7: $\Omega_m h^2 = 0.133 \pm 0.006 \to 4.5\%$
- Distance to recombination D_* determined to $\frac{1}{4}4.5\% \approx 1\%$
- Expansion rate during any redshift in the deceleration epoch determined to 4.5%
- Distance to any redshift in the deceleration epoch determined as

$$D(z) = D_* - \int_z^{z_*} \frac{dz}{H(z)}$$

- Volumes determined by a combination $dV = D_A^2 d\Omega dz / H(z)$
- Structure also determined by growth of fluctuations from z_*
- $\Omega_m h^2$ can be determined to $\sim 1\%$ from Planck.

Doppler Effect

- Bulk motion of fluid changes the observed temperature via Doppler shifts

$$\left(\frac{\Delta T}{T}\right)_{\rm dop} = \hat{\mathbf{n}} \cdot \mathbf{v}_\gamma$$

- Averaged over directions

$$\left(\frac{\Delta T}{T}\right)_{\rm rms} = \frac{v_\gamma}{\sqrt{3}}$$

- Acoustic solution

$$\frac{v_\gamma}{\sqrt{3}} = -\frac{\sqrt{3}}{k}\dot{\Theta} = \frac{\sqrt{3}}{k} k c_s \, \Theta(0) \sin(ks)$$
$$= \Theta(0) \sin(ks)$$

Doppler Peaks?

- Doppler effect for the photon dominated system is of equal amplitude and $\pi/2$ out of phase: extrema of temperature are turning points of velocity
- Effects add in quadrature:

$$\left(\frac{\Delta T}{T}\right)^2 = \Theta^2(0)[\cos^2(ks) + \sin^2(ks)] = \Theta^2(0)$$

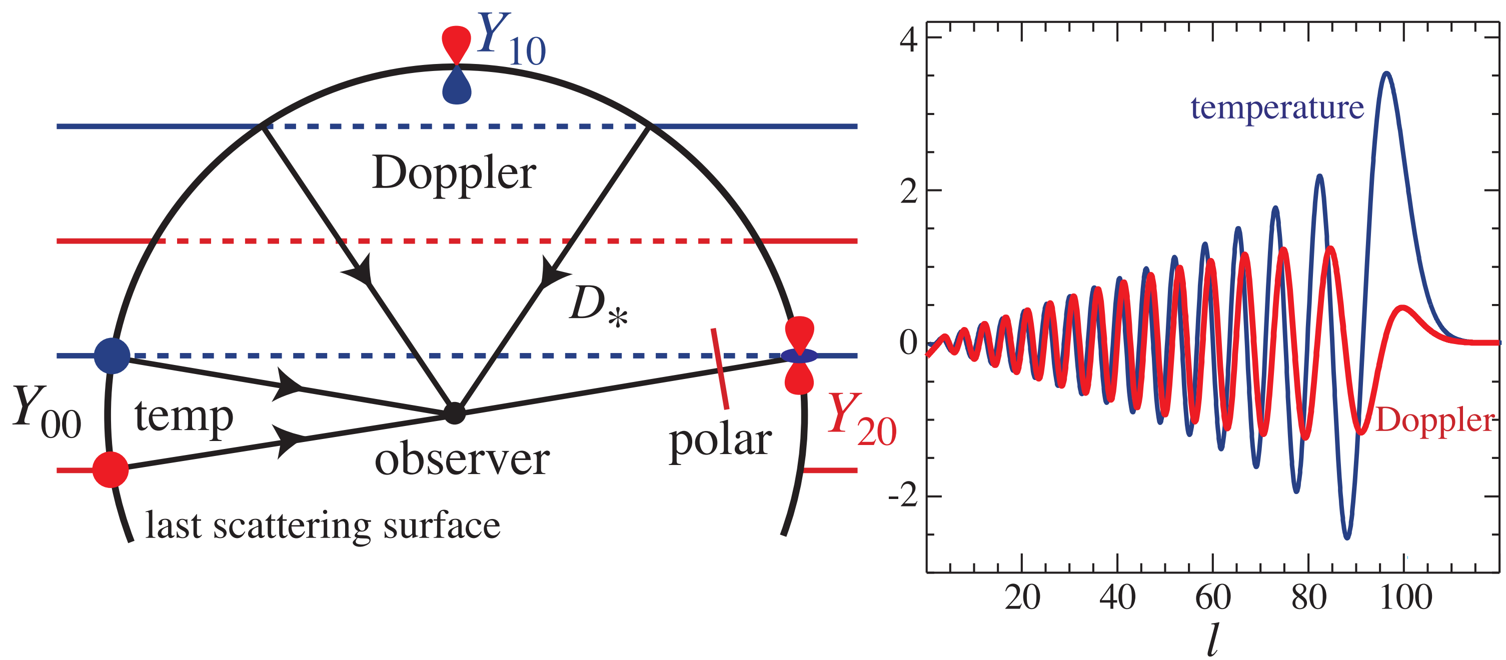
- No peaks in k spectrum! However the Doppler effect carries an angular dependence that changes its projection on the sky $\hat{\mathbf{n}} \cdot \mathbf{v}_\gamma \propto \hat{\mathbf{n}} \cdot \hat{\mathbf{k}}$

Doppler Peaks?

- Coordinates where $\hat{\mathbf{z}} \parallel \hat{\mathbf{k}}$

$$Y_{10} Y_{\ell 0} \rightarrow Y_{\ell \pm 1\, 0}$$

recoupling $j'_\ell Y_{\ell 0}$: no peaks in Doppler effect



Restoring Gravity

- Take a simple photon dominated system with gravity
- Continuity altered since a gravitational potential represents a stretching of the spatial fabric that dilutes number densities – formally a spatial curvature perturbation
- Think of this as a perturbation to the scale factor $a \to a(1+\Phi)$ so that the cosmogical redshift is generalized to

$$\frac{\dot{a}}{a} \to \frac{\dot{a}}{a} + \dot{\Phi}$$

so that the continuity equation becomes

$$\dot{\Theta} = -\frac{1}{3}kv_\gamma - \dot{\Phi}$$

Restoring Gravity

- Gravitational force in momentum conservation $\mathbf{F} = -m\nabla\Psi$ generalized to momentum density modifies the Euler equation to

$$\dot{v}_\gamma = k(\Theta + \Psi)$$

- General relativity says that Φ and Ψ are the relativistic analogues of the Newtonian potential and that $\Phi \approx -\Psi$.
- In our matter-dominated approximation, Φ represents matter density fluctuations through the cosmological Poisson equation

$$k^2\Phi = 4\pi G a^2 \rho_m \Delta_m$$

where the difference comes from the use of comoving coordinates for k (a^2 factor), the removal of the background density into the background expansion $(\rho\Delta_m)$ and finally a coordinate subtlety that enters into the definition of Δ_m

Constant Potentials

- In the matter dominated epoch potentials are constant because infall generates velocities as $v_m \sim k\eta\Psi$
- Velocity divergence generates density perturbations as $\Delta_m \sim -k\eta v_m \sim -(k\eta)^2\Psi$
- And density perturbations generate potential fluctuations

$$\Phi = \frac{4\pi G a^2 \rho \Delta}{k^2} \approx \frac{3}{2}\frac{H^2 a^2}{k}^2 \Delta \sim \frac{\Delta}{(k\eta)^2} \sim -\Psi$$

keeping them constant. Note that because of the expansion, density perturbations must grow to keep potentials constant.

Constant Potentials

- More generally, if stress perturbations are negligible compared with density perturbations ($\delta p \ll \delta\rho$) then potential will remain roughly constant
- More specifically a variant called the Bardeen or comoving curvature is strictly constant

$$\mathcal{R} = \mathrm{const} \approx \frac{5+3w}{3+3w}\Phi$$

where the approximation holds when $w \approx$const.

Oscillator: Take Two

- Combine these to form the simple harmonic oscillator equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - \ddot{\Phi}$$

- In a CDM dominated expansion $\dot{\Phi} = \dot{\Psi} = 0$. Also for photon domination $c_s^2 = 1/3$ so the oscillator equation becomes

$$\ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0$$

- Solution is just an offset version of the original

$$[\Theta + \Psi](\eta) = [\Theta + \Psi](0) \cos(ks)$$

- $\Theta + \Psi$ is also the observed temperature fluctuation since photons lose energy climbing out of gravitational potentials at recombination

Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or effective temperature

$$\Theta + \Psi$$

- Effective temperature oscillates around zero with amplitude given by the initial conditions
- Note: initial conditions are set when the perturbation is outside of horizon, need inflation or other modification to matter-radiation FRW universe.
- GR says that initial temperature is given by initial potential

Sachs-Wolfe Effect and the Magic 1/3

- A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]

$$\frac{\delta t}{t} = \Psi$$

- Convert this to a perturbation in the scale factor,

$$t = \int \frac{da}{aH} \propto \int \frac{da}{a\rho^{1/2}} \propto a^{3(1+w)/2}$$

where $w \equiv p/\rho$ so that during matter domination

$$\frac{\delta a}{a} = \frac{2}{3}\frac{\delta t}{t}$$

- CMB temperature is cooling as $T \propto a^{-1}$ so

$$\Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3}\Psi$$

Sachs-Wolfe Normalization

- Use measurements of $\Delta T/T \approx 10^{-5}$ in the Sachs-Wolfe effect to infer $\Delta^2_{\mathcal{R}}$
- Recall in matter domination $\Psi = -3\mathcal{R}/5$

$$\frac{\ell(\ell+1)C_\ell}{2\pi} \approx \Delta_T^2 \approx \frac{1}{25}\Delta_R^2$$

- So that the amplitude of initial curvature fluctuations is $\Delta_R \approx 5 \times 10^{-5}$
- Modern usage: WMAP's measurement of 1st peak plus known radiation transfer function is used to convert $\Delta T/T$ to Δ_R.

Baryon Loading

- Baryons add extra mass to the photon-baryon fluid
- Controlling parameter is the momentum density ratio:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}}\right)$$

of order unity at recombination

- Momentum density of the joint system is conserved

$$(\rho_\gamma + p_\gamma)v_\gamma + (\rho_b + p_b)v_b \approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma)v_\gamma$$
$$= (1 + R)(\rho_\gamma + p_\gamma)v_{\gamma b}$$

where the controlling parameter is the momentum density ratio:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}}\right)$$

of order unity at recombination

New Euler Equation

- Momentum density ratio enters as

$$[(1+R)v_{\gamma b}]^{\cdot} = k\Theta + (1+R)k\Psi$$

- Photon continuity remains the same

$$\dot{\Theta} = -\frac{k}{3}v_{\gamma b} - \dot{\Phi}$$

- Modification of oscillator equation

$$[(1+R)\dot{\Theta}]^{\cdot} + \frac{1}{3}k^2\Theta = -\frac{1}{3}k^2(1+R)\Psi - [(1+R)\dot{\Phi}]^{\cdot}$$

Oscillator: Take Three

- Combine these to form the not-quite-so simple harmonic oscillator equation

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

where $c_s^2 \equiv \dot{p}_{\gamma b}/\dot{\rho}_{\gamma b}$

$$c_s^2 = \frac{1}{3}\frac{1}{1+R}$$

- In a CDM dominated expansion $\dot{\Phi} = \dot{\Psi} = 0$ and the adiabatic approximation $\dot{R}/R \ll \omega = kc_s$

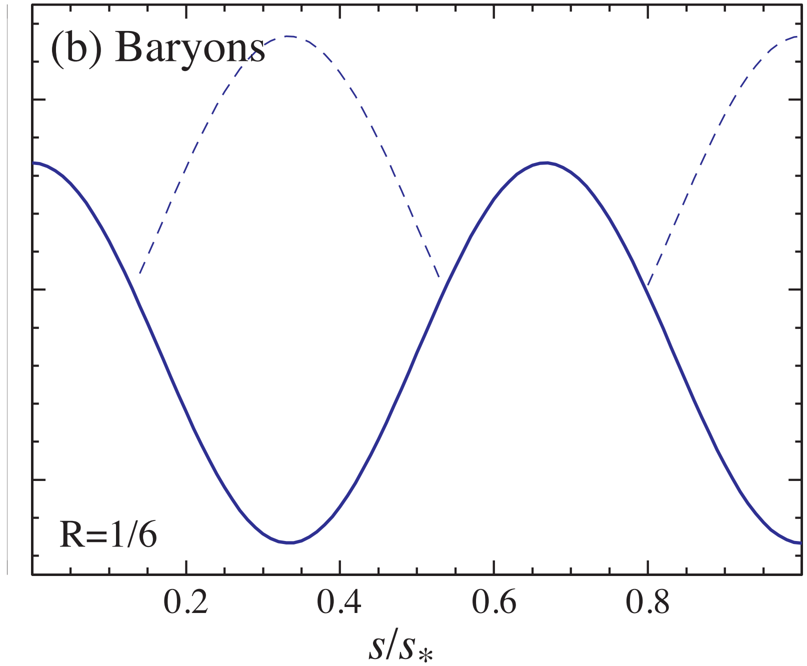
$$[\Theta + (1+R)\Psi](\eta) = [\Theta + (1+R)\Psi](0)\cos(ks)$$

Baryon Peak Phenomenology

- Photon-baryon ratio enters in three ways
- Overall larger amplitude:

$$[\Theta + (1+R)\Psi](0) = \frac{1}{3}(1+3R)\Psi(0)$$

- Even-odd peak modulation of effective temperature



$$[\Theta + \Psi]_{\rm peaks} = [\pm(1+3R) - 3R]\,\frac{1}{3}\Psi(0)$$

$$[\Theta + \Psi]_1 - [\Theta + \Psi]_2 = [-6R]\frac{1}{3}\Psi(0)$$

- Shifting of the sound horizon down or ℓ_A up

$$\ell_A \propto \sqrt{1+R}$$

Photon Baryon Ratio Evolution

- Actual effects smaller since R evolves
- Oscillator equation has time evolving mass

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = 0$$

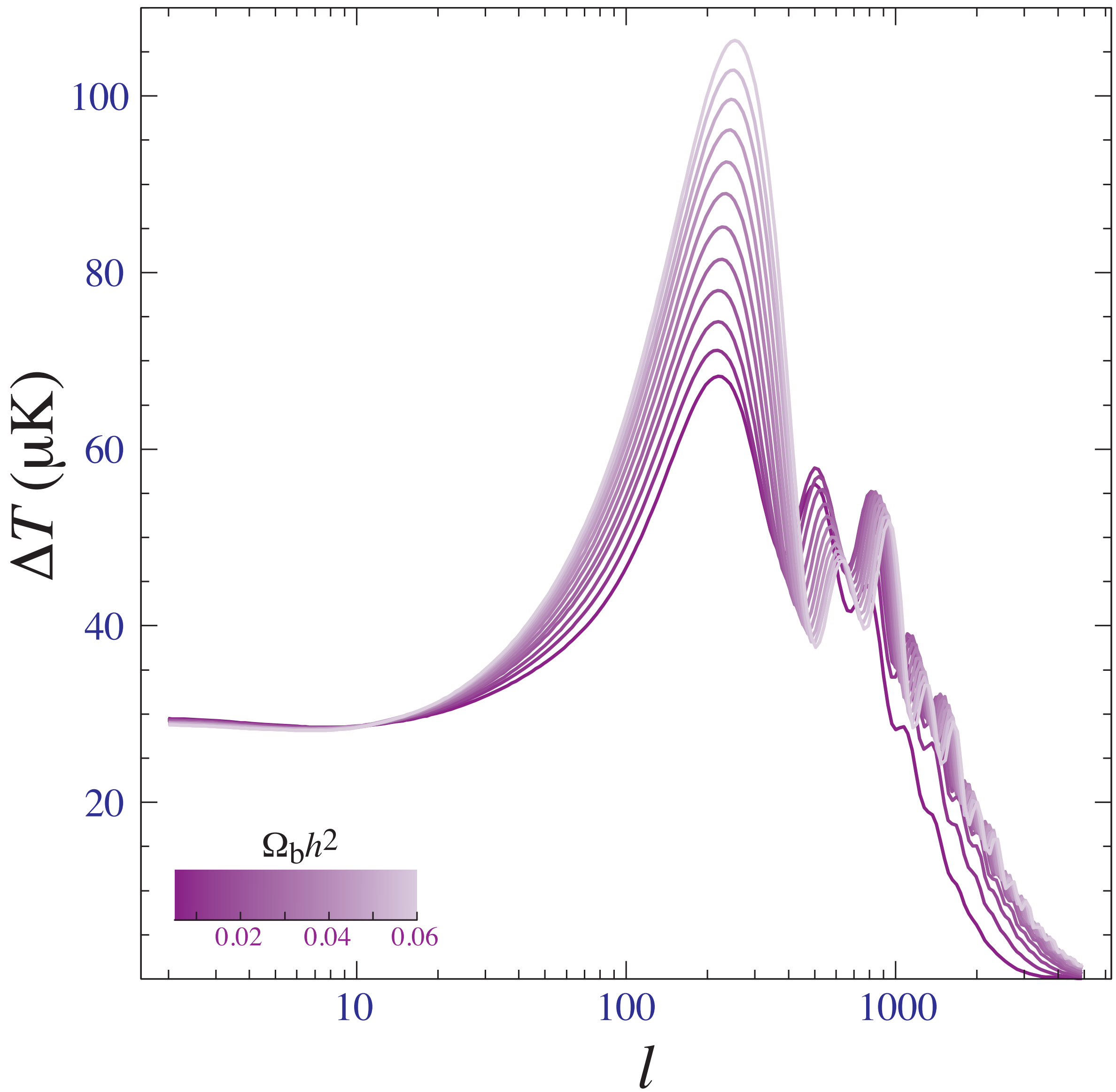
- Effective mass is is $m_{\rm eff} = 3c_s^{-2} = (1 + R)$
- Adiabatic invariant

$$\frac{E}{\omega} = \frac{1}{2} m_{\rm eff} \omega A^2 = \frac{1}{2} 3 c_s^{-2} k c_s A^2 \propto A^2 (1 + R)^{1/2} = const.$$

- Amplitude of oscillation $A \propto (1 + R)^{-1/4}$ decays adiabatically as the photon-baryon ratio changes

Baryons in the Power Spectrum

- Relative heights of peaks



Oscillator: Take Three and a Half

- The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\Phi)$$

  changes in the gravitational potentials alter the form of the acoustic oscillations

- If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator
- Term involving Ψ is the ordinary gravitational force
- Term involving Φ involves the $\dot{\Phi}$ term in the continuity equation as a (curvature) perturbation to the scale factor

Potential Decay

- Matter-to-radiation ratio

$$\frac{\rho_m}{\rho_r} \approx 24\Omega_m h^2 \left(\frac{a}{10^{-3}}\right)$$

  of order unity at recombination in a low Ω_m universe

- Radiation is not stress free and so impedes the growth of structure

$$k^2\Phi = 4\pi G a^2 \rho_r \Delta_r$$

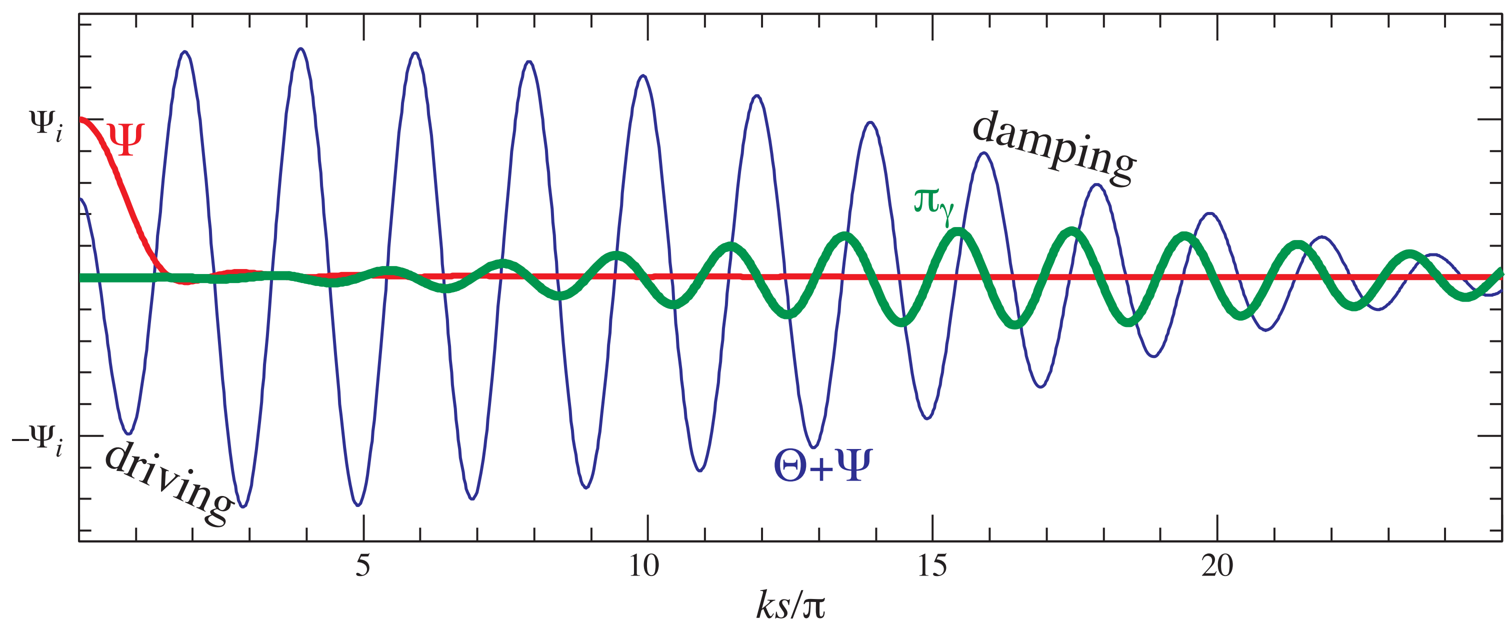
  $\Delta_r \sim 4\Theta$ oscillates around a constant value, $\rho_r \propto a^{-4}$ so the Netwonian curvature decays.

- General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale

Radiation Driving

- Decay is timed precisely to drive the oscillator - close to fully coherent

$$
\begin{aligned}
|[\Theta + \Psi](\eta)| &= |[\Theta + \Psi](0) + \Delta\Psi - \Delta\Phi| \\
&= |\frac{1}{3}\Psi(0) - 2\Psi(0)| = |\frac{5}{3}\Psi(0)|
\end{aligned}
$$



- $5\times$ the amplitude of the Sachs-Wolfe effect!

External Potential Approach

- Solution to homogeneous equation

$$(1+R)^{-1/4}\cos(ks)\,, \qquad (1+R)^{-1/4}\sin(ks)$$

- Give the general solution for an external potential by propagating impulsive forces

$$(1+R)^{1/4}\Theta(\eta) = \Theta(0)\cos(ks) + \frac{\sqrt{3}}{k}\left[\dot\Theta(0) + \frac{1}{4}\dot R(0)\Theta(0)\right]\sin ks + \frac{\sqrt{3}}{k}\int_0^\eta d\eta'(1+R')^{3/4}\sin[ks - ks']F(\eta')$$

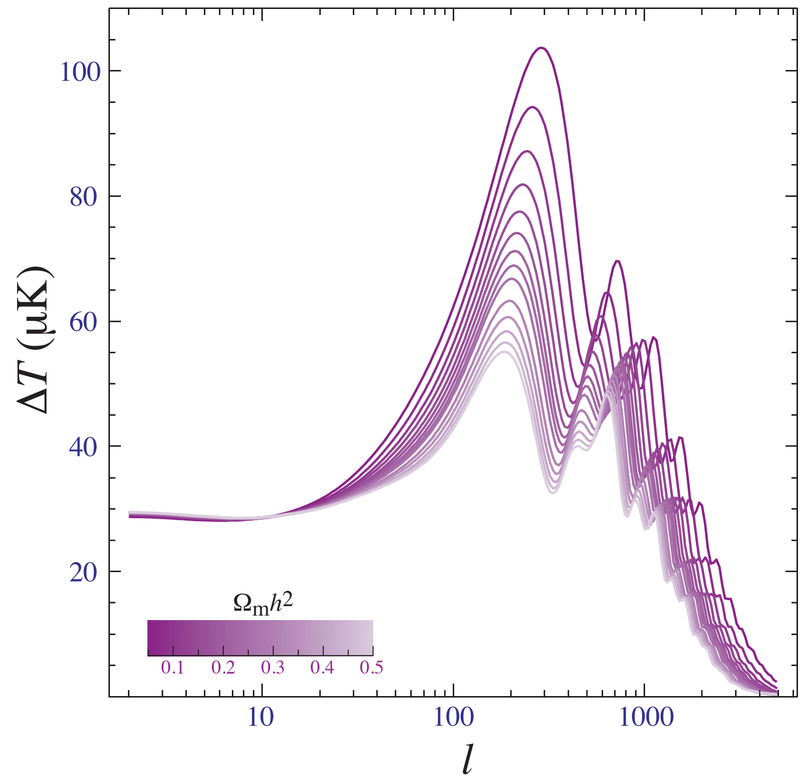
where

$$F = -\ddot\Phi - \frac{\dot R}{1+R}\dot\Phi - \frac{k^2}{3}\Psi$$

- Useful if general form of potential evolution is known

Matter-Radiation in the Power Spectrum

- Coherent approximation is exact for a photon-baryon fluid but reality is reduced to $\sim 4\times$ because of neutrino contribution to radiation
- Actual initial conditions are $\Theta + \Psi = \Psi/2$ for radiation domination but comparison to matter dominated SW correct



Damping

- Tight coupling equations assume a perfect fluid: no viscosity, no heat conduction
- Fluid imperfections are related to the mean free path of the photons in the baryons

$$\lambda_C = \dot{\tau}^{-1} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a$$

  is the conformal opacity to Thomson scattering
- Dissipation is related to the diffusion length: random walk approximation

$$\lambda_D = \sqrt{N} \lambda_C = \sqrt{\eta / \lambda_C}\, \lambda_C = \sqrt{\eta \lambda_C}$$

  the geometric mean between the horizon and mean free path
- $\lambda_D / \eta_* \sim$ few %, so expect the peaks > 3 to be affected by dissipation

Equations of Motion

- Continuity

$$\dot{\Theta} = -\frac{k}{3} v_\gamma - \dot{\Phi} \,, \quad \dot{\delta}_b = -k v_b - 3\dot{\Phi}$$

where the photon equation remains unchanged and the baryons follow number conservation with $\rho_b = m_b n_b$

- Navier-Stokes (Euler + heat conduction, viscosity)

$$\begin{aligned} \dot{v}_\gamma &= k(\Theta + \Psi) - \frac{k}{6}\pi_\gamma - \dot{\tau}(v_\gamma - v_b) \\ \dot{v}_b &= -\frac{\dot{a}}{a} v_b + k\Psi + \dot{\tau}(v_\gamma - v_b)/R \end{aligned}$$

where the photons gain an anisotropic stress term π_γ from radiation viscosity and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation

Viscosity

- Viscosity is generated from radiation streaming from hot to cold regions
- Expect

$$\pi_\gamma \sim v_\gamma \frac{k}{\dot{\tau}}$$

generated by streaming, suppressed by scattering in a wavelength of the fluctuation. Radiative transfer says

$$\pi_\gamma \approx 2 A_v v_\gamma \frac{k}{\dot{\tau}}$$

where $A_v = 16/15$

$$\dot{v}_\gamma = k(\Theta + \Psi) - \frac{k}{3} A_v \frac{k}{\dot{\tau}} v_\gamma$$

Oscillator: Penultimate Take

- Adiabatic approximation ($\omega \gg \dot{a}/a$)

$$\dot{\Theta} \approx -\frac{k}{3} v_\gamma$$

- Oscillator equation contains a $\dot{\Theta}$ damping term

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} A_v \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

- Heat conduction term similar in that it is proportional to v_γ and is suppressed by scattering $k/\dot{\tau}$. Expansion of Euler equations to leading order in $k\dot{\tau}$ gives

$$A_h = \frac{R^2}{1+R}$$

since the effects are only significant if the baryons are dynamically important

Oscillator: Final Take

- Final oscillator equation

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}}[A_v + A_h]\dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

- Solve in the adiabatic approximation

$$\Theta \propto \exp(i \int \omega d\eta)$$

$$-\omega^2 + \frac{k^2 c_s^2}{\dot{\tau}}(A_v + A_h) i\omega + k^2 c_s^2 = 0$$

Dispersion Relation

- Solve

$$\omega^2 = k^2 c_s^2 \left[1 + i\frac{\omega}{\dot\tau}(A_v + A_h)\right]$$

$$\omega = \pm k c_s \left[1 + \frac{i}{2}\frac{\omega}{\dot\tau}(A_v + A_h)\right]$$

$$= \pm k c_s \left[1 \pm \frac{i}{2}\frac{k c_s}{\dot\tau}(A_v + A_h)\right]$$

- Exponentiate

$$\exp(i\int \omega d\eta) = e^{\pm iks} \exp[-k^2 \int d\eta \frac{1}{2}\frac{c_s^2}{\dot\tau}(A_v + A_h)]$$

$$= e^{\pm iks} \exp[-(k/k_D)^2]$$

- Damping is exponential under the scale k_D

Diffusion Scale

- Diffusion wavenumber

$$k_D^{-2} = \int d\eta \frac{1}{\dot\tau} \frac{1}{6(1+R)} \left(\frac{16}{15} + \frac{R^2}{(1+R)} \right)$$

- Limiting forms

$$\lim_{R \to 0} k_D^{-2} = \frac{1}{6} \frac{16}{15} \int d\eta \frac{1}{\dot\tau}$$

$$\lim_{R \to \infty} k_D^{-2} = \frac{1}{6} \int d\eta \frac{1}{\dot\tau}$$

- Geometric mean between horizon and mean free path as expected from a random walk

$$\lambda_D = \frac{2\pi}{k_D} \sim \frac{2\pi}{\sqrt{6}} (\eta \dot\tau^{-1})^{1/2}$$

Thomson Scattering

- Polarization state of radiation in direction $\hat{\mathbf{n}}$ described by the intensity matrix $\langle E_i(\hat{\mathbf{n}})E_j^*(\hat{\mathbf{n}})\rangle$, where $\mathbf{E}$ is the electric field vector and the brackets denote time averaging.
- Differential cross section

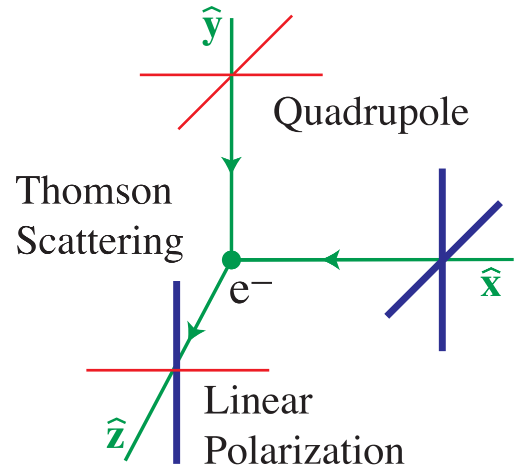
$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi}|\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \,,$$

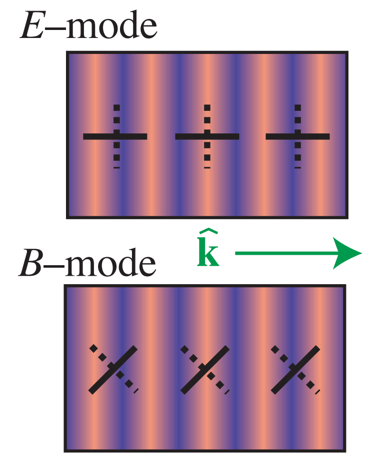
  where $\sigma_T = 8\pi\alpha^2/3m_e$ is the Thomson cross section, $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.
- Summed over angle and incoming polarization

$$\sum_{i=1,2} \int d\hat{\mathbf{n}}' \frac{d\sigma}{d\Omega} = \sigma_T$$

Polarization Generation

- Heuristic:
  incoming radiation shakes
  an electron in direction
  of electric field vector $\hat{\mathbf{E}}'$





- Radiates photon with
  polarization also in direction $\hat{\mathbf{E}}'$
- But photon cannot be longitudinally polarized so that scattering into 90° can only pass one polarization
- Linearly polarized radiation like polarization by reflection
- Unlike reflection of sunlight, incoming radiation is nearly isotropic
- Missing from direction orthogonal to original incoming direction
- Only quadrupole anisotropy generates polarization by Thomson scattering

Acoustic Polarization

- Break down of tight-coupling leads to quadrupole anisotropy of

$$\pi_\gamma \approx \frac{k}{\dot\tau} v_\gamma$$

- Scaling $k_D = (\dot\tau/\eta_*)^{1/2} \rightarrow \dot\tau = k_D^2 \eta_*$
- Know: $k_D s_* \approx k_D \eta_* \approx 10$
- So:

$$\pi_\gamma \approx \frac{k}{k_D}\frac{1}{10} v_\gamma$$

$$\Delta_P \approx \frac{\ell}{\ell_D}\frac{1}{10}\Delta_T$$

Acoustic Polarization

- Gradient of velocity is along direction of wavevector, so polarization is pure E-mode
- Velocity is 90° out of phase with temperature – turning points of oscillator are zero points of velocity:

$$\Theta + \Psi \propto \cos(ks); \quad v_\gamma \propto \sin(ks)$$

- Polarization peaks are at troughs of temperature power

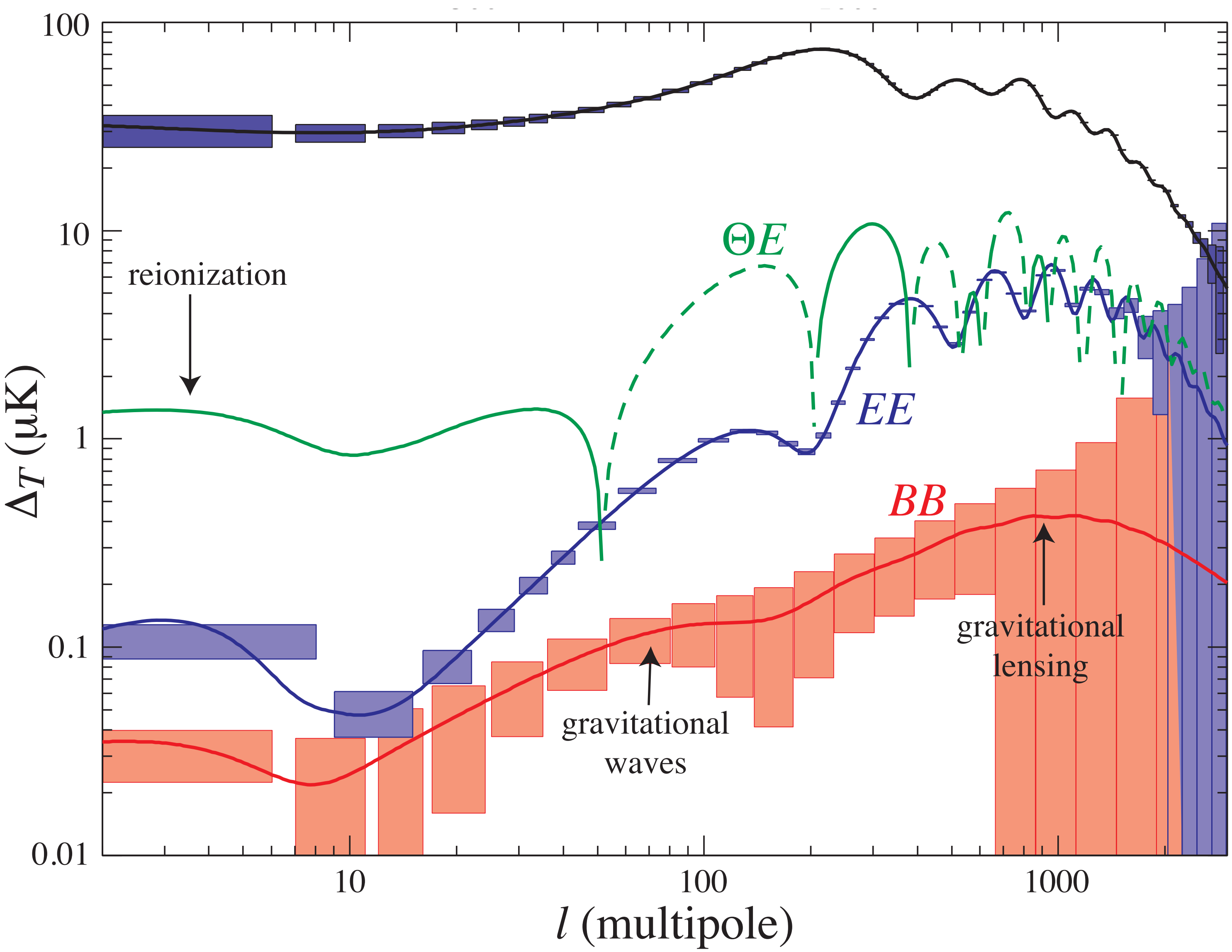
Cross Correlation

- Cross correlation of temperature and polarization

$$(\Theta + \Psi)(v_\gamma) \propto \cos(ks)\sin(ks) \propto \sin(2ks)$$

- Oscillation at twice the frequency
- Correlation: radial or tangential around hot spots
- Partial correlation: easier to measure if polarization data is noisy, harder to measure if polarization data is high S/N or if bands do not resolve oscillations
- Good check for systematics and foregrounds
- Comparison of temperature and polarization is proof against features in initial conditions mimicking acoustic features

Polarization Power



Angular Moments

- Define the angularly dependent Stokes perturbation

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta), \quad Q(\mathbf{x}, \hat{\mathbf{n}}, \eta), \quad U(\mathbf{x}, \hat{\mathbf{n}}, \eta)$$

- Decompose into normal modes: plane waves for spatial part and spherical harmonics for angular part

$$G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k}\cdot\mathbf{x})$$
$${}_{\pm 2}G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} {}_{\pm 2}Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k}\cdot\mathbf{x})$$

- In a spatially curved universe generalize the plane wave part

Normal Modes

- Temperature and polarization fields

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} \Theta_\ell^{(m)} G_\ell^m$$

$$[Q \pm iU](\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} [E_\ell^{(m)} \pm iB_\ell^{(m)}]_{\pm 2} G_\ell^m$$

- For each $\mathbf{k}$ mode, work in coordinates where $\mathbf{k} \parallel \mathbf{z}$ and so $m = 0$ represents scalar modes, $m = \pm 1$ vector modes, $m = \pm 2$ tensor modes, $|m| > 2$ vanishes. Since modes add incoherently and $Q \pm iU$ is invariant up to a phase, rotation back to a fixed coordinate system is trivial.

Liouville Equation

- In absence of scattering, the phase space distribution of photons in each polarization state a is conserved along the propagation path
- Rewrite variables in terms of the photon propagation direction $\mathbf{q} = q\hat{\mathbf{n}}$, so $f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta)$ and

$$\frac{d}{d\eta} f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta) = 0$$

$$= \left(\frac{\partial}{\partial \eta} + \frac{d\mathbf{x}}{d\eta} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{d\hat{\mathbf{n}}}{d\eta} \cdot \frac{\partial}{\partial \hat{\mathbf{n}}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a$$

- For simplicity, assume spatially flat universe $K = 0$ then $d\hat{\mathbf{n}}/d\eta = 0$ and $d\mathbf{x} = \hat{\mathbf{n}} d\eta$

$$\dot{f}_a + \hat{\mathbf{n}} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0$$

Scalar, Vector, Tensor

- Normalization of modes is chosen so that the lowest angular mode for scalars, vectors and tensors are normalized in the same way as the mode function

$$G_0^0 = Q^{(0)} \quad G_1^0 = n^i Q_i^{(0)} \quad G_2^0 \propto n^i n^j Q_{ij}^{(0)}$$
$$G_1^{\pm 1} = n^i Q_i^{(\pm 1)} \quad G_2^{\pm 1} \propto n^i n^j Q_{ij}^{(\pm 1)}$$
$$G_2^{\pm 2} = n^i n^j Q_{ij}^{(\pm 2)}$$

where recall

$$\begin{aligned} Q^{(0)} &= \exp(i\mathbf{k}\cdot\mathbf{x}) \\ Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k}\cdot\mathbf{x}) \\ Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k}\cdot\mathbf{x}) \end{aligned}$$

Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies
- Spatial gradient term hits plane wave:

$$\hat{\mathbf{n}} \cdot \nabla e^{i\mathbf{k}\cdot\mathbf{x}} = i\hat{\mathbf{n}} \cdot \mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} = i\sqrt{\frac{4\pi}{3}} k Y_1^0(\hat{\mathbf{n}}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

- Dipole term adds to angular dependence through the addition of angular momentum

$$\sqrt{\frac{4\pi}{3}} Y_1^0 Y_\ell^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell+1)(2\ell-1)}} Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell+1)(2\ell+3)}} Y_{\ell+1}^m$$

where $\kappa_\ell^m = \sqrt{\ell^2 - m^2}$ is given by Clebsch-Gordon coefficients.

Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

$$\dot{\Theta}_\ell^{(m)} = k\left[\frac{\kappa_\ell^m}{2\ell+1}\Theta_{\ell-1}^{(m)} - \frac{\kappa_{\ell+1}^m}{2\ell+3}\Theta_{\ell+1}^{(m)}\right] - \dot{\tau}\Theta_\ell^{(m)} + S_\ell^{(m)}$$

  where $S_\ell^{(m)}$ are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic $\ell = 0$ temperature perturbation will eventually become a high order anisotropy by "free streaming" or simple projection
- Original CMB codes solved the full hierarchy equations out to the ℓ of interest.

Integral Solution

- Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface
- In general, the solution describes the decomposition of the source $S_\ell^{(m)}$ with its local angular dependence as seen at a distance $\mathbf{x} = D\hat{\mathbf{n}}$.
- Proceed by decomposing the angular dependence of the plane wave

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} j_\ell(kD) Y_\ell^0(\hat{\mathbf{n}})$$

- Recouple to the local angular dependence of G_ℓ^m

$$G_{\ell_s}^m = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} \alpha_{\ell_s \ell}^{(m)}(kD) Y_\ell^m(\hat{\mathbf{n}})$$

Integral Solution

- Projection kernels:

$$\ell_s = 0, \quad m = 0 \qquad\qquad \alpha^{(0)}_{0\ell} \equiv j_\ell$$

$$\ell_s = 1, \quad m = 0 \qquad\qquad \alpha^{(0)}_{1\ell} \equiv j'_\ell$$

- Integral solution:

$$\frac{\Theta^{(m)}_\ell(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S^{(m)}_{\ell_s} \alpha^{(m)}_{\ell_s \ell}(k(\eta_0-\eta))$$

- Power spectrum:

$$C_\ell = \frac{2}{\pi}\int \frac{dk}{k} \sum_m \frac{k^3 \langle \Theta^{(m)*}_\ell \Theta^{(m)}_\ell \rangle}{(2\ell+1)^2}$$

- Solving for C_ℓ reduces to solving for the behavior of a handful of sources

Polarization Hierarchy

- In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hierarchy:

$$\dot{E}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell+1)}B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}E_{\ell+1}^{(m)}\right] - \dot{\tau}E_\ell^{(m)} + \mathcal{E}_\ell^{(m)}$$

$$\dot{B}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell+1)}E_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}B_{\ell+1}^{(m)}\right] - \dot{\tau}B_\ell^{(m)} + \mathcal{B}_\ell^{(m)}$$

  where ${}_2\kappa_\ell^m = \sqrt{(\ell^2-m^2)(\ell^2-4)/\ell^2}$ is given by the Clebsch-Gordon coefficients and $\mathcal{E}$, $\mathcal{B}$ are the sources (scattering only).

- Note that for vectors and tensors $|m|>0$ and B modes may be generated from E modes by projection. Cosmologically $\mathcal{B}_\ell^{(m)} = 0$

Polarization Integral Solution

- Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

$$\frac{E_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \epsilon_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

$$\frac{B_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \beta_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

- The only source to the polarization is from the quadrupole anisotropy so we only need $\ell_s = 2$, e.g. for scalars

$$\epsilon_{2\ell}^{(0)}(x) = \sqrt{\frac{3}{8}\frac{(\ell+2)!}{(\ell-2)!}}\frac{j_\ell(x)}{x^2} \qquad \beta_{2\ell}^{(0)} = 0$$

Gravitational Terms

- As in our Newtonian gauge calculation, gravitational terms - now including vectors and tensors in an arbitrary gauge, come from the geodesic equation
- First define the slicing (lapse function A, shift function B^i)

$$
\begin{aligned}
g^{00} &= -a^{-2}(1-2A)\,, \\
g^{0i} &= -a^{-2}B^i\,,
\end{aligned}
$$

A defines the lapse of proper time between 3-surfaces whereas B^i defines the threading or relationship between the 3-coordinates of the surfaces

Gravitational Terms

- This absorbs 1+3=4 degrees of freedom in the metric, remaining 6 is in the spatial surfaces which we parameterize as

$$g^{ij} = a^{-2}(\gamma^{ij} - 2H_L\gamma^{ij} - 2H_T^{ij})\,.$$

  here (1) H_L a perturbation to the spatial curvature; (5) H_T^{ij} a trace-free distortion to spatial metric (which also can perturb the curvature)

- Geodesic equation gives the redshifting term

$$\frac{\dot q}{q} = -\frac{\dot a}{a} - \frac{1}{2}n^i n^j \dot H_{Tij} - \dot H_L + n^i \dot B_i - \hat{\mathbf{n}} \cdot \nabla A$$

  which is incorporated in the conservation and gauge transformation equations

Source Terms

- Temperature source terms $S_l^{(m)}$ (rows $\pm|m|$; flat assumption

$$\left(\begin{array}{lll} \dot\tau\Theta_0^{(0)} - \dot H_L^{(0)} & \dot\tau v_b^{(0)} + \dot B^{(0)} & \dot\tau P^{(0)} - \frac{2}{3}\dot H_T^{(0)} \\ 0 & \dot\tau v_b^{(\pm1)} + \dot B^{(\pm1)} & \dot\tau P^{(\pm1)} - \frac{\sqrt{3}}{3}\dot H_T^{(\pm1)} \\ 0 & 0 & \dot\tau P^{(\pm2)} - \dot H_T^{(\pm2)} \end{array} \right)$$

where

$$P^{(m)} \equiv \frac{1}{10}(\Theta_2^{(m)} - \sqrt{6}E_2^{(m)})$$

- Polarization source term

$$\mathcal{E}_\ell^{(m)} = -\dot\tau\sqrt{6}P^{(m)}\delta_{\ell,2}$$

$$\mathcal{B}_\ell^{(m)} = 0$$

Truncated Hierarchy

- CMBFast introduced the hybrid truncated hierarchy, integral solution technique
- Formal integral solution contains sources that are not external to system but defined through the Boltzmann hierarchy itself
- Solution: recall that we used this technique in the tight coupling regime by applying a closure condition from tight coupling
- CMBFast extends this idea by solving a truncated hierarchy of equations, e.g. out to $\ell = 25$ with non-reflecting boundary conditions
- For completeness, we explicitly derive the scattering source term via polarized radiative transfer in the last part of the notes

Polarized Radiative Transfer

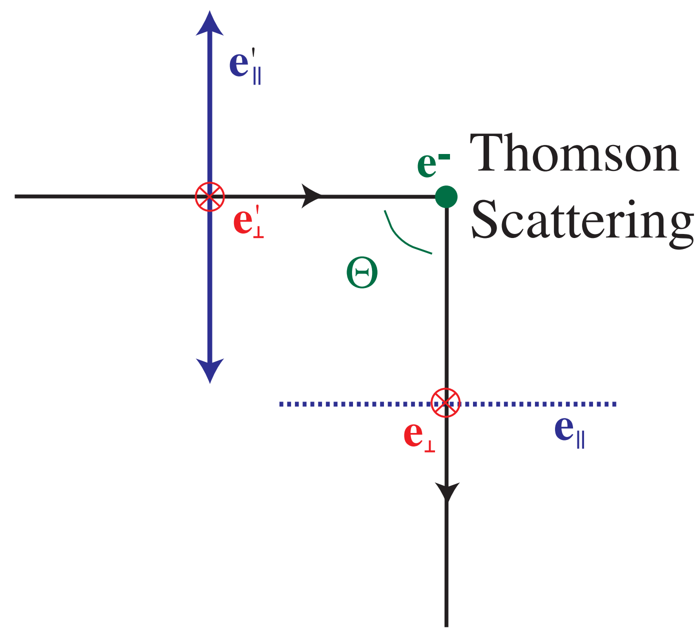
- Define a specific intensity "vector": $\mathbf{I}_\nu = (\Theta_\parallel, \Theta_\perp, U, V)$ where $\Theta = \Theta_\parallel + \Theta_\perp$, $Q = \Theta_\parallel - \Theta_\perp$

$$\frac{d\mathbf{I}_\nu}{d\eta} = \dot{\tau}(\mathbf{S}_\nu - \mathbf{I}_\nu)$$

- Thomson collision

  based on differential cross section

$$\frac{d\sigma_T}{d\Omega} = \frac{3}{8\pi}|\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \ ,$$



Polarized Radiative Transfer

- $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.
- Thomson scattering by 90 deg: $\Theta_\perp \to \Theta_\perp$ but $\Theta_{||}$ does not scatter
- More generally if Θ is the scattering angle

$$\mathbf{S}_\nu = \frac{3}{8\pi}\int d\Omega' \begin{pmatrix} \cos^2\Theta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\Theta & 0 \\ 0 & 0 & 0 & \cos\Theta \end{pmatrix} \mathbf{I}'_\nu$$

- But to calculate Stokes parameters in a fixed coordinate system must rotate into the scattering basis, scatter and rotate back out to the fixed coordinate system

Thomson Collision Term

- The $U \to U'$ transfer follows by writing down the polarization vectors in the $45°$ rotated basis

$$\hat{\mathbf{E}}_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_\parallel + \hat{\mathbf{E}}_\perp), \qquad \hat{\mathbf{E}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_\parallel - \hat{\mathbf{E}}_\perp)$$

- Define the temperature in this basis

$$\begin{aligned}
\Theta_1 &\propto |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_1|^2 \Theta_1' + |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2|^2 \Theta_2' \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta_1' + \frac{1}{4}(\cos\beta - 1)^2 \Theta_2' \\
\Theta_2 &\propto |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_2|^2 \Theta_2' + |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_1|^2 \Theta_1' \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta_2' + \frac{1}{4}(\cos\beta - 1)^2 \Theta_1'
\end{aligned}$$

or $\Theta_1 - \Theta_2 \propto \cos\beta(\Theta_1' - \Theta_2')$

Scattering Matrix

- Transfer matrix of Stokes state $\mathbf{T} \equiv (\Theta, Q + iU, Q - iU)$

$$\mathbf{T} \propto \mathbf{S}(\beta)\mathbf{T}'$$

$$\mathbf{S}(\beta) = \frac{3}{4}\begin{pmatrix} \cos^2\beta + 1 & -\frac{1}{2}\sin^2\beta & -\frac{1}{2}\sin^2\beta \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta + 1)^2 & \frac{1}{2}(\cos\beta - 1)^2 \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta - 1)^2 & \frac{1}{2}(\cos\beta + 1)^2 \end{pmatrix}$$

normalization factor of 3 is set by photon conservation in scattering

Scattering Matrix

- Transform to a fixed basis, by a rotation of the incoming and outgoing states $\mathbf{T} = \mathbf{R}(\psi)\mathbf{T}$ where

$$\mathbf{R}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & e^{2i\psi} \end{pmatrix}$$

giving the scattering matrix

$$\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha) = \tag{1}$$

$$\frac{1}{2}\sqrt{\frac{4\pi}{5}} \begin{pmatrix} Y_2^0(\beta,\alpha) + 2\sqrt{5}Y_0^0(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^{-2}(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^{2}(\beta,\alpha) \\ -\sqrt{6}\,{}_2Y_2^0(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^{-2}(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^{2}(\beta,\alpha)e^{2i\gamma} \\ -\sqrt{6}\,{}_{-2}Y_2^0(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^{-2}(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^{2}(\beta,\alpha)e^{-2i\gamma} \end{pmatrix} \tag{2}$$

Addition Theorem for Spin Harmonics

- Spin harmonics are related to rotation matrices as

$$ {}_sY_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\mathcal{D}^\ell_{-ms}(\phi,\theta,0) $$

Note: for explicit evaluation sign convention differs from usual (e.g. Jackson) by $(-1)^m$

- Multiplication of rotations

$$ \sum_{m''} \mathcal{D}^\ell_{mm''}(\alpha_2,\beta_2,\gamma_2)\mathcal{D}^\ell_{m''m}(\alpha_1,\beta_1,\gamma_1) = \mathcal{D}^\ell_{mm'}(\alpha,\beta,\gamma) $$

- Implies

$$ \sum_m {}_{s_1}Y_\ell^{m*}(\theta',\phi')\ {}_{s_2}Y_\ell^m(\theta,\phi) = (-1)^{s_1-s_2}\sqrt{\frac{2\ell+1}{4\pi}}\ {}_{s_2}Y_\ell^{-s_1}(\beta,\alpha)e^{is_2\gamma} $$

Sky Basis

- Scattering into the state (rest frame)

$$C_{\rm in}[\mathbf{T}] = \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi}\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha)\mathbf{T}(\hat{\mathbf{n}}')\,,$$

$$= \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi}(\Theta', 0, 0) + \frac{1}{10}\dot\tau \int d\hat{\mathbf{n}}' \sum_{m=-2}^{2} \mathbf{P}^{(m)}(\hat{\mathbf{n}},\hat{\mathbf{n}}')\mathbf{T}(\hat{\mathbf{n}}')\,.$$

where the quadrupole coupling term is $\mathbf{P}^{(m)}(\hat{\mathbf{n}},\hat{\mathbf{n}}') =$

$$\begin{pmatrix} Y_2^{m*}(\hat{\mathbf{n}}')\,Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\,{}_2Y_2^{m*}(\hat{\mathbf{n}}')\,Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\,{}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\,Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_2Y_2^m(\hat{\mathbf{n}}) & 3\,{}_2Y_2^{m*}(\hat{\mathbf{n}}')\,{}_2Y_2^m(\hat{\mathbf{n}}) & 3\,{}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_2Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\,{}_2Y_2^{m*}(\hat{\mathbf{n}}')\,{}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\,{}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_{-2}Y_2^m(\hat{\mathbf{n}}) \end{pmatrix},$$

expression uses angle addition relation above. We call this term C_Q.

Scattering Matrix

- Full scattering matrix involves difference of scattering into and out of state

$$C[\mathbf{T}] = C_{\rm in}[\mathbf{T}] - C_{\rm out}[\mathbf{T}]$$

- In the electron rest frame

$$C[\mathbf{T}] = \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi}(\Theta', 0, 0) - \dot\tau \mathbf{T} + C_Q[\mathbf{T}]$$

which describes isotropization in the rest frame. All moments have $e^{-\tau}$ suppression except for isotropic temperature Θ_0. Transformation into the background frame simply induces a dipole term

$$C[\mathbf{T}] = \dot\tau \left(\hat{\mathbf{n}} \cdot \mathbf{v}_b + \int \frac{d\hat{\mathbf{n}}'}{4\pi}\Theta', 0, 0 \right) - \dot\tau \mathbf{T} + C_Q[\mathbf{T}]$$