Astro 448

Set 5: Polarization

Wayne Hu

Stokes Parameters

- Specific intensity is related to quadratic combinations of the field.
- Define the intensity matrix (time averaged over oscillations) $\langle \mathbf{E}\,\mathbf{E}^\dagger \rangle$
- Hermitian matrix can be decomposed into Pauli matrices

$$\mathbf{P} = \langle \mathbf{E}\,\mathbf{E}^\dagger \rangle = \frac{1}{2}\left(I\boldsymbol{\sigma}_0 + Q\,\boldsymbol{\sigma}_3 + U\,\boldsymbol{\sigma}_1 - V\,\boldsymbol{\sigma}_2 \right) ,$$

where

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Stokes parameters recovered as $\mathrm{Tr}(\sigma_i \mathbf{P})$

Stokes Parameters

- Consider a general plane wave solution

$$\mathbf{E}(t,z) = E_1(t,z)\hat{\mathbf{e}}_1 + E_2(t,z)\hat{\mathbf{e}}_2$$

$$E_1(t,z) = A_1 e^{i\phi_1} e^{i(kz-\omega t)}$$

$$E_2(t,z) = A_2 e^{i\phi_2} e^{i(kz-\omega t)}$$

- Explicitly:

$$I = \langle E_1 E_1^* + E_2 E_2^* \rangle = A_1^2 + A_2^2$$

$$Q = \langle E_1 E_1^* - E_2 E_2^* \rangle = A_1^2 - A_2^2$$

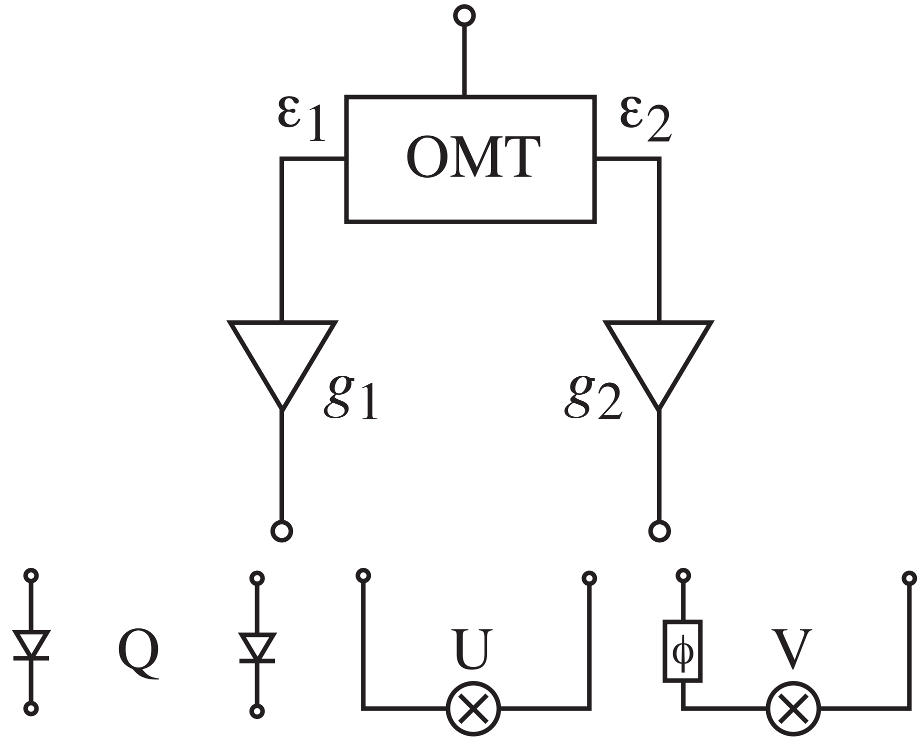
$$U = \langle E_1 E_2^* + E_2 E_1^* \rangle = 2A_1 A_2 \cos(\phi_2 - \phi_1)$$

$$V = -i \langle E_1 E_2^* - E_2 E_1^* \rangle = 2A_1 A_2 \sin(\phi_2 - \phi_1)$$

so that the Stokes parameters define the state up to an unobservable overall phase of the wave

Detection

- This suggests that abstractly there are two different ways to detect polarization: separate and difference orthogonal modes (bolometers I, Q) or correlate the separated components (U, V).



- In the correlator example the natural output would be U but one can recover V by introducing a phase lag $\phi = \pi/2$ on one arm, and Q by having the OMT pick out directions rotated by $\pi/4$.
- Likewise, in the bolometer example, one can rotate the polarizer and also introduce a coherent front end to change V to U.

Detection

- Techniques also differ in the systematics that can convert unpolarized sky to fake polarization
- Differencing detectors are sensitive to relative gain fluctuations
- Correlation detectors are sensitive to cross coupling between the arms
- More generally, the intended block diagram and systematic problems map components of the polarization matrix onto others and are kept track of through "Jones" or instrumental response matrices $\mathbf{E}_{\rm det} = \mathbf{J}\mathbf{E}_{\rm in}$

$$\mathbf{P}_{\rm det} = \mathbf{J}\mathbf{P}_{\rm in}\mathbf{J}^\dagger$$

where the end result is either a differencing or a correlation of the $\mathbf{P}_{\rm det}$.

Polarization

- Radiation field involves a directed quantity, the electric field vector, which defines the polarization
- Consider a general plane wave solution

$$\mathbf{E}(t,z) = E_1(t,z)\hat{\mathbf{e}}_1 + E_2(t,z)\hat{\mathbf{e}}_2$$

$$E_1(t,z) = \mathrm{Re} A_1 e^{i\phi_1} e^{i(kz-\omega t)}$$

$$E_2(t,z) = \mathrm{Re} A_2 e^{i\phi_2} e^{i(kz-\omega t)}$$

or at $z = 0$ the field vector traces out an ellipse

$$\mathbf{E}(t,0) = A_1 \cos(\omega t - \phi_1)\hat{\mathbf{e}}_1 + A_2 \cos(\omega t - \phi_2)\hat{\mathbf{e}}_2$$

with principal axes defined by

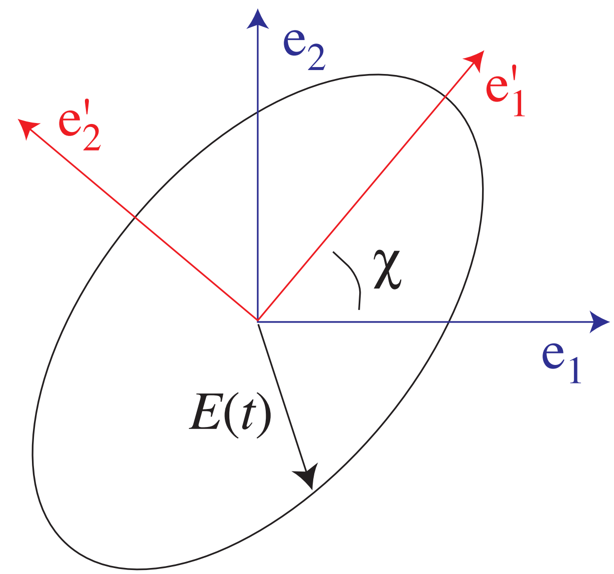
$$\mathbf{E}(t,0) = A_1' \cos(\omega t)\hat{\mathbf{e}}_1' - A_2' \sin(\omega t)\hat{\mathbf{e}}_2'$$

so as to trace out a clockwise rotation for $A_1', A_2' > 0$

Polarization

- Define polarization angle

$$\hat{\mathbf{e}}'_1 = \cos\chi\hat{\mathbf{e}}_1 + \sin\chi\hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}'_2 = -\sin\chi\hat{\mathbf{e}}_1 + \cos\chi\hat{\mathbf{e}}_2$$



- Match

$$\begin{aligned}\mathbf{E}(t,0) &= A'_1\cos\omega t[\cos\chi\hat{\mathbf{e}}_1 + \sin\chi\hat{\mathbf{e}}_2] \\ &\quad - A'_2\cos\omega t[-\sin\chi\hat{\mathbf{e}}_1 + \cos\chi\hat{\mathbf{e}}_2] \\ &= A_1[\cos\phi_1\cos\omega t + \sin\phi_1\sin\omega t]\hat{\mathbf{e}}_1 \\ &\quad + A_2[\cos\phi_2\cos\omega t + \sin\phi_2\sin\omega t]\hat{\mathbf{e}}_2\end{aligned}$$

Polarization

- Define relative strength of two principal states

$$A_1' = E_0 \cos\beta \quad A_2' = E_0 \sin\beta$$

- Characterize the polarization by two angles

$$A_1 \cos\phi_1 = E_0 \cos\beta \cos\chi, \qquad A_1 \sin\phi_1 = E_0 \sin\beta \sin\chi,$$
$$A_2 \cos\phi_2 = E_0 \cos\beta \sin\chi, \qquad A_2 \sin\phi_2 = -E_0 \sin\beta \cos\chi$$

Or Stokes parameters by

$$I = E_0^2\,, \quad Q = E_0^2 \cos 2\beta \cos 2\chi$$
$$U = E_0^2 \cos 2\beta \sin 2\chi\,, \quad V = E_0^2 \sin 2\beta$$

- So $I^2 = Q^2 + U^2 + V^2$, double angles reflect the spin 2 field or headless vector nature of polarization

Polarization

Special cases

- If $\beta = 0, \pi/2, \pi$ then only one principal axis, ellipse collapses to a line and $V = 0 \rightarrow$ linear polarization oriented at angle χ

  If $\chi = 0, \pi/2, \pi$ then $I = \pm Q$ and $U = 0$

  If $\chi = \pi/4, 3\pi/4 ...$ then $I = \pm U$ and $Q = 0$ - so U is Q in a frame rotated by 45 degrees

- If $\beta = \pi/4, 3\pi/4$, then principal components have equal strength and E field rotates on a circle: $I = \pm V$ and $Q = U = 0 \rightarrow$ circular polarization

- $U/Q = \tan 2\chi$ defines angle of linear polarization and $V/I = \sin 2\beta$ defines degree of circular polarization

Natural Light

- A monochromatic plane wave is completely polarized
  $I^2 = Q^2 + U^2 + V^2$
- Polarization matrix is like a density matrix in quantum mechanics and allows for pure (coherent) states and mixed states
- Suppose the total $\mathbf{E}_{\rm tot}$ field is composed of different (frequency) components

$$\mathbf{E}_{\rm tot} = \sum_i \mathbf{E}_i$$

- Then components decorrelate in time average

$$\left\langle \mathbf{E}_{\rm tot}\mathbf{E}_{\rm tot}^\dagger \right\rangle = \sum_{ij} \left\langle \mathbf{E}_i \mathbf{E}_j^\dagger \right\rangle = \sum_i \left\langle \mathbf{E}_i \mathbf{E}_i^\dagger \right\rangle$$

Natural Light

- So Stokes parameters of incoherent contributions add

$$I = \sum_i I_i \quad Q = \sum_i Q_i \quad U = \sum_i U_i \quad V = \sum_i V_i$$

and since individual Q, U and V can have either sign:
$I^2 \geq Q^2 + U^2 + V^2$, all 4 Stokes parameters needed

Linear Polarization

- $Q \propto \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle$, $U \propto \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle$.
- Counterclockwise rotation of axes by $\theta = 45°$

$$E_1 = (E_1' - E_2')/\sqrt{2}\,, \quad E_2 = (E_1' + E_2')/\sqrt{2}$$

- $U \propto \langle E_1' E_1'^* \rangle - \langle E_2' E_2'^* \rangle$, difference of intensities at $45°$ or Q'
- More generally, **P** transforms as a tensor under rotations and

$$Q' = \cos(2\theta) Q + \sin(2\theta) U$$
$$U' = -\sin(2\theta) Q + \cos(2\theta) U$$

or

$$Q' \pm iU' = e^{\mp 2i\theta}[Q \pm iU]$$

acquires a phase under rotation and is a spin ± 2 object

Coordinate Independent Representation

- Two directions: orientation of polarization and change in amplitude, i.e. Q and U in the basis of the Fourier wavevector (pointing with angle ϕ_l) for small sections of sky are called E and B components

$$E(\mathbf{l}) \pm iB(\mathbf{l}) = -\int d\hat{\mathbf{n}}[Q'(\hat{\mathbf{n}}) \pm iU'(\hat{\mathbf{n}})]e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}}$$

$$= -e^{\mp 2i\phi_l}\int d\hat{\mathbf{n}}[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})]e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}}$$

- For the B-mode to not vanish, the polarization must point in a direction not related to the wavevector - not possible for density fluctuations in linear theory
- Generalize to all-sky: plane waves are eigenmodes of the Laplace operator on the tensor $\mathbf{P}$.

Spin Harmonics

- Laplace Eigenfunctions

$$\nabla^2 {}_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1] = -[l(l+1)-4]_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1]$$

- Spin s spherical harmonics: orthogonal and complete

$$\int d\hat{\mathbf{n}} {}_sY_{\ell m}^*(\hat{\mathbf{n}}) {}_sY_{\ell m}(\hat{\mathbf{n}}) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\sum_{\ell m} {}_sY_{\ell m}^*(\hat{\mathbf{n}}) {}_sY_{\ell m}(\hat{\mathbf{n}}') = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta')$$

where the ordinary spherical harmonics are $Y_{\ell m} = {}_0Y_{\ell m}$

- Given in terms of the rotation matrix

$${}_sY_{\ell m}(\beta\alpha) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell}_{-ms}(\alpha\beta 0)$$

Statistical Representation

- All-sky decomposition

$$[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})] = \sum_{\ell m} [E_{\ell m} \pm iB_{\ell m}]_{\pm 2}Y_{\ell m}(\hat{\mathbf{n}})$$

- Power spectra

$$\langle E^*_{\ell m} E_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{EE}$$

$$\langle B^*_{\ell m} B_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{BB}$$

- Cross correlation

$$\langle E^*_{\ell m} E_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{\Theta E}$$

others vanish if parity is conserved

Thomson Scattering

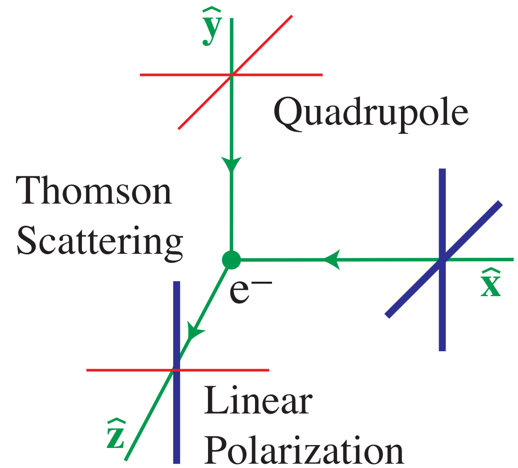
- Polarization state of radiation in direction $\hat{\mathbf{n}}$ described by the intensity matrix $\langle E_i(\hat{\mathbf{n}})E_j^*(\hat{\mathbf{n}})\rangle$, where $\mathbf{E}$ is the electric field vector and the brackets denote time averaging.
- Differential cross section

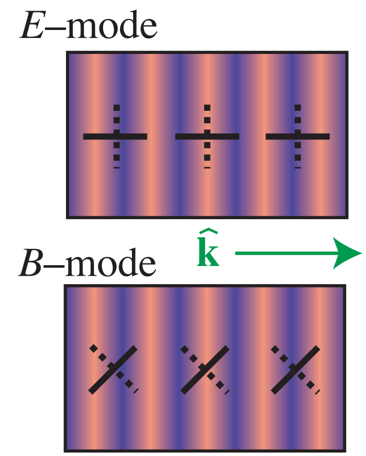
$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi}|\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \,,$$

  where $\sigma_T = 8\pi\alpha^2/3m_e$ is the Thomson cross section, $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.
- Summed over angle and incoming polarization

$$\sum_{i=1,2} \int d\hat{\mathbf{n}}' \frac{d\sigma}{d\Omega} = \sigma_T$$

Polarization Generation





- Heuristic:
  incoming radiation shakes
  an electron in direction
  of electric field vector $\hat{\mathbf{E}}'$
- Radiates photon with
  polarization also in direction $\hat{\mathbf{E}}'$
- But photon cannot be longitudinally polarized so that scattering into 90° can only pass one polarization
- Linearly polarized radiation like polarization by reflection
- Unlike reflection of sunlight, incoming radiation is nearly isotropic
- Missing from direction orthogonal to original incoming direction
- Only quadrupole anisotropy generates polarization by Thomson scattering

Acoustic Polarization

- Break down of tight-coupling leads to quadrupole anisotropy of

$$\pi_\gamma \approx \frac{k}{\dot\tau} v_\gamma$$

- Scaling $k_D = (\dot\tau/\eta_*)^{1/2} \to \dot\tau = k_D^2 \eta_*$
- Know: $k_D s_* \approx k_D \eta_* \approx 10$
- So:

$$\pi_\gamma \approx \frac{k}{k_D}\frac{1}{10} v_\gamma$$

$$\Delta_P \approx \frac{\ell}{\ell_D}\frac{1}{10}\Delta_T$$

Acoustic Polarization

- Gradient of velocity is along direction of wavevector, so polarization is pure E-mode
- Velocity is 90° out of phase with temperature – turning points of oscillator are zero points of velocity:

$$\Theta + \Psi \propto \cos(ks); \quad v_\gamma \propto \sin(ks)$$

- Polarization peaks are at troughs of temperature power

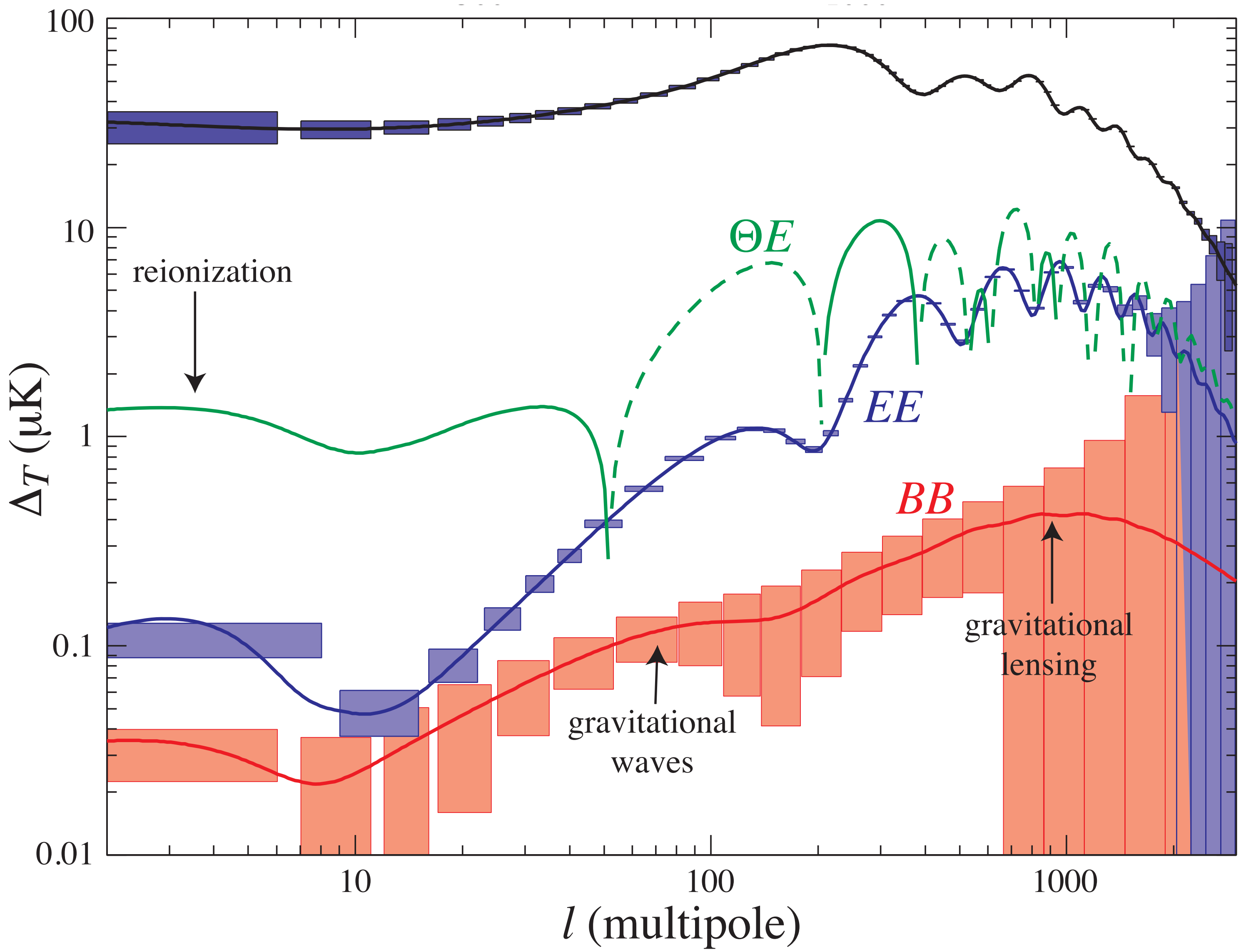
Cross Correlation

- Cross correlation of temperature and polarization

$$(\Theta + \Psi)(v_\gamma) \propto \cos(ks)\sin(ks) \propto \sin(2ks)$$

- Oscillation at twice the frequency
- Correlation: radial or tangential around hot spots
- Partial correlation: easier to measure if polarization data is noisy, harder to measure if polarization data is high S/N or if bands do not resolve oscillations
- Good check for systematics and foregrounds
- Comparison of temperature and polarization is proof against features in initial conditions mimicking acoustic features

Polarization Power



Angular Moments

- Define the angularly dependent Stokes perturbation

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta), \quad Q(\mathbf{x}, \hat{\mathbf{n}}, \eta), \quad U(\mathbf{x}, \hat{\mathbf{n}}, \eta)$$

- Decompose into normal modes: plane waves for spatial part and spherical harmonics for angular part

$$G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$

$${}_{\pm 2}G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} {}_{\pm 2}Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$

- In a spatially curved universe generalize the plane wave part

Normal Modes

- Temperature and polarization fields

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} \Theta_\ell^{(m)} G_\ell^m$$

$$[Q \pm iU](\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} [E_\ell^{(m)} \pm iB_\ell^{(m)}]_{\pm 2} G_\ell^m$$

- For each $\mathbf{k}$ mode, work in coordinates where $\mathbf{k} \parallel \mathbf{z}$ and so $m = 0$ represents scalar modes, $m = \pm 1$ vector modes, $m = \pm 2$ tensor modes, $|m| > 2$ vanishes. Since modes add incoherently and $Q \pm iU$ is invariant up to a phase, rotation back to a fixed coordinate system is trivial.

Liouville Equation

- In absence of scattering, the phase space distribution of photons in each polarization state a is conserved along the propagation path
- Rewrite variables in terms of the photon propagation direction $\mathbf{q} = q\hat{\mathbf{n}}$, so $f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta)$ and

$$\frac{d}{d\eta} f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta) = 0$$
$$= \left(\frac{\partial}{\partial \eta} + \frac{d\mathbf{x}}{d\eta} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{d\hat{\mathbf{n}}}{d\eta} \cdot \frac{\partial}{\partial \hat{\mathbf{n}}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a$$

- For simplicity, assume spatially flat universe $K = 0$ then $d\hat{\mathbf{n}}/d\eta = 0$ and $d\mathbf{x} = \hat{\mathbf{n}} d\eta$

$$\dot{f}_a + \hat{\mathbf{n}} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0$$

Scalar, Vector, Tensor

- Normalization of modes is chosen so that the lowest angular mode for scalars, vectors and tensors are normalized in the same way as the mode function

$$G_0^0 = Q^{(0)} \quad G_1^0 = n^i Q_i^{(0)} \quad G_2^0 \propto n^i n^j Q_{ij}^{(0)}$$

$$G_1^{\pm 1} = n^i Q_i^{(\pm 1)} \quad G_2^{\pm 1} \propto n^i n^j Q_{ij}^{(\pm 1)}$$

$$G_2^{\pm 2} = n^i n^j Q_{ij}^{(\pm 2)}$$

where recall

$$\begin{aligned} Q^{(0)} &= \exp(i\mathbf{k}\cdot\mathbf{x}) \\ Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k}\cdot\mathbf{x}) \\ Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k}\cdot\mathbf{x}) \end{aligned}$$

Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies
- Spatial gradient term hits plane wave:

$$\hat{\mathbf{n}} \cdot \nabla e^{i\mathbf{k}\cdot\mathbf{x}} = i\hat{\mathbf{n}} \cdot \mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} = i\sqrt{\frac{4\pi}{3}} k Y_1^0(\hat{\mathbf{n}}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

- Dipole term adds to angular dependence through the addition of angular momentum

$$\sqrt{\frac{4\pi}{3}} Y_1^0 Y_\ell^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell+1)(2\ell-1)}} Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell+1)(2\ell+3)}} Y_{\ell+1}^m$$

where $\kappa_\ell^m = \sqrt{\ell^2 - m^2}$ is given by Clebsch-Gordon coefficients.

Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

$$\dot{\Theta}_\ell^{(m)} = k\left[\frac{\kappa_\ell^m}{2\ell+1}\Theta_{\ell-1}^{(m)} - \frac{\kappa_{\ell+1}^m}{2\ell+3}\Theta_{\ell+1}^{(m)}\right] - \dot{\tau}\Theta_\ell^{(m)} + S_\ell^{(m)}$$

  where $S_\ell^{(m)}$ are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic $\ell = 0$ temperature perturbation will eventually become a high order anisotropy by "free streaming" or simple projection
- Original CMB codes solved the full hierarchy equations out to the ℓ of interest.

Integral Solution

- Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface
- In general, the solution describes the decomposition of the source $S_\ell^{(m)}$ with its local angular dependence as seen at a distance $\mathbf{x} = D\hat{\mathbf{n}}$.
- Proceed by decomposing the angular dependence of the plane wave

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} j_\ell(kD) Y_\ell^0(\hat{\mathbf{n}})$$

- Recouple to the local angular dependence of G_ℓ^m

$$G_{\ell_s}^m = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} \alpha_{\ell_s \ell}^{(m)}(kD) Y_\ell^m(\hat{\mathbf{n}})$$

Integral Solution

- Projection kernels:

$$\ell_s = 0, \quad m = 0 \qquad\qquad \alpha^{(0)}_{0\ell} \equiv j_\ell$$

$$\ell_s = 1, \quad m = 0 \qquad\qquad \alpha^{(0)}_{1\ell} \equiv j'_\ell$$

- Integral solution:

$$\frac{\Theta_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S_{\ell_s}^{(m)} \alpha_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

- Power spectrum:

$$C_\ell = \frac{2}{\pi} \int \frac{dk}{k} \sum_m \frac{k^3 \langle \Theta_\ell^{(m)*} \Theta_\ell^{(m)} \rangle}{(2\ell+1)^2}$$

- Solving for C_ℓ reduces to solving for the behavior of a handful of sources

Polarization Hierarchy

- In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hierarchy:

$$\dot{E}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell+1)}B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}E_{\ell+1}^{(m)}\right] - \dot{\tau}E_\ell^{(m)} + \mathcal{E}_\ell^{(m)}$$

$$\dot{B}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell+1)}E_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}B_{\ell+1}^{(m)}\right] - \dot{\tau}B_\ell^{(m)} + \mathcal{B}_\ell^{(m)}$$

  where ${}_2\kappa_\ell^m = \sqrt{(\ell^2 - m^2)(\ell^2-4)/\ell^2}$ is given by the Clebsch-Gordon coefficients and $\mathcal{E}$, $\mathcal{B}$ are the sources (scattering only).

- Note that for vectors and tensors $|m| > 0$ and B modes may be generated from E modes by projection. Cosmologically $\mathcal{B}_\ell^{(m)} = 0$

Polarization Integral Solution

- Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

$$\frac{E_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \epsilon_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

$$\frac{B_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \beta_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

- The only source to the polarization is from the quadrupole anisotropy so we only need $\ell_s = 2$, e.g. for scalars

$$\epsilon_{2\ell}^{(0)}(x) = \sqrt{\frac{3}{8}\frac{(\ell+2)!}{(\ell-2)!}} \frac{j_\ell(x)}{x^2} \qquad \beta_{2\ell}^{(0)} = 0$$

Gravitational Terms

- As in our Newtonian gauge calculation, gravitational terms - now including vectors and tensors in an arbitrary gauge, come from the geodesic equation
- First define the slicing (lapse function A, shift function B^i)

$$
\begin{aligned}
g^{00} &= -a^{-2}(1-2A)\,, \\
g^{0i} &= -a^{-2}B^i\,,
\end{aligned}
$$

A defines the lapse of proper time between 3-surfaces whereas B^i defines the threading or relationship between the 3-coordinates of the surfaces

Gravitational Terms

- This absorbs 1+3=4 degrees of freedom in the metric, remaining 6 is in the spatial surfaces which we parameterize as

$$g^{ij} = a^{-2}(\gamma^{ij} - 2H_L\gamma^{ij} - 2H_T^{ij}) \, .$$

here (1) H_L a perturbation to the spatial curvature; (5) H_T^{ij} a trace-free distortion to spatial metric (which also can perturb the curvature)

- Geodesic equation gives the redshifting term

$$\frac{\dot q}{q} = -\frac{\dot a}{a} - \frac{1}{2} n^i n^j \dot H_{Tij} - \dot H_L + n^i \dot B_i - \hat{\mathbf{n}} \cdot \nabla A$$

which is incorporated in the conservation and gauge transformation equations

Source Terms

- Temperature source terms $S_l^{(m)}$ (rows $\pm|m|$; flat assumption

$$\left(\begin{array}{lll} \dot\tau\Theta_0^{(0)} - \dot H_L^{(0)} & \dot\tau v_b^{(0)} + \dot B^{(0)} & \dot\tau P^{(0)} - \frac{2}{3}\dot H_T^{(0)} \\ 0 & \dot\tau v_b^{(\pm1)} + \dot B^{(\pm1)} & \dot\tau P^{(\pm1)} - \frac{\sqrt{3}}{3}\dot H_T^{(\pm1)} \\ 0 & 0 & \dot\tau P^{(\pm2)} - \dot H_T^{(\pm2)} \end{array} \right)$$

where

$$P^{(m)} \equiv \frac{1}{10}(\Theta_2^{(m)} - \sqrt{6}E_2^{(m)})$$

- Polarization source term

$$\mathcal{E}_\ell^{(m)} = -\dot\tau\sqrt{6}P^{(m)}\delta_{\ell,2}$$

$$\mathcal{B}_\ell^{(m)} = 0$$

Truncated Hierarchy

- CMBFast introduced the hybrid truncated hierarchy, integral solution technique
- Formal integral solution contains sources that are not external to system but defined through the Boltzmann hierarchy itself
- Solution: recall that we used this technique in the tight coupling regime by applying a closure condition from tight coupling
- CMBFast extends this idea by solving a truncated hierarchy of equations, e.g. out to $\ell = 25$ with non-reflecting boundary conditions

- For completeness, we explicitly derive the scattering source term via polarized radiative transfer in the last part of the notes

Polarized Radiative Transfer

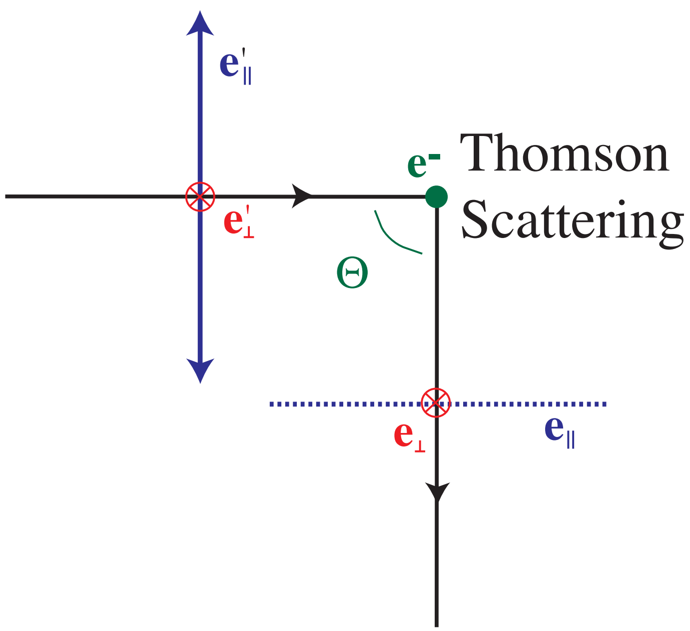
- Define a specific intensity "vector": $\mathbf{I}_\nu = (\Theta_\parallel, \Theta_\perp, U, V)$ where $\Theta = \Theta_\parallel + \Theta_\perp$, $Q = \Theta_\parallel - \Theta_\perp$

$$\frac{d\mathbf{I}_\nu}{d\eta} = \dot{\tau}(\mathbf{S}_\nu - \mathbf{I}_\nu)$$

- Thomson collision

  based on differential cross section

$$\frac{d\sigma_T}{d\Omega} = \frac{3}{8\pi}|\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \ ,$$



Polarized Radiative Transfer

- $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.
- Thomson scattering by 90 deg: $\Theta_\perp \to \Theta_\perp$ but $\Theta_{||}$ does not scatter
- More generally if Θ is the scattering angle

$$\mathbf{S}_\nu = \frac{3}{8\pi}\int d\Omega' \begin{pmatrix} \cos^2\Theta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\Theta & 0 \\ 0 & 0 & 0 & \cos\Theta \end{pmatrix} \mathbf{I}'_\nu$$

- But to calculate Stokes parameters in a fixed coordinate system must rotate into the scattering basis, scatter and rotate back out to the fixed coordinate system

Thomson Collision Term

- The $U \to U'$ transfer follows by writing down the polarization vectors in the 45° rotated basis

$$\hat{\mathbf{E}}_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_\parallel + \hat{\mathbf{E}}_\perp), \qquad \hat{\mathbf{E}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_\parallel - \hat{\mathbf{E}}_\perp)$$

- Define the temperature in this basis

$$\begin{aligned}
\Theta_1 &\propto |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_1|^2 \Theta_1' + |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2|^2 \Theta_2' \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta_1' + \frac{1}{4}(\cos\beta - 1)^2 \Theta_2' \\
\Theta_2 &\propto |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_2|^2 \Theta_2' + |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_1|^2 \Theta_1' \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta_2' + \frac{1}{4}(\cos\beta - 1)^2 \Theta_1'
\end{aligned}$$

or $\Theta_1 - \Theta_2 \propto \cos\beta(\Theta_1' - \Theta_2')$

Scattering Matrix

- Transfer matrix of Stokes state $\mathbf{T} \equiv (\Theta, Q + iU, Q - iU)$

$$\mathbf{T} \propto \mathbf{S}(\beta)\mathbf{T}'$$

$$\mathbf{S}(\beta) = \frac{3}{4}\begin{pmatrix} \cos^2\beta + 1 & -\frac{1}{2}\sin^2\beta & -\frac{1}{2}\sin^2\beta \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta + 1)^2 & \frac{1}{2}(\cos\beta - 1)^2 \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta - 1)^2 & \frac{1}{2}(\cos\beta + 1)^2 \end{pmatrix}$$

normalization factor of 3 is set by photon conservation in scattering

Scattering Matrix

- Transform to a fixed basis, by a rotation of the incoming and outgoing states $\mathbf{T} = \mathbf{R}(\psi)\mathbf{T}$ where

$$\mathbf{R}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & e^{2i\psi} \end{pmatrix}$$

giving the scattering matrix

$$\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha) =$$

$$\frac{1}{2}\sqrt{\frac{4\pi}{5}} \begin{pmatrix} Y_2^0(\beta,\alpha) + 2\sqrt{5}Y_0^0(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^{-2}(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^2(\beta,\alpha) \\ -\sqrt{6}\,{}_2Y_2^0(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^{-2}(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^2(\beta,\alpha)e^{2i\gamma} \\ -\sqrt{6}\,{}_{-2}Y_2^0(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^{-2}(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^2(\beta,\alpha)e^{-2i\gamma} \end{pmatrix}$$

Addition Theorem for Spin Harmonics

- Spin harmonics are related to rotation matrices as

$$ {}_sY_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\mathcal{D}^\ell_{-ms}(\phi,\theta,0) $$

  Note: for explicit evaluation sign convention differs from usual (e.g. Jackson) by $(-1)^m$

- Multiplication of rotations

$$ \sum_{m''} \mathcal{D}^\ell_{mm''}(\alpha_2,\beta_2,\gamma_2)\mathcal{D}^\ell_{m''m}(\alpha_1,\beta_1,\gamma_1) = \mathcal{D}^\ell_{mm'}(\alpha,\beta,\gamma) $$

- Implies

$$ \sum_m {}_{s_1}Y_\ell^{m*}(\theta',\phi')\ {}_{s_2}Y_\ell^m(\theta,\phi) = (-1)^{s_1-s_2}\sqrt{\frac{2\ell+1}{4\pi}}\ {}_{s_2}Y_\ell^{-s_1}(\beta,\alpha)e^{is_2\gamma} $$

Sky Basis

- Scattering into the state (rest frame)

$$C_{\rm in}[\mathbf{T}] = \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi} \mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha)\mathbf{T}(\hat{\mathbf{n}}') \,,$$

$$= \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi} (\Theta', 0, 0) + \frac{1}{10}\dot\tau \int d\hat{\mathbf{n}}' \sum_{m=-2}^{2} \mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}')\mathbf{T}(\hat{\mathbf{n}}') \,.$$

where the quadrupole coupling term is $\mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') =$

$$\begin{pmatrix} Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\, {}_2Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\, {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}')\, {}_2Y_2^m(\hat{\mathbf{n}}) & 3\, {}_2Y_2^{m*}(\hat{\mathbf{n}}')\, {}_2Y_2^m(\hat{\mathbf{n}}) & 3\, {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\, {}_2Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}')\, {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\, {}_2Y_2^{m*}(\hat{\mathbf{n}}')\, {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\, {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\, {}_{-2}Y_2^m(\hat{\mathbf{n}}) \end{pmatrix} ,$$

expression uses angle addition relation above. We call this term C_Q.

Scattering Matrix

- Full scattering matrix involves difference of scattering into and out of state

$$C[\mathbf{T}] = C_{\rm in}[\mathbf{T}] - C_{\rm out}[\mathbf{T}]$$

- In the electron rest frame

$$C[\mathbf{T}] = \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi}(\Theta', 0, 0) - \dot\tau\mathbf{T} + C_Q[\mathbf{T}]$$

which describes isotropization in the rest frame. All moments have $e^{-\tau}$ suppression except for isotropic temperature Θ_0.
Transformation into the background frame simply induces a dipole term

$$C[\mathbf{T}] = \dot\tau \left(\hat{\mathbf{n}} \cdot \mathbf{v}_b + \int \frac{d\hat{\mathbf{n}}'}{4\pi}\Theta', 0, 0 \right) - \dot\tau\mathbf{T} + C_Q[\mathbf{T}]$$