Astro 4PT

Lecture Notes Set 1

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References

- Relativistic Cosmological Perturbation Theory

  Inflation

  Dark Energy

  Modified Gravity

  Cosmic Microwave Background

  Large Scale Structure

- Bardeen (1980), PRD 22 1882
- Kodama & Sasaki (1984), Prog. Th. Phys. Supp., 78, 1
- Mukhanov, Feldman, Brandenberger (1992), Phys. Reports, 215, 203
- Malik & Wands (2009), Phys. Reports, 475, 1

Covariant Perturbation Theory

- Covariant = takes same form in all coordinate systems
- Invariant = takes the same value in all coordinate systems
- Fundamental equations: Einstein equations, covariant conservation of stress-energy tensor:

$$
\begin{aligned}
G_{\mu\nu} &= 8\pi G T_{\mu\nu} \\
\nabla_\mu T^{\mu\nu} &= 0
\end{aligned}
$$

- Preserve general covariance by keeping all free variables: 10 for each symmetric 4×4 tensor

1	2	3	4
	5	6	7
		8	9
			10

Metric Tensor

- Useful to think in a $3+1$ language since there are preferred spatial surfaces where the stress tensor is nearly homogeneous
- In general this is an Arnowitt-Deser-Misner (ADM) split
- Specialize to the case of a nearly FRW metric

$$g_{00} = -a^2, \qquad g_{ij} = a^2 \gamma_{ij} \, .$$

where the "0" component is conformal time $\eta = dt/a$ and γ_{ij} is a spatial metric of constant curvature $K = H_0^2(\Omega_{\rm tot} - 1)$.

$$^{(3)}R = \frac{6K}{a^2}$$

Metric Tensor

- First define the slicing (lapse function A, shift function B^i)

$$
\begin{aligned}
g^{00} &= -a^{-2}(1-2A)\,, \\
g^{0i} &= -a^{-2}B^i\,,
\end{aligned}
$$

  A defines the lapse of proper time between 3-surfaces whereas B^i defines the threading or relationship between the 3-coordinates of the surfaces

- This absorbs 1+3=4 free variables in the metric, remaining 6 is in the spatial surfaces which we parameterize as

$$
g^{ij} = a^{-2}(\gamma^{ij} - 2H_L\gamma^{ij} - 2H_T^{ij})\,.
$$

  here (1) H_L a perturbation to the spatial curvature; (5) H_T^{ij} a trace-free distortion to spatial metric (which also can perturb the curvature)

Curvature Perturbation

- Curvature perturbation on the 3D slice

$$\delta[^{(3)}R] = -\frac{4}{a^2}\left(\nabla^2 + 3K\right) H_L + \frac{2}{a^2}\nabla_i\nabla_j H_T^{ij}$$

- Note that we will often loosely refer to H_L as the "curvature perturbation"
- We will see that many representations have $H_T = 0$
- It is easier to work with a dimensionless quantity
- First example of a 3-scalar - SVT decomposition

Matter Tensor

- Likewise expand the matter stress energy tensor around a homogeneous density ρ and pressure p:

$$\begin{aligned} T^0{}_0 &= -\rho - \delta\rho\,, \\ T^0{}_i &= (\rho + p)(v_i - B_i)\,, \\ T_0{}^i &= -(\rho + p)v^i\,, \\ T^i{}_j &= (p + \delta p)\delta^i{}_j + p\Pi^i{}_j\,, \end{aligned}$$

- (1) $\delta\rho$ a density perturbation; (3) v_i a vector velocity, (1) δp a pressure perturbation; (5) Π_{ij} an anisotropic stress perturbation
- So far this is fully general and applies to any type of matter or coordinate choice including non-linearities in the matter, e.g. scalar fields, cosmological defects, exotic dark energy.

Counting Variables

20	Variables (10 metric; 10 matter)
-10	Einstein equations
-4	Conservation equations
$+4$	Bianchi identities
-4	Gauge (coordinate choice 1 time, 3 space)
—	
6	Free Variables

- Without loss of generality these can be taken to be the 6 components of the matter stress tensor
- For the background, specify $p(a)$ or equivalently $w(a) \equiv p(a)/\rho(a)$ the equation of state parameter.

Homogeneous Einstein Equations

- Einstein (Friedmann) equations:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = -\frac{K}{a^2} + \frac{8\pi G}{3}\rho \quad [= \left(\frac{1}{a}\frac{\dot{a}}{a}\right)^2]$$

$$\frac{1}{a}\frac{d^2 a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p) \quad [= \frac{1}{a^2}\frac{d}{d\eta}\frac{\dot{a}}{a}]$$

so that $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

overdots are conformal time but equally true with coordinate time

Homogeneous Einstein Equations

- Counting exercise:

20	Variables (10 metric; 10 matter)
-17	Homogeneity and Isotropy
-2	Einstein equations
-1	Conservation equations
$+1$	Bianchi identities
1	Free Variables

without loss of generality choose ratio of homogeneous & isotropic component of the stress tensor to the density $w(a) = p(a)/\rho(a)$.

Acceleration Implies Negative Pressure

- Role of stresses in the background cosmology
- Homogeneous Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ imply the two Friedmann equations (flat universe, or associating curvature $\rho_K = -3K/8\pi G a^2$)

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho$$

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p)$$

  so that the total equation of state $w \equiv p/\rho < -1/3$ for acceleration
- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}$$

  so that ρ must scale more slowly than a^{-2}

Scalar, Vector, Tensor

- In linear perturbation theory, perturbations may be separated by their transformation properties under 3D rotation and translation.
- The eigenfunctions of the Laplacian operator form a complete set

$$\begin{aligned} \nabla^2 Q^{(0)} &= -k^2 Q^{(0)} && \mathrm{S}\,, \\ \nabla^2 Q_i^{(\pm 1)} &= -k^2 Q_i^{(\pm 1)} && \mathrm{V}\,, \\ \nabla^2 Q_{ij}^{(\pm 2)} &= -k^2 Q_{ij}^{(\pm 2)} && \mathrm{T}\,, \end{aligned}$$

- Vector and tensor modes satisfy divergence-free and transverse-traceless conditions

$$\nabla^i Q_i^{(\pm 1)} = 0$$

$$\nabla^i Q_{ij}^{(\pm 2)} = 0$$

$$\gamma^{ij} Q_{ij}^{(\pm 2)} = 0$$

Vector and Tensor Quantities

- A scalar mode carries with it associated vector (curl-free) and tensor (longitudinal) quantities
- A vector mode carries and associated tensor (trace and divergence free) quantities
- A tensor mode has only a tensor (trace and divergence free)
- These are built from the mode basis out of covariant derivatives and the metric

$$
\begin{aligned}
Q_i^{(0)} &= -k^{-1}\nabla_i Q^{(0)} \,, \\
Q_{ij}^{(0)} &= (k^{-2}\nabla_i\nabla_j + \frac{1}{3}\gamma_{ij})Q^{(0)} \,, \\
Q_{ij}^{(\pm 1)} &= -\frac{1}{2k}[\nabla_i Q_j^{(\pm 1)} + \nabla_j Q_i^{(\pm 1)}] \,,
\end{aligned}
$$

Perturbation k-Modes

- For the kth eigenmode, the scalar components become

$$A(\mathbf{x}) = A(k)\,Q^{(0)}\,, \qquad H_L(\mathbf{x}) = H_L(k)\,Q^{(0)}\,,$$
$$\delta\rho(\mathbf{x}) = \delta\rho(k)\,Q^{(0)}\,, \qquad \delta p(\mathbf{x}) = \delta p(k)\,Q^{(0)}\,,$$

the vectors components become

$$B_i(\mathbf{x}) = \sum_{m=-1}^{1} B^{(m)}(k)\,Q_i^{(m)}\,, \qquad v_i(\mathbf{x}) = \sum_{m=-1}^{1} v^{(m)}(k)\,Q_i^{(m)}\,,$$

and the tensors components

$$H_{Tij}(\mathbf{x}) = \sum_{m=-2}^{2} H_T^{(m)}(k)\,Q_{ij}^{(m)}\,, \quad \Pi_{ij}(\mathbf{x}) = \sum_{m=-2}^{2} \Pi^{(m)}(k)\,Q_{ij}^{(m)}\,,$$

- Note that the curvature perturbation only involves scalars

$$\delta[{}^{(3)}R] = \frac{4}{a^2}(k^2 - 3K)(H_L^{(0)} + \frac{1}{3}H_T^{(0)})Q^{(0)}$$

Spatially Flat Case

- For a spatially flat background metric, harmonics are related to plane waves:

$$
\begin{aligned}
Q^{(0)} &= \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_i^{(\pm1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_{ij}^{(\pm2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k}\cdot\mathbf{x})
\end{aligned}
$$

  where $\hat{\mathbf{e}}_3 \parallel \mathbf{k}$. Chosen as spin states, c.f. polarization.

- For vectors, the harmonic points in a direction orthogonal to $\mathbf{k}$ suitable for the vortical component of a vector

Spatially Flat Case

- Tensor harmonics are the transverse traceless gauge representation
- Tensor amplitude related to the more traditional

$$h_+[(\mathbf{e}_1)_i(\mathbf{e}_1)_j - (\mathbf{e}_2)_i(\mathbf{e}_2)_j]\,, \qquad h_\times[(\mathbf{e}_1)_i(\mathbf{e}_2)_j + (\mathbf{e}_2)_i(\mathbf{e}_1)_j]$$

as

$$h_+ \pm i h_\times = -\sqrt{6} H_T^{(\mp 2)}$$

- $H_T^{(\pm 2)}$ proportional to the right and left circularly polarized amplitudes of gravitational waves with a normalization that is convenient to match the scalar and vector definitions

Covariant Scalar Equations

- DOF counting exercise

8	Variables (4 metric; 4 matter)
-4	Einstein equations
-2	Conservation equations
$+2$	Bianchi identities
-2	Gauge (coordinate choice 1 time, 1 space)
———	
2	Free Variables

without loss of generality choose scalar components of the stress tensor δp, Π .

Covariant Scalar Equations

- Einstein equations (suppressing 0) superscripts

$$(k^2 - 3K)[H_L + \frac{1}{3}H_T] - 3(\frac{\dot{a}}{a})^2 A + 3\frac{\dot{a}}{a}\dot{H}_L + \frac{\dot{a}}{a}kB =$$

$$= 4\pi G a^2 \delta\rho \,, \quad \text{00 Poisson Equation}$$

$$k^2(A + H_L + \frac{1}{3}H_T) + \left(\frac{d}{d\eta} + 2\frac{\dot{a}}{a}\right)(kB - \dot{H}_T)$$

$$= 8\pi G a^2 p\Pi \,, \quad ij \text{ Anisotropy Equation}$$

$$\frac{\dot{a}}{a}A - \dot{H}_L - \frac{1}{3}\dot{H}_T - \frac{K}{k^2}(kB - \dot{H}_T)$$

$$= 4\pi G a^2(\rho + p)(v - B)/k \,, \quad 0i \text{ Momentum Equation}$$

$$\left[2\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\frac{d}{d\eta} - \frac{k^2}{3}\right]A - \left[\frac{d}{d\eta} + \frac{\dot{a}}{a}\right](\dot{H}_L + \frac{1}{3}kB)$$

$$= 4\pi G a^2(\delta p + \frac{1}{3}\delta\rho) \,, \quad ii \text{ Acceleration Equation}$$

Covariant Scalar Equations

- Conservation equations: continuity and Navier Stokes

$$\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho + 3\frac{\dot a}{a}\delta p = -(\rho + p)(kv + 3\dot H_L)\,,$$

$$\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right]\left[(\rho + p)\frac{(v - B)}{k}\right] = \delta p - \frac{2}{3}(1 - 3\frac{K}{k^2})p\Pi + (\rho + p)A\,,$$

- Equations are not independent since $\nabla_\mu G^{\mu\nu} = 0$ via the Bianchi identities.
- Related to the ability to choose a coordinate system or "gauge" to represent the perturbations.

Covariant Vector Equations

- Einstein equations

$$(1 - 2K/k^2)(kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) = 16\pi G a^2 (\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k \,,$$

$$\left[\frac{d}{d\eta} + 2\frac{\dot{a}}{a}\right](kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) = -8\pi G a^2 p \Pi^{(\pm 1)} \,.$$

- Conservation Equations

$$\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a}\right][(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k] = -\frac{1}{2}(1 - 2K/k^2) p \Pi^{(\pm 1)} \,,$$

- Gravity provides no source to vorticity $\rightarrow$ decay

Covariant Vector Equations

- DOF counting exercise

8	Variables (4 metric; 4 matter)
-4	Einstein equations
-2	Conservation equations
$+2$	Bianchi identities
-2	Gauge (coordinate choice 1 time, 1 space)
——	
2	Free Variables

without loss of generality choose vector components of the stress tensor $\Pi^{(\pm 1)}$.

Covariant Tensor Equation

- Einstein equation

$$\left[\frac{d^2}{d\eta^2} + 2\frac{\dot{a}}{a}\frac{d}{d\eta} + (k^2 + 2K)\right] H_T^{(\pm 2)} = 8\pi G a^2 p \Pi^{(\pm 2)} \,.$$

- DOF counting exercise

4	Variables (2 metric; 2 matter)
-2	Einstein equations
-0	Conservation equations
$+0$	Bianchi identities
-0	Gauge (coordinate choice 1 time, 1 space)
2	Free Variables

wlog choose tensor components of the stress tensor $\Pi^{(\pm 2)}$.

Arbitrary Dark Components

- Total stress energy tensor can be broken up into individual pieces
- Dark components interact only through gravity and so satisfy separate conservation equations
- Einstein equation source remains the sum of components.
- To specify an arbitrary dark component, give the behavior of the stress tensor: 6 components: δp, $\Pi^{(i)}$, where $i = -2, ..., 2$.
- Many types of dark components (dark matter, scalar fields, massive neutrinos,..) have simple forms for their stress tensor in terms of the energy density, i.e. described by equations of state.
- An equation of state for the background $w = p/\rho$ is *not* sufficient to determine the behavior of the perturbations.

Separate Universes

- Geometry of the gauge or time slicing and spatial threading
- For perturbations larger than the horizon, a local observer should just see a different (separate) FRW universe
- Scalar equations should be equivalent to an appropriately remapped Friedmann equation
- Unit normal vector N^μ to constant time hypersurfaces $N_\mu dx^\mu = N_0 d\eta$, $N^\mu N_\mu = -1$, to linear order in metric

$$\begin{aligned} N_0 &= -a(1+AQ), \qquad N_i = 0 \\ N^0 &= a^{-1}(1-AQ), \qquad N^i = -BQ^i \end{aligned}$$

- Expansion of spatial volume per proper time is given by 4-divergence

$$\nabla_\mu N^\mu \equiv \theta = 3H(1-AQ) + \frac{k}{a}BQ + \frac{3}{a}\dot{H}_L Q$$

Shear and Acceleration

- Other pieces of $\nabla_\nu N_\mu$ give the vorticity, shear and acceleration

$$\nabla_\nu N_\mu \equiv \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\theta P_{\mu\nu} - a_\mu N_\nu$$

with

$$\begin{aligned}
P_{\mu\nu} &= g_{\mu\nu} + N_\mu N_\nu \\
\omega_{\mu\nu} &= P_\mu{}^\alpha P_\nu{}^\beta (\nabla_\beta N_\alpha - \nabla_\alpha N_\beta) \\
\sigma_{\mu\nu} &= \frac{1}{2} P_\mu{}^\alpha P_\nu{}^\beta (\nabla_\beta N_\alpha + \nabla_\alpha N_\beta) - \frac{1}{3}\theta P_{\mu\nu} \\
a_\mu &= (\nabla_\alpha N_\mu) N^\alpha
\end{aligned}$$

projection $P_{\mu\nu}N^\nu = 0$, trace free antisymmetric vorticity, symmetric shear and acceleration

Shear and Acceleration

- Vorticity $\omega_{\mu\nu} = 0$, $\sigma_{00} = \sigma_{0i} = 0 = a_0$
- Remaining perturbed quantities are the spatial shear and acceleration

$$\begin{aligned}\sigma_{ij} &= a(\dot{H}_T - kB)Q_{ij} \\ a_i &= -kAQ_i\end{aligned}$$

- A convenient choice of coordinates might be shear free $\dot{H}_T - kB = 0$
- A alone is related to the perturbed acceleration

Separate Universes

- So the e-foldings of the expansion are given by $d\tau = (1 + AQ)ad\eta$

$$
\begin{aligned}
N &= \int d\tau \frac{1}{3}\theta \\
&= \int d\eta \left(\frac{\dot a}{a} + \dot H_L Q + \frac{1}{3} kBQ \right)
\end{aligned}
$$

  Thus if kB can be ignored as $k \to 0$ then H_L plays the role of a local change in the scale factor, more generally B plays the role of Eulerian $\to$ Lagrangian coordinates.

- Change in H_L between separate universes related to change in number of e-folds: so called δN approach, simplifying equations by using N as time variable to absorb local scale factor effects
- We shall see that for adiabatic perturbations $p(\rho)$ that $\dot H_L = 0$ outside horizon for an appropriate choice of slicing – plays an important role in simplifying calculations

Separate Universes

- Choosing coordinates where $\dot{H}_L + kB/3 = 0$ (defines the slicing), the e-folding remains unperturbed, we get that the 00 Einstein equations at $k \to 0$ are

$$-\left(\frac{\dot{a}}{a}\right)^2 A + \frac{1}{3}\frac{k^2 - 3K}{a^2}(H_L + H_T/3) = \frac{4\pi G}{3} a^2 \delta\rho$$

which is to be compared to the Friedmann equation

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

Noting that $H = \bar{H}(1 - AQ)$ and using the perturbation to ${}^{(3)}\mathcal{R}$

$$\begin{aligned}
2\delta H \bar{H} + \frac{\delta K}{a^2} &= \frac{8\pi G}{3}\delta\rho Q \\
-2AQ\bar{H}^2 + \frac{2}{3}\frac{k^2 - 3K}{a^2}(H_L + H_T/3)Q &= \frac{8\pi G}{3}\delta\rho Q \\
-\left(\frac{\dot{a}}{a}\right)^2 A + \frac{1}{3}\frac{k^2 - 3K}{a^2}(H_L + H_T/3) &= \frac{4\pi G}{3}\delta\rho
\end{aligned}$$

Separate Universes

- And the space-space piece

$$\left[2\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\frac{d}{d\eta}\right] A = 4\pi G a^2(\delta p + \delta\rho/3)$$

which is to be compared with the acceleration equation

$$\frac{d}{d\eta}(aH) = -4\pi G a^2(p + \rho/3)$$

again expanding $H = \bar{H}(1 - AQ)$ and also $d\eta = (1 + AQ)d\bar{\eta}$

$$\begin{aligned}
\frac{d}{d\eta}(aH) &= (1 - AQ)\frac{d}{d\bar{\eta}}(a\bar{H})[1 - AQ] \\
&\approx \frac{d}{d\bar{\eta}}(a\bar{H}) - 2AQ\frac{d}{d\bar{\eta}}\frac{\dot{a}}{a} - \frac{\dot{a}}{a}\frac{d}{d\bar{\eta}}AQ
\end{aligned}$$

Separate Universes

- Finally the continuity equation (using slicing with $\dot{H}_L = -kB/3$)

$$\dot{\delta\rho} + 3\frac{\dot{a}}{a}(\delta\rho + \delta p) = -(\rho + p)k(v - B)$$

is to be compared to

$$d\rho/d\eta = -3(aH)(\rho + p)$$

which again with the substitutions becomes

$$\begin{aligned} (1 - AQ)\frac{d}{d\bar{\eta}}(\bar{\rho} + \delta\rho Q) &= -3(aH)(1 - AQ)[\bar{\rho} + \bar{p}] - 3(aH)[\delta\rho + \delta p]Q \\ \frac{d}{d\bar{\eta}}\delta\rho &= -3\frac{\dot{a}}{a}(\delta\rho + \delta p) \end{aligned}$$

- $\delta\rho/\rho$ constant in $\dot{H}_L + kB/3 = 0$ slicing outside horizon where peculiar velocity cannot move matter (cf. Newtonian gauge below).
- Note also that $v - B$ has a special interpretation as well: setting $v = B$ gives a comoving slicing since $N^i \propto v^i$, $N_i \propto v_i - B_i = 0$

Separate Universes

- There are other possible choices what to hold fixed on constant time slices besides $N = \ln a$. While separate universe statements still hold a must be perturbed and the simplest gauge to see these identifications with the Friedmann equations changes.
- More generally the analysis of the normal to constant time surfaces has identified geometric quantities associated with the metric perturbations
- Uniform efolding: $\dot{H}_L + kB/3 = 0$
- Shear free: $\dot{H}_T - kB = 0$
- Zero acceleration, coordinate and proper time coincide: $A = 0$
- Uniform expansion: $-3HA + (3\dot{H}_L + kB) = 0$
- Comoving: $v = B$

Gauge

- Metric and matter fluctuations take on different values in different coordinate system
- No such thing as a "gauge invariant" density perturbation!
- General coordinate transformation:

$$
\begin{aligned}
\tilde{\eta} &= \eta + T \\
\tilde{x}^i &= x^i + L^i
\end{aligned}
$$

  free to choose (T, L^i) to simplify equations or physics - corresponds to a choice of slicing and threading in ADM.

- Decompose these into scalar T, $L^{(0)}$ and vector harmonics $L^{(\pm 1)}$.

Gauge

- $g_{\mu\nu}$ and $T_{\mu\nu}$ transform as tensors, so components in different frames can be related

$$
\begin{aligned}
\tilde{g}_{\mu\nu}(\tilde{\eta}, \tilde{x}^i) &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\eta, x^i) \\
&= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\tilde{\eta} - TQ, \tilde{x}^i - LQ^i)
\end{aligned}
$$

- Fluctuations are compared at the same coordinate positions (not same space time positions) between the two gauges
- For example with a TQ perturbation, an event labeled with $\tilde{\eta} =$const. and $\tilde{x} =$const. represents a different time with respect to the underlying homogeneous and isotropic background

Gauge Transformation

- Scalar Metric:

$$\tilde{A} = A - \dot{T} - \frac{\dot{a}}{a}T\,,$$
$$\tilde{B} = B + \dot{L} + kT\,,$$
$$\tilde{H}_L = H_L - \frac{k}{3}L - \frac{\dot{a}}{a}T\,,$$
$$\tilde{H}_T = H_T + kL\,, \qquad \tilde{H}_L + \frac{1}{3}\tilde{H}_T = H_L + \frac{1}{3}H_T - \frac{\dot{a}}{a}T$$

curvature perturbation depends on slicing not threading

- Scalar Matter (Jth component):

$$\delta\tilde{\rho}_J = \delta\rho_J - \dot{\rho}_J T\,,$$
$$\delta\tilde{p}_J = \delta p_J - \dot{p}_J T\,,$$
$$\tilde{v}_J = v_J + \dot{L}\,,$$

density and pressure likewise depend on slicing only

Gauge Transformation

- Vector:

$$
\begin{aligned}
\tilde{B}^{(\pm 1)} &= B^{(\pm 1)} + \dot{L}^{(\pm 1)}, \\
\tilde{H}_T^{(\pm 1)} &= H_T^{(\pm 1)} + kL^{(\pm 1)}, \\
\tilde{v}_J^{(\pm 1)} &= v_J^{(\pm 1)} + \dot{L}^{(\pm 1)},
\end{aligned}
$$

- Spatial vector has no background component hence no dependence on slicing at first order

  Tensor: no dependence on slicing or threading at first order

- Gauge transformations and covariant representation can be extended to higher orders

- A coordinate system is fully specified if there is an explicit prescription for (T, L^i) or for scalars (T, L)

Slicing

Common choices for slicing T: set something geometric to zero

- Proper time slicing $A = 0$: proper time between slices corresponds to coordinate time – T allows c/a freedom
- Comoving (velocity orthogonal) slicing: $v - B = 0$, matter 4 velocity is related to N^ν and orthogonal to slicing - T fixed
- Newtonian (shear free) slicing: $\dot{H}_T - kB = 0$, expansion rate is isotropic, shear free, T fixed
- Uniform expansion slicing: $-(\dot{a}/a)A + \dot{H}_L + kB/3 = 0$, perturbation to the volume expansion rate θ vanishes, T fixed
- Flat (constant curvature) slicing, $\delta^{(3)}R = 0$, $(H_L + H_T/3 = 0)$, T fixed
- Constant density slicing, $\delta\rho_I = 0$, T fixed

Threading

- Threading specifies the relationship between constant spatial coordinates between slices and is determined by L

  Typically involves a condition on v, B, H_T

  - Orthogonal threading $B = 0$, constant spatial coordinates orthogonal to slicing (zero shift), allows $\delta L = c$ translational freedom
  - Comoving threading $v = 0$, allows $\delta L = c$ translational freedom.
  - Isotropic threading $H_T = 0$, fully fixes L

Newtonian (Longitudinal) Gauge

- Newtonian (shear free slicing, isotropic threading):

$$
\begin{aligned}
\tilde{B} &= \tilde{H}_T = 0 \\
\Psi &\equiv \tilde{A} \quad \text{(Newtonian potential)} \\
\Phi &\equiv \tilde{H}_L \quad \text{(Newtonian curvature)} \\
L &= -H_T/k \\
T &= -B/k + \dot{H}_T/k^2
\end{aligned}
$$

Good: intuitive Newtonian like gravity; matter and metric algebraically related; commonly chosen for analytic CMB and lensing work

Bad: numerically unstable

Newtonian (Longitudinal) Gauge

- Newtonian (shear free) slicing, isotropic threading $B = H_T = 0$:

$$(k^2 - 3K)\Phi = 4\pi G a^2 \left[\delta\rho + 3\frac{\dot a}{a}(\rho + p)v/k\right] \quad \text{Poisson + Momentum}$$

$$k^2(\Psi + \Phi) = 8\pi G a^2 p \Pi \quad \text{Anisotropy}$$

so $\Psi = -\Phi$ if anisotropic stress $\Pi = 0$ and

$$\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho + 3\frac{\dot a}{a}\delta p = -(\rho + p)(kv + 3\dot\Phi)\,,$$

$$\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho + p)v = k\delta p - \frac{2}{3}(1 - 3\frac{K}{k^2})p\,k\Pi + (\rho + p)\,k\Psi\,,$$

- Newtonian competition between stress (pressure and viscosity) and potential gradients
- Note: Poisson source is the density perturbation on comoving slicing

Total Matter Gauge

- Total matter: (comoving slicing, isotropic threading)

$$
\begin{aligned}
\tilde{B} &= \tilde{v} \quad (T^0_i = 0) \\
H_T &= 0 \\
\xi &= \tilde{A} \\
\mathcal{R} &= \tilde{H}_L \quad \text{(comoving curvature)} \\
\Delta &= \tilde{\delta} \quad \text{(total density pert)} \\
T &= (v - B)/k \\
L &= -H_T/k
\end{aligned}
$$

Good: Algebraic relations between matter and metric; comoving curvature perturbation obeys conservation law

Bad: Non-intuitive threading involving v

Total Matter Gauge

- Euler equation becomes an algebraic relation between stress and potential

$$(\rho + p)\xi = -\delta p + \frac{2}{3}\left(1 - \frac{3K}{k^2}\right) p\Pi$$

- Einstein equation lacks momentum density source

$$\frac{\dot{a}}{a}\xi - \dot{\mathcal{R}} - \frac{K}{k^2} kv = 0$$

Combine: $\mathcal{R}$ is conserved if stress fluctuations negligible, e.g. above the horizon if $|K| \ll H^2$

$$\dot{\mathcal{R}} + Kv/k = \frac{\dot{a}}{a}\left[-\frac{\delta p}{\rho + p} + \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\frac{p}{\rho + p}\Pi\right] \to 0$$

“Gauge Invariant” Approach

- Gauge transformation rules allow variables which take on a geometric meaning in one choice of slicing and threading to be accessed from variables on another choice
- Functional form of the relationship between the variables is gauge invariant (*not* the variable values themselves! – i.e. equation is *covariant*)
- E.g. comoving curvature and density perturbations

$$
\begin{aligned}
\mathcal{R} &= H_L + \frac{1}{3}H_T - \frac{\dot{a}}{a}(v - B)/k \\
\Delta\rho &= \delta\rho + 3(\rho + p)\frac{\dot{a}}{a}(v - B)/k
\end{aligned}
$$

Newtonian-Total Matter Hybrid

- With the gauge in(*or co*)variant approach, express variables of one gauge in terms of those in another – allows a mixture in the equations of motion
- Example: Newtonian curvature and comoving density

$$(k^2 - 3K)\Phi = 4\pi G a^2 \rho \Delta$$

  ordinary Poisson equation then implies Φ approximately constant if stresses negligible.
- Example: Exact Newtonian curvature above the horizon derived through comoving curvature conservation

  Gauge transformation

$$\Phi = \mathcal{R} + \frac{\dot{a}}{a}\frac{v}{k}$$

Hybrid "Gauge Invariant" Approach

Einstein equation to eliminate velocity

$$\frac{\dot{a}}{a}\Psi - \dot{\Phi} = 4\pi G a^2 (\rho + p) v/k$$

Friedmann equation with no spatial curvature

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} a^2 \rho$$

With $\dot{\Phi} = 0$ and $\Psi \approx -\Phi$

$$\frac{\dot{a}}{a}\frac{v}{k} = -\frac{2}{3(1+w)}\Phi$$

Newtonian-Total Matter Hybrid

Combining gauge transformation with velocity relation

$$\Phi = \frac{3+3w}{5+3w}\mathcal{R}$$

Usage: calculate $\mathcal{R}$ from inflation determines Φ for any choice of matter content or causal evolution.

- Example: Scalar field ("quintessence" dark energy) equations in total matter gauge imply a sound speed $\delta p/\delta\rho = 1$ independent of potential $V(\phi)$. Solve in synchronous gauge.

Synchronous Gauge

- Synchronous: (proper time slicing, orthogonal threading)

$$
\begin{aligned}
\tilde{A} &= \tilde{B} = 0 \\
\eta_T &\equiv -\tilde{H}_L - \frac{1}{3}\tilde{H}_T \\
h_L &\equiv 6H_L \\
T &= a^{-1}\int d\eta a A + c_1 a^{-1} \\
L &= -\int d\eta(B + kT) + c_2
\end{aligned}
$$

Good: stable, the choice of numerical codes

Bad: residual gauge freedom in constants c_1, c_2 must be specified as an initial condition, intrinsically relativistic, threading conditions breaks down beyond linear regime if c_1 is fixed to CDM comoving.

Synchronous Gauge

- The Einstein equations give

$$
\begin{aligned}
\dot\eta_T - \frac{K}{2k^2}(\dot h_L + 6\dot\eta_T) &= 4\pi G a^2(\rho+p)\frac{v}{k}\,,\\
\ddot h_L + \frac{\dot a}{a}\dot h_L &= -8\pi G a^2(\delta\rho + 3\delta p)\,,\\
(k^2-3K)\eta_T + \frac{1}{2}\frac{\dot a}{a}\dot h_L &= 4\pi G a^2\delta\rho
\end{aligned}
$$

[choose (1 & 2) or (1 & 3)] while the conservation equations give

$$
\begin{aligned}
\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho_J + 3\frac{\dot a}{a}\delta p_J &= -(\rho_J+p_J)(kv_J + \frac{1}{2}\dot h_L)\,,\\
\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho_J+p_J)\frac{v_J}{k} &= \delta p_J - \frac{2}{3}(1-3\frac{K}{k^2})p_J\Pi_J\,.
\end{aligned}
$$

Synchronous Gauge

- Lack of a lapse A implies no gravitational forces in Navier-Stokes equation. Hence for stress free matter like cold dark matter, zero velocity initially implies zero velocity always.
- Choosing the momentum and acceleration Einstein equations is good since for CDM domination, curvature η_T is conserved and $\dot{h}_L$ is simple to solve for.
- Choosing the momentum and Poisson equations is good when the equation of state of the matter is complicated since δp is not involved. This is the choice of CAMB.

  Caution: since the curvature η_T appears and it has zero CDM source, subtle effects like dark energy perturbations are important everywhere

Spatially Flat Gauge

- Spatially Flat (flat slicing, isotropic threading):

$$\begin{aligned} \tilde{H}_L &= \tilde{H}_T = 0 \\ L &= -H_T/k \\ \tilde{A}\,,\tilde{B} &= \quad \text{metric perturbations} \\ T &= \left(\frac{\dot{a}}{a}\right)^{-1}\left(H_L + \frac{1}{3}H_T\right) \end{aligned}$$

  Good: eliminates spatial metric in evolution equations; useful in inflationary calculations (Mukhanov et al)

  Bad: non-intuitive slicing (no curvature!) and threading

- Caution: perturbation evolution is governed by the behavior of stress fluctuations and an isotropic stress fluctuation δp is gauge dependent.

Uniform Density Gauge

- Uniform density: (constant density slicing, isotropic threading)

$$
\begin{aligned}
H_T &= 0\,, \\
\zeta_I &\equiv H_L \\
B_I &\equiv B \\
A_I &\equiv A \\
T &= \frac{\delta\rho_I}{\dot\rho_I} \\
L &= -H_T/k
\end{aligned}
$$

Good: Curvature conserved involves only stress energy conservation; simplifies isocurvature treatment

Bad: non intuitive slicing (no density pert! problems beyond linear regime) and threading

Uniform Density Gauge

- Einstein equations with I as the total or dominant species

$$(k^2 - 3K)\zeta_I - 3\left(\frac{\dot a}{a}\right)^2 A_I + 3\frac{\dot a}{a}\dot\zeta_I + \frac{\dot a}{a}kB_I = 0\,,$$

$$\frac{\dot a}{a}A_I - \dot\zeta_I - \frac{K}{k}B_I = 4\pi G a^2(\rho + p)\frac{v - B_I}{k}\,,$$

- The conservation equations (if $J = I$ then $\delta\rho_J = 0$)

$$\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho_J + 3\frac{\dot a}{a}\delta p_J = -(\rho_J + p_J)(kv_J + 3\dot\zeta_I)\,,$$

$$\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho_J + p_J)\frac{v_J - B_I}{k} = \delta p_J - \frac{2}{3}(1 - 3\frac{K}{k^2})p_J\Pi_J + (\rho_J + p_J)A_I\,.$$

Uniform Density Gauge

- Conservation of curvature - single component I

$$\dot{\zeta}_I = -\frac{\dot{a}}{a}\frac{\delta p_I}{\rho_I + p_I} - \frac{1}{3}kv_I \, .$$

- Since $\delta\rho_I = 0$, δp_I is the non-adiabatic stress and curvature is constant as $k \to 0$ for adiabatic fluctuations $p_I(\rho_I)$.

- Note that this conservation law does not involve the Einstein equations at all: just local energy momentum conservation so it is valid for alternate theories of gravity

- Curvature on comoving slices $\mathcal{R}$ and ζ_I related by

$$\zeta_I = \mathcal{R} + \frac{1}{3}\frac{\rho\Delta_I}{(\rho_I + p_I)}\Big|_{\rm comoving} \, .$$

  and coincide above the horizon for adiabatic fluctuations

Uniform Density Gauge

- Simple relationship to density fluctuations in the spatially flat gauge

$$\zeta_I = \frac{1}{3} \frac{\delta\tilde{\rho}_I}{(\rho_I + p_I)}\Big|_{\rm flat} .$$

- For each particle species $\delta\rho/(\rho+p) = \delta n/n$, the number density fluctuation
- Multiple ζ_J carry information about number density fluctuations between species
- ζ_J constant component by component outside horizon if each component is adiabatic $p_J(\rho_J)$.

Vector Gauges

- Vector gauge depends only on threading L
- Poisson gauge: orthogonal threading $B^{(\pm 1)} = 0$, leaves constant L translational freedom
- Isotropic gauge: isotropic threading $H_T^{(\pm 1)} = 0$, fixes L
- To first order scalar and vector gauge conditions can be chosen separately
- More care required for second and higher order where scalars and vectors mix