Astro 449

Spherical Collapse & Halo Model
Wayne Hu
Closed Universe

- **Friedmann equation in a closed universe**

\[
\frac{1}{a} \frac{da}{dt} = H_0 \left( \Omega_m a^{-3} + (1 - \Omega_m) a^{-2} \right)^{1/2}
\]

- **Parametric solution in terms of a development angle**

\[\theta = H_0 \eta (\Omega_m - 1)^{1/2}, \text{ scaled conformal time } \eta\]

\[r(\theta) = A(1 - \cos \theta)\]
\[t(\theta) = B(\theta - \sin \theta)\]

where \(A = r_0 \Omega_m / 2(\Omega_m - 1), \ B = H_0^{-1} \Omega_m / 2(\Omega_m - 1)^{3/2}\).

- Turn around at \(\theta = \pi, \ r = 2A, \ t = B\pi\).

- Collapse at \(\theta = 2\pi, \ r \to 0, \ t = 2\pi B\)
Spherical Collapse

- Parametric Solution:

![Graph showing Spherical Collapse](image)
Correspondence

- Eliminate cosmological correspondence in $A$ and $B$ in terms of enclosed mass $M$

$$M = \frac{4\pi}{3} r_0^3 \Omega_m \rho_c = \frac{4\pi}{3} r_0^3 \Omega_m \frac{3H_0^2}{8\pi G}$$

- Related as $A^3 = GM B^2$, and to initial perturbation

$$\lim_{\theta \to 0} r(\theta) = A \left( \frac{1}{2} \theta^2 - \frac{1}{24} \theta^4 \right)$$

$$\lim_{\theta \to 0} t(\theta) = B \left( \frac{1}{6} \theta^3 - \frac{1}{120} \theta^5 \right)$$

- Leading Order: $r = A \theta^2 / 2$, $t = B \theta^3 / 6$

$$r = \frac{A}{2} \left( \frac{6t}{B} \right)^{2/3}$$
Next Order

- Leading order is unperturbed matter dominated expansion
  \[ r \propto a \propto t^{2/3} \]

- Iterate \( r \) and \( t \) solutions

\[
\lim_{\theta \to 0} t(\theta) = \frac{\theta^3}{6 B} \left[ 1 - \frac{1}{20} \left( \frac{6t}{B} \right)^{2/3} \right]
\]

\[
\theta \approx \left( \frac{6t}{B} \right)^{1/3} \left[ 1 + \frac{1}{60} \left( \frac{6t}{B} \right)^{2/3} \right]
\]
Next Order

- Substitute back into $r(\theta)$

\[
r(\theta) \quad = \quad A \frac{\theta^2}{2} \left( 1 - \frac{\theta^2}{12} \right)
\]

\[
= \quad A \left( \frac{6t}{B} \right)^{2/3} \left[ 1 - \frac{1}{20} \left( \frac{6t}{B} \right)^{2/3} \right]
\]

\[
= \quad \frac{1}{2} (6t)^{2/3} (GM)^{1/3} \left[ 1 - \frac{1}{20} \left( \frac{6t}{B} \right)^{2/3} \right]
\]
Density Correspondence

- Density

\[
\rho_m = \frac{M}{\frac{4}{3} \pi r^3}
\]

\[
= \frac{1}{6\pi t^2 G} \left[ 1 + \frac{3}{20} \left( \frac{6t}{B} \right)^{2/3} \right]
\]

- Density perturbation

\[
\delta \equiv \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m} \approx \frac{3}{20} \left( \frac{6t}{B} \right)^{2/3}
\]
Density Correspondence

- Time $\rightarrow$ scale factor

\[
t = \frac{2}{3H_0 \Omega_m^{1/2}} a^{3/2}
\]

\[
\delta = \frac{3}{20} a \left( \frac{4}{BH_0 \Omega_m^{1/2}} \right)^{2/3}
\]

- $A$ and $B$ constants $\rightarrow$ initial cond.

\[
B = \frac{1}{2H_0 \Omega_m^{1/2}} \left( \frac{3 a_i}{5 \delta_i} \right)^{3/2}
\]

\[
A = \frac{3}{10} \frac{r_i}{\delta_i}
\]
Spherical Collapse Relations

- Scale factor $a \propto t^{2/3}$
  
  $$a = \left( \frac{3}{4} \right)^{2/3} \left( \frac{3}{5} \frac{a_i}{\delta_i} \right) (\theta - \sin \theta)^{2/3}$$

- At collapse $\theta = 2\pi$
  
  $$a_{\text{col}} = \left( \frac{3}{4} \right)^{2/3} \left( \frac{3}{5} \frac{a_i}{\delta_i} \right) (2\pi)^{2/3} \approx 1.686 \frac{a_i}{\delta_i}$$

- Perturbation collapses when linear theory predicts $\delta_c \equiv 1.686$
**Virialization**

- A real density perturbation is neither spherical nor homogeneous
- **Shell crossing** if $\delta_i$ doesn’t monotonically decrease
- Collapse does not proceed to a point but reaches **virial equilibrium**

\[ U = -2K, \quad E = U + K = U(r_{\text{max}}) = \frac{1}{2}U(r_{\text{vir}}) \quad (1) \]

\[ r_{\text{vir}} = \frac{1}{2}r_{\text{max}} \text{ since } U \propto r^{-1}. \text{ Thus } \theta_{\text{vir}} = \frac{3}{2}\pi \]

- **Overdensity** at virialization

\[ \frac{\rho_m(\theta = 3\pi/2)}{\bar{\rho}_m(\theta = 2\pi)} = 18\pi^2 \approx 178 \]

- **Threshold** $\Delta_v = 178$ often used to define a **collapsed object**

- Equivalently relation between virial mass, radius, overdensity:

\[ M_v = \frac{4\pi}{3} r_v^3 \rho_m \Delta_v \]
Virialization

- Schematic Picture:
Generalization Beyond Matter

- In a universe with smooth components like dark energy driving the expansion but not participating in collapse we cannot consider spherical collapse to be a separate universe.

- Go back to the continuity and Euler equation to derive the general equation:

\[
\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot (1 + \delta) \mathbf{v} = 0
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a} (\mathbf{v} \cdot \nabla) \mathbf{v} + H \mathbf{v} = -\frac{1}{a} \nabla \Psi
\]

which is true for any type of dark energy or even metric modified gravity.
Generalization Beyond Matter

• For a tophat density perturbation \( \nu = A(t)r \) interior given the continuity equation and so

\[
\frac{d^2 \delta}{dt^2} - \frac{4}{3} \frac{1}{1 + \delta} \left( \frac{d\delta}{dt} \right)^2 + 2H \frac{d\delta}{dt} = \frac{(1 + \delta)}{a^2} \nabla^2 \Psi
\]

• Under ordinary gravity \( \nabla^2 \Psi = 4\pi G a^2 \bar{\rho}_m \delta \) and so a tophat remains a tophat

• Thus use conservation of the dark matter mass

\[
M = (4\pi/3)r^3 \bar{\rho}_m (1 + \delta)
\]

\( \delta \to R \) to trade the density for the tophat radius.
Generalization Beyond Matter

- Using the Friedmann equations for the evolution of the background

\[ H^2 = \frac{8\pi G}{3}(\bar{\rho}_m + \bar{\rho}_{\text{eff}}) \]

we obtain using the Poisson equation

\[ \frac{1}{r} \frac{d^2 r}{dt^2} = H^2 + \dot{H} - \frac{1}{3} \nabla^2 \Psi \]

\[ = -\frac{4\pi G}{3} [\rho_m + (1 + 3w_{\text{eff}})\bar{\rho}_{\text{eff}}] \]

where \( \rho_m = \bar{\rho}_m(1 + \delta) \) includes the tophat fluctuation whereas \( \bar{\rho}_{\text{eff}} \) is a smooth background contribution to the Friedmann equation

- In other words \( H^2 + \dot{H} \) carries the acceleration effect of background total density but \( \Psi \) carries only that of the collapsing component - alters the collapse relations
Generalization Beyond Matter

• Similarly virial equilibrium altered to include smooth contribution to acceleration or effective potential

\[ U = -2K \]

where

\[ U = - \frac{3GM^2}{5R} - \frac{4\pi G}{5}(1 + 3w_{\text{eff}})\bar{\rho}_{\text{eff}}MR^2 \]

• Note that virial equilibrium is defined in terms of the trace of the potential tensor and is a statement of force balance

\[ U \equiv -\int d^3x \rho_m \mathbf{x} \cdot \nabla \Psi_{\text{tot}} \]
Generalization Beyond Matter

- Hence $U$ is well defined even in cases where energy is not conserved in the usual manner (though still convariantly conserved), e.g. if $\rho_{\text{eff}}$ is not constant during collapse.

- In general keep track of the kinetic energy during collapse and finding the virial radius as the point at which

$$U(r_{\text{vir}}) = -2K(r_{\text{vir}})$$

- Rather than using energy conservation (important if $w_{\text{eff}} \neq -1$)
The Mass Function

- **Spherical collapse** predicts the end state as virialized **halos** given an initial density perturbation.
- Initial density perturbation is a **Gaussian random field**.
- Compare the variance in the linear density field to **threshold** $\delta_c = 1.686$ to determine collapse fraction.
- Combine to form the **mass function**, the number density of halos in a range $dM$ around $M$.
- Halo density defined entirely by linear theory.
- **Fudge the result** to get the right answer compared with simulations (a la Press-Schechter)!
Press-Schechter Formalism

- **Smooth** linear density field on mass scale $M$ with tophat

$$R = \left( \frac{3M}{4\pi} \right)^{1/3}$$

- Result is a Gaussian random field with variance $\sigma^2(M)$

- Fluctuations above the threshold $\delta_c$ correspond to collapsed regions. The fraction in halos $> M$ becomes

$$\frac{1}{\sqrt{2\pi}\sigma(M)} \int_{\delta_c}^{\infty} d\delta \exp \left( -\frac{\delta^2}{2\sigma^2(M)} \right) = \frac{1}{2} \text{erfc} \left( \frac{\nu}{\sqrt{2}} \right)$$

where $\nu \equiv \delta_c / \sigma(M)$

- **Problem:** even as $\sigma(M) \to \infty$, $\nu \to 0$, collapse fraction $\to 1/2$ – only overdense regions participate in spherical collapse.

- Multiply by an ad hoc factor of 2!
Press-Schechter Mass Function

- Differentiate in $M$ to find fraction in range $dM$ and multiply by $\rho_m/M$ the number density of halos if all of the mass were composed of such halos $\rightarrow$ differential number density of halos

\[
\frac{dn}{d \ln M} = \frac{\rho_m}{M} \frac{d}{d \ln M} \text{erfc} \left( \frac{\nu}{\sqrt{2}} \right)
\]

\[
= \sqrt{\frac{2}{\pi}} \frac{\rho_m}{M} \frac{d \ln \sigma^{-1}}{d \ln M} \nu \exp(-\nu^2/2)
\]

- High mass: exponential cut off above $M_\ast$ where $\sigma(M_\ast) = \delta_c$

\[
M_\ast \sim 10^{13} h^{-1} M_\odot \quad \text{today}
\]

- Low mass divergence: (too many for the observations?)

\[
\frac{dn}{d \ln M} \propto \sim M^{-1}
\]
Extended Press-Schechter Formalism

- A region that is \textit{underdense} when smoothed on the scale $M$ may be \textit{overdense} on a scale of a larger $M$

- If smoothing is a tophat in $k$-space, independence of $k$-modes implies fluctuation executes a \textit{random walk}

\begin{center}
\begin{figure}
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Press-Schechter prescription}
\label{fig:diagram}
\end{figure}
\end{center}
Extended Press-Schechter Formalism

- For each trajectory that lies above threshold at $M_2$, there is an equivalent trajectory that is its mirror image reflected around $\delta_c$

- Press-Schechter ignored this branch. It supplies the missing factor of 2
Conditional Mass Function

- Extended Press-Schechter also gives the conditional mass function, useful for merger histories.
- Given a halo of mass $M_1$ exists at $z_1$, what is the probability that it was part of a halo of mass $M_2$ at $z_2$
Conditional Mass Function

- Same as before but with the origin translated.
- Conditional mass function is mass function with $\delta_c$ and $\sigma^2(M)$ shifted.

![Diagram showing a graph with $\delta$, $M_1$, $M_2$, and $R(M)$ axes, with horizontal lines at $(1+z_1)\delta_c$, $(1+z_2)\delta_c$, $\delta$, $M_1$, and $M_2$.]
Magic “2” resolved?

- Spherical collapse is defined for a real-space not $k$-space smoothing. Random walk is only a qualitative explanation.

- Modern approach: think of spherical collapse as motivating a fitting form for the mass function

$$\nu \exp(-\nu^2/2) \rightarrow A[1 + (a\nu^2)^{-p}]\sqrt{a\nu^2} \exp(-a\nu^2/2)$$

Sheth-Torman 1999, $a = 0.75$, $p = 0.3$. or a completely empirical fitting

$$\frac{dn}{d\ln M} = 0.301 \frac{\rho_m}{M} \frac{d\ln \sigma^{-1}}{d\ln M} \exp[-|\ln \sigma^{-1} + 0.64|^{3.82}]$$

Jenkins et al 2001. Choice is tied up with the question: what is the mass of a halo?
Numerical Mass Function

- Example of difference in mass definition (from Hu & Kravstov 2002)

\[ \Omega_m = \Gamma = 0.15; \text{ flat; } h = 0.65; \sigma_8 = 1.07 \]
Halo Bias

- If halos are formed without regard to the underlying density fluctuation and move under the gravitational field then their number density is an unbiased tracer of the dark matter density fluctuation

\[
\left( \frac{\delta n}{n} \right)_{\text{halo}} = \left( \frac{\delta \rho}{\rho} \right)
\]

- However spherical collapse says the probability of forming a halo depends on the initial density field

- Large scale density field acts as “background” enhancement of probability of forming a halo or “peak”

- Peak-Background Split (Efstathiou 1998; Cole & Kaiser 1989; Mo & White 1997)
Peak-Background Split

- Schematic Picture:
Perturbed Mass Function

- Density fluctuation split

\[ \delta = \delta_b + \delta_p \]

- Lowers the threshold for collapse

\[ \delta_{cp} = \delta_c - \delta_b \]

so that \( \nu = \delta_{cp}/\sigma \)

- Taylor expand number density \( n_M \equiv dn/d\ln M \)

\[ n_M + \frac{dn_M}{d\nu} \frac{d\nu}{d\delta_b} \delta_b \ldots = n_M \left[ 1 + \frac{(\nu^2 - 1)}{\sigma \nu} \delta_b \right] \]

if mass function is given by Press-Schechter

\[ n_M \propto \nu \exp(-\nu^2/2) \]
Halo Bias

- Halos are biased tracers of the “background” dark matter field with a bias $b(M)$ that is given by spherical collapse and the form of the mass function.

- Combine the enhancement with the original unbiased expectation

$$\frac{\delta n_M}{n_M} = b(M)\delta_b$$

- For Press-Schechter

$$b(M) = 1 + \frac{\nu^2 - 1}{\delta_c}$$

- Improved by the Sheth-Torman mass function

$$b(M) = 1 + \frac{a\nu^2 - 1}{\delta_c} + \frac{2p}{\delta_c[1 + (a\nu^2)^p]}$$

with $a = 0.75$ and $p = 0.3$ to match simulations.
Numerical Bias

- Example of halo bias from a simulation (from Hu & Kravstov 2002)
What is a Halo?

- Mass function and halo bias depend on the definition of mass of a halo.
- Agreement with simulations depend on how halos are identified.
- Other observables (associated galaxies, X-ray, SZ) depend on the details of the density profile.
- Fortunately, simulations have shown that halos take on a near universal form in their density profile at least on large scales.
• Density profile well-described by (Navarro, Frenk & White 1997)

\[ \rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2} \]
Einasto Profile

- Current best simulations find that the inner slope runs rather than asymptotes to a cuspy constant
- This form is better fit by the Einasto profile (c.f. Sersic profile)

\[
\ln \frac{\rho(r)}{\rho_s} = -2 \frac{1}{\alpha} \left[ \left( \frac{r}{r_s} \right)^\alpha - 1 \right]
\]

- The local slope is given by

\[
\frac{d \ln \rho}{d \ln r} = -2 \left( \frac{r}{r_s} \right)^\alpha
\]

and continues to decrease as \( r/r_s \to 0 \)
Whence Universal Profile?

- Recent investigations by Dalal, Lithwick, Kuhlen (2010) suggests that the universal halo profile arises generically from peaks in a Gaussian random field.
- Outer $r^{-3}$ profile predicted from slow accretion of material at low initial overdensity compared with peak.
- Inner profile comes from adiabatic contraction (i.e. preserving adiabatic invariants during collapse) and depends on the initial density profile of peak.
- Dynamical friction implies that the centroid of the initial density peak will settle to the center of the final halo.
Transforming the Masses

- NFW profile gives a way of transforming different mass definitions

\[ n(M) (\text{Mpc}^{-3}) \]

\[ M (h^{-1} M_\odot) \]

\[ \Delta_{180} \] (wrt mean)

\[ \Delta_{666} \]

- Jenkins et al.

- Rescaled

\[ W_m = G = 0.15; \text{ flat; } h = 0.65; s_8 = 1.07 \]
Lack of Concentration?

- NFW parameters may be recast into $M_v$, the mass of a halo out to the virial radius $r_v$ where the overdensity wrt mean reaches $\Delta_v = 180$.
- Concentration parameter

$$c \equiv \frac{r_v}{r_s}$$

- CDM predicts $c \sim 10$ for $M_\star$ halos. Too centrally concentrated for galactic rotation curves?
- Possible discrepancy has lead to the exploration of dark matter alternatives: warm ($m \sim \text{keV}$) dark matter, self-interacting dark-matter, annihilating dark matter, ultra-light “fuzzy” dark matter, . . .
The Halo Model

- NFW halos, of abundance $n_M$ given by mass function, clustered according to the halo bias $b(M)$ and the linear theory $P(k)$
- Power spectrum example:
Non-Linear Power Spectrum

- Non-linear power spectrum is composed of dark matter halos that are clustered according to the halo bias $+\, the\, clustering\, due\, to\, the\, halo\, density\, profile$

\[ P_{nl}(k, z) = I_2^2(k, z)P(k, z) + I_1(k, z) \]

where

\[ I_2(k, z) = \int d\ln M \left( \frac{M}{\rho_m(z = 0)} \right) \frac{dn}{d\ln M} b(M)y(k, M) \]

\[ I_1(k, z) = \int d\ln M \left( \frac{M}{\rho_m(z = 0)} \right)^2 \frac{dn}{d\ln M} y^2(k, M) \]

and $y$ is the Fourier transform of the halo profile with $y(0, M) = 1$

\[ y(k, M) = \frac{1}{M} \int_0^{r_h} dr 4\pi r^2 \rho(r, M) \frac{\sin(kr)}{kr} \]
Galaxy Power Spectrum

- For galaxies, one defines a halo occupation distribution which determines the number of galaxies (satisfying a certain observational criteria) that can occupy a halo of mass $M$.
- Take a simple example of a mass selection on the galaxies, then $N(M) = 0$ for $M < M_{th}$ and above threshold.
  $N(M) = C + S(M)$ where $C = 1$ accounts for the central galaxy and satellite galaxies follow a poisson distribution with mean $S(M) \approx M/30M_{th}$.
Galaxy Power Spectrum

- Then assuming that satellites are distributed according to the mass profile

\[ P_{\text{gal}}(k, z) = I_2^2(k, z)P(k, z) + I_1(k, z) \]

where

\[ I_2(k, z) = \frac{1}{n_{\text{gal}}} \int d\ln M \frac{dn}{d\ln M} b(M)[C + y(k, M)S(M)] \]

\[ I_1(k, z) = \frac{1}{n_{\text{gal}}^2} \int d\ln M \frac{dn}{d\ln M} [S^2(M)y^2(k, M) + 2CS(M)y(k, M)] \]

- Break between the one and two halo regime first seen by SDSS
Galaxy Power Spectrum

- Example (Seljak 2001)

- An explanation of the nearly power law galaxy spectrum
Incredible, Extensible Halo Model

- An industry developed to build semi-analytic models for wide variety of cosmological observables based on the halo model
- Idea: associate an observable (galaxies, gas, ...) with dark matter halos
- Let the halo model describe the statistics of the observable
- The overextended halo model?
Halo Temperature

- Motivate with *isothermal distribution*, correct from simulations

\[ \rho(r) = \frac{\sigma^2}{2\pi Gr^2} \]

- Express in terms of *virial mass* \( M_v \) enclosed at *virial radius* \( r_v \)

\[ M_v = \frac{4\pi}{3} r_v^3 \rho_m \Delta_v = \frac{2}{G} r_v \sigma^2 \]

- Eliminate \( r_v \), temperature \( T \propto \sigma^2 \) velocity dispersion\(^2\)

- Then \( T \propto M_v^{2/3} (\rho_m \Delta_v)^{1/3} \) or

\[
\left( \frac{M_v}{10^{15} h^{-1} M_\odot} \right) = \left[ \frac{f}{(1 + z)(\Omega_m \Delta_v)^{1/3}} \frac{T}{1 \text{keV}} \right]^{3/2}
\]

- Theory (*X*-ray weighted): \( f \sim 0.75 \); observations \( f \sim 0.54 \). Difference is *crucial* in determining cosmology from *cluster counts*!
Summary

- **Dark matter simulations** well-understood and can be modelled with dark matter halos

- Halo formation modelled by *spherical collapse*, two magic numbers $\delta_c = 1.686$ and $\Delta_v = 178$

- Halo abundance described by a mass function with exponential high mass cutoff – *rare clusters* extremely sensitive to power spectrum amplitude and growth rate $\rightarrow$ dark energy
  
  Possibly too many small halos or *sub-structure*?

- Halo clustering modelled with peak-background split leading to *halo bias*

- **Halo profile** described by NFW halos
  
  Possibly too high central concentration

- Associate an *observable* with a halo $\rightarrow$ a halo model