

*Astro 449*

# Spherical Collapse & Halo Model

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# Closed Universe

- Friedmann equation in a closed universe

$$\frac{1}{a} \frac{da}{dt} = H_0 \left( \Omega_m a^{-3} + (1 - \Omega_m) a^{-2} \right)^{1/2}$$

- Parametric solution in terms of a **development angle**

$$\theta = H_0 \eta (\Omega_m - 1)^{1/2}, \text{ scaled conformal time } \eta$$

$$r(\theta) = A(1 - \cos \theta)$$

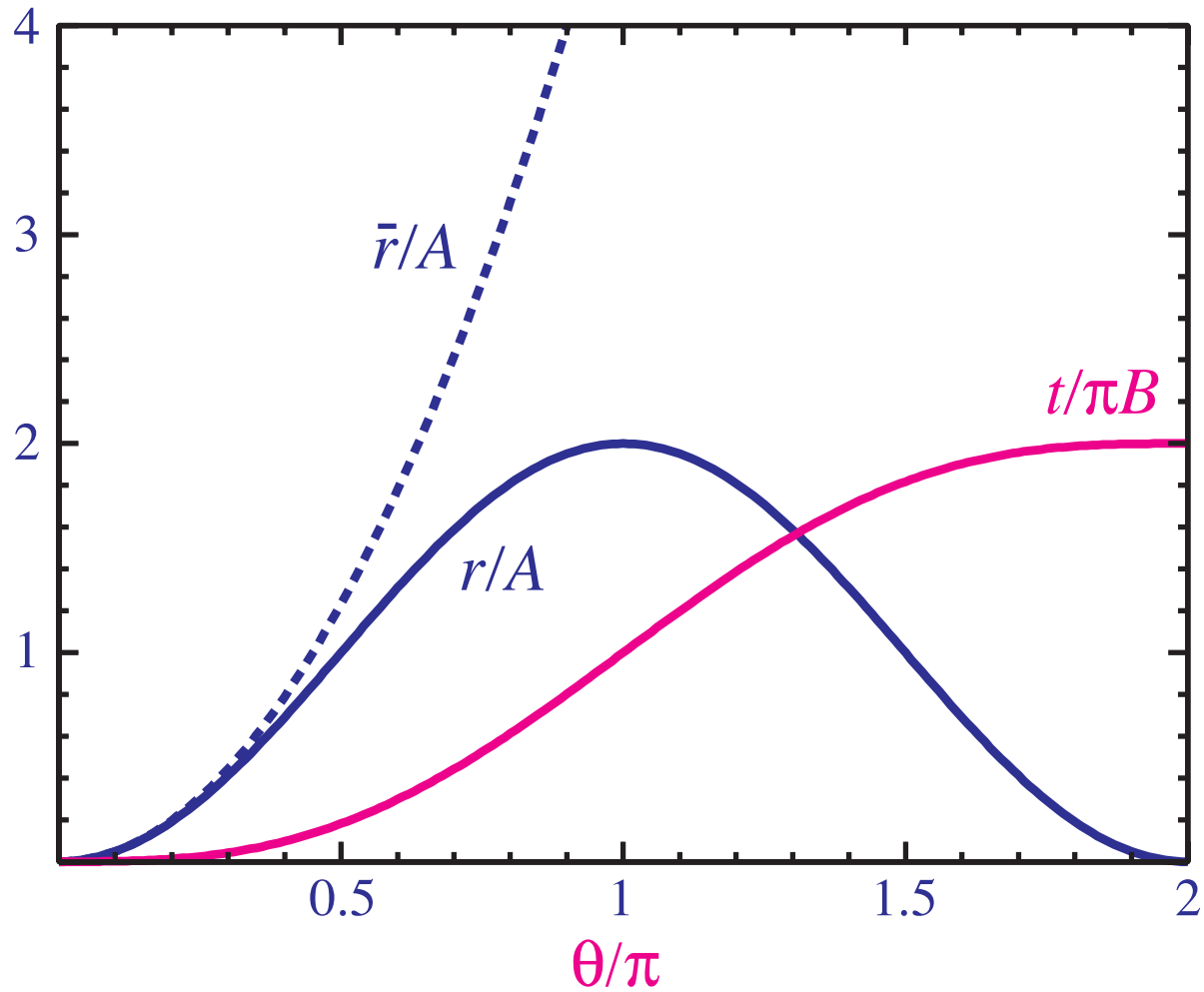
$$t(\theta) = B(\theta - \sin \theta)$$

where  $A = r_0 \Omega_m / 2(\Omega_m - 1)$ ,  $B = H_0^{-1} \Omega_m / 2(\Omega_m - 1)^{3/2}$ .

- Turn around at  $\theta = \pi$ ,  $r = 2A$ ,  $t = B\pi$ .
- Collapse at  $\theta = 2\pi$ ,  $r \rightarrow 0$ ,  $t = 2\pi B$

# Spherical Collapse

- Parametric Solution:



# Correspondence

- Eliminate cosmological correspondence in  $A$  and  $B$  in terms of enclosed mass  $M$

$$M = \frac{4\pi}{3} r_0^3 \Omega_m \rho_c = \frac{4\pi}{3} r_0^3 \Omega_m \frac{3H_0^2}{8\pi G}$$

- Related as  $A^3 = GM B^2$ , and to initial perturbation

$$\lim_{\theta \rightarrow 0} r(\theta) = A \left( \frac{1}{2} \theta^2 - \frac{1}{24} \theta^4 \right)$$

$$\lim_{\theta \rightarrow 0} t(\theta) = B \left( \frac{1}{6} \theta^3 - \frac{1}{120} \theta^5 \right)$$

- Leading Order:  $r = A\theta^2/2$ ,  $t = B\theta^3/6$

$$r = \frac{A}{2} \left( \frac{6t}{B} \right)^{2/3}$$

# Next Order

- Leading order is unperturbed matter dominated expansion

$$r \propto a \propto t^{2/3}$$

- Iterate  $r$  and  $t$  solutions

$$\lim_{\theta \rightarrow 0} t(\theta) = \frac{\theta^3}{6} B \left[ 1 - \frac{1}{20} \left( \frac{6t}{B} \right)^{2/3} \right]$$

$$\theta \approx \left( \frac{6t}{B} \right)^{1/3} \left[ 1 + \frac{1}{60} \left( \frac{6t}{B} \right)^{2/3} \right]$$

# Next Order

- Substitute back into  $r(\theta)$

$$\begin{aligned}r(\theta) &= A \frac{\theta^2}{2} \left( 1 - \frac{\theta^2}{12} \right) \\&= \frac{A}{2} \left( \frac{6t}{B} \right)^{2/3} \left[ 1 - \frac{1}{20} \left( \frac{6t}{B} \right)^{2/3} \right] \\&= \frac{1}{2} (6t)^{2/3} (GM)^{1/3} \left[ 1 - \frac{1}{20} \left( \frac{6t}{B} \right)^{2/3} \right]\end{aligned}$$

# Density Correspondence

- Density

$$\begin{aligned}\rho_m &= \frac{M}{\frac{4}{3}\pi r^3} \\ &= \frac{1}{6\pi t^2 G} \left[ 1 + \frac{3}{20} \left( \frac{6t}{B} \right)^{2/3} \right]\end{aligned}$$

- Density perturbation

$$\delta \equiv \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m} \approx \frac{3}{20} \left( \frac{6t}{B} \right)^{2/3}$$

# Density Correspondence

- Time  $\rightarrow$  scale factor

$$t = \frac{2}{3H_0\Omega_m^{1/2}} a^{3/2}$$

$$\delta = \frac{3}{20} a \left( 4/B H_0 \Omega_m^{1/2} \right)^{2/3}$$

- $A$  and  $B$  constants  $\rightarrow$  initial cond.

$$B = \frac{1}{2H_0\Omega_m^{1/2}} \left( \frac{3 a_i}{5 \delta_i} \right)^{3/2}$$

$$A = \frac{3 r_i}{10 \delta_i}$$



# Spherical Collapse Relations

- Scale factor  $a \propto t^{2/3}$

$$a = \left(\frac{3}{4}\right)^{2/3} \left(\frac{3 a_i}{5 \delta_i}\right) (\theta - \sin \theta)^{2/3}$$

- At collapse  $\theta = 2\pi$

$$a_{\text{col}} = \left(\frac{3}{4}\right)^{2/3} \left(\frac{3 a_i}{5 \delta_i}\right) (2\pi)^{2/3} \approx 1.686 \frac{a_i}{\delta_i}$$

- Perturbation collapses when **linear theory** predicts  $\delta_c \equiv 1.686$

# Virialization

- A real density perturbation is neither spherical nor homogeneous
- **Shell crossing** if  $\delta_i$  doesn't monotonically decrease
- Collapse does not proceed to a point but reaches **virial equilibrium**

$$U = -2K, \quad E = U + K = U(r_{\max}) = \frac{1}{2}U(r_{\text{vir}}) \quad (1)$$

$$r_{\text{vir}} = \frac{1}{2}r_{\max} \text{ since } U \propto r^{-1}. \text{ Thus } \theta_{\text{vir}} = \frac{3}{2}\pi$$

- **Overdensity** at virialization

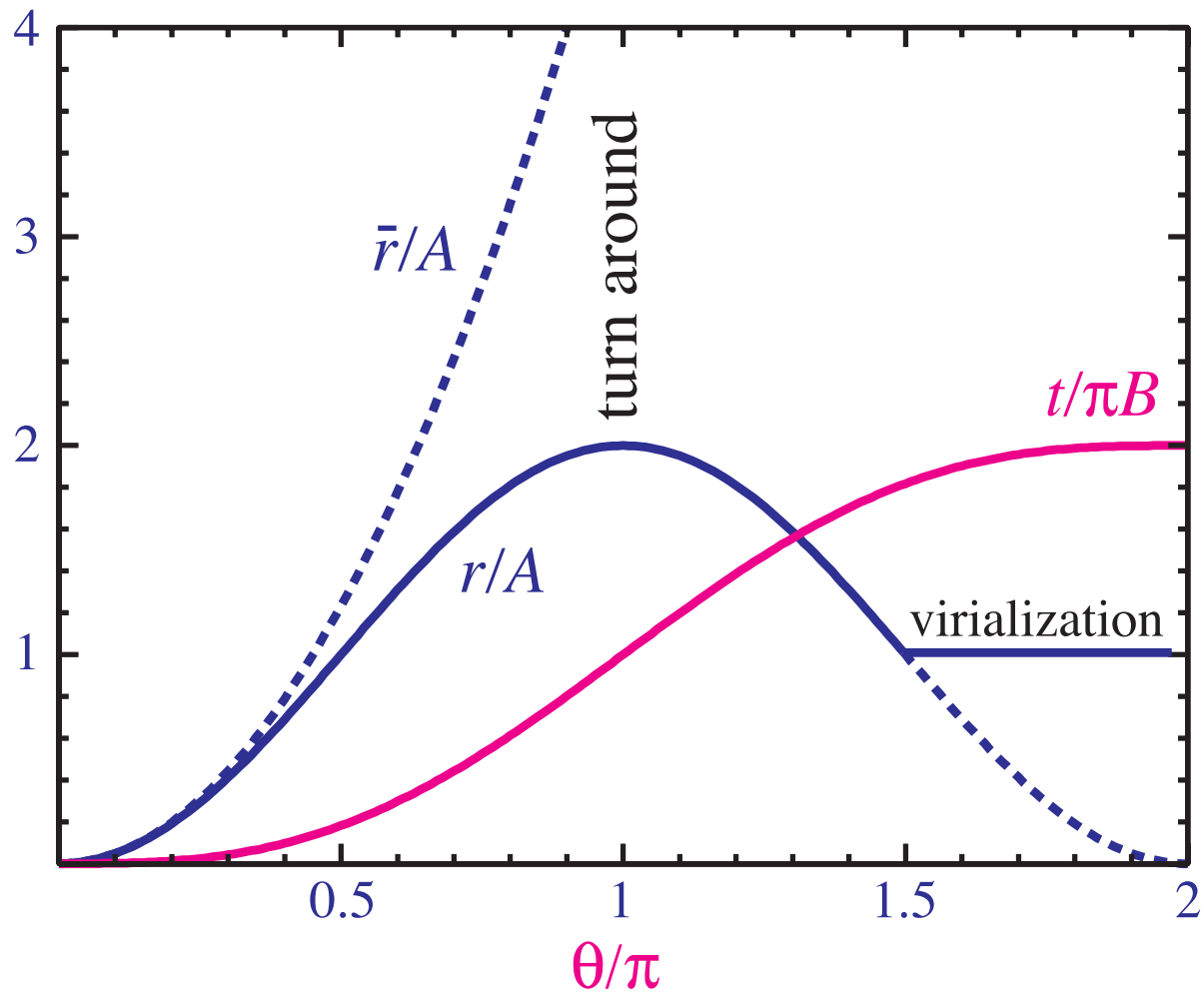
$$\frac{\rho_m(\theta = 3\pi/2)}{\bar{\rho}_m(\theta = 2\pi)} = 18\pi^2 \approx 178$$

- Threshold  $\Delta_v = 178$  often used to define a **collapsed object**
- Equivalently relation between virial mass, radius, overdensity:

$$M_v = \frac{4\pi}{3}r_v^3\rho_m\Delta_v$$

# Virialization

- Schematic Picture:



# Generalization Beyond Matter

- In a universe with smooth components like dark energy driving the expansion but not participating in collapse we cannot consider spherical collapse to be a separate universe
- Go back to the continuity and Euler equation to derive the general equation

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot (1 + \delta) \mathbf{v} = 0$$
$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a} (\mathbf{v} \cdot \nabla) \mathbf{v} + H \mathbf{v} = -\frac{1}{a} \nabla \Psi$$

which is true for any type of dark energy or even metric modified gravity

# Generalization Beyond Matter

- For a tophat density perturbation  $\mathbf{v} = A(t)\mathbf{r}$  interior given the continuity equation and so

$$\frac{d^2\delta}{dt^2} - \frac{4}{3} \frac{1}{1+\delta} \left(\frac{d\delta}{dt}\right)^2 + 2H \frac{d\delta}{dt} = \frac{(1+\delta)}{a^2} \nabla^2 \Psi$$

- Under ordinary gravity  $\nabla^2 \Psi = 4\pi G a^2 \bar{\rho}_m \delta$  and so a tophat remains a tophat
- Thus use conservation of the dark matter mass

$$M = (4\pi/3)r^3 \bar{\rho}_m (1 + \delta)$$

to trade the density for the tophat radius  $\delta \rightarrow R$

# Generalization Beyond Matter

- Using the Friedmann equations for the evolution of the background

$$H^2 = \frac{8\pi G}{3}(\bar{\rho}_m + \bar{\rho}_{\text{eff}})$$

we obtain using the Poisson equation

$$\begin{aligned}\frac{1}{r} \frac{d^2 r}{dt^2} &= H^2 + \dot{H} - \frac{1}{3} \nabla^2 \Psi \\ &= -\frac{4\pi G}{3} [\rho_m + (1 + 3w_{\text{eff}})\bar{\rho}_{\text{eff}}]\end{aligned}$$

where  $\rho_m = \bar{\rho}_m(1 + \delta)$  includes the tophat fluctuation whereas  $\bar{\rho}_{\text{eff}}$  is a smooth background contribution to the Friedmann equation

- In other words  $H^2 + \dot{H}$  carries the acceleration effect of background total density but  $\Psi$  carries only that of the collapsing component - alters the collapse relations

# Generalization Beyond Matter

- Similarly virial equilibrium altered to include smooth contribution to acceleration or effective potential

$$U = -2K$$

where

$$U = -\frac{3}{5} \frac{GM^2}{R} - \frac{4\pi G}{5} (1 + 3w_{\text{eff}}) \bar{\rho}_{\text{eff}} MR^2$$

- Note that virial equilibrium is defined in terms of the trace of the potential tensor and is a statement of force balance

$$U \equiv - \int d^3x \rho_m \mathbf{x} \cdot \nabla \Psi_{\text{tot}}$$

# Generalization Beyond Matter

- Hence  $U$  is well defined even in cases where energy is not conserved in the usual manner (though still covariantly conserved), e.g. if  $\rho_{\text{eff}}$  is not constant during collapse
- In general keep track of the kinetic energy during collapse and finding the virial radius as the point at which

$$U(r_{\text{vir}}) = -2K(r_{\text{vir}})$$

- Rather than using energy conservation (important if  $w_{\text{eff}} \neq -1$ )



# The Mass Function

- Spherical collapse predicts the end state as virialized halos given an initial density perturbation
- Initial density perturbation is a Gaussian random field
- Compare the variance in the linear density field to threshold  $\delta_c = 1.686$  to determine collapse fraction
- Combine to form the mass function, the number density of halos in a range  $dM$  around  $M$ .
- Halo density defined entirely by linear theory
- Fudge the result to get the right answer compared with simulations (a la Press-Schechter)!

# Press-Schechter Formalism

- Smooth linear density field on mass scale  $M$  with tophat

$$R = \left( \frac{3M}{4\pi} \right)^{1/3}$$

- Result is a Gaussian random field with variance  $\sigma^2(M)$
- Fluctuations above the threshold  $\delta_c$  correspond to collapsed regions. The fraction in halos  $> M$  becomes

$$\frac{1}{\sqrt{2\pi}\sigma(M)} \int_{\delta_c}^{\infty} d\delta \exp\left(-\frac{\delta^2}{2\sigma^2(M)}\right) = \frac{1}{2} \operatorname{erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$

where  $\nu \equiv \delta_c/\sigma(M)$

- **Problem:** even as  $\sigma(M) \rightarrow \infty$ ,  $\nu \rightarrow 0$ , collapse fraction  $\rightarrow 1/2$  – only overdense regions participate in spherical collapse.
- Multiply by an ad hoc factor of 2!

# Press-Schechter Mass Function

- Differentiate in  $M$  to find fraction in range  $dM$  and multiply by  $\rho_m/M$  the number density of halos if all of the mass were composed of such halos  $\rightarrow$  differential number density of halos

$$\begin{aligned}\frac{dn}{d \ln M} &= \frac{\rho_m}{M} \frac{d}{d \ln M} \operatorname{erfc} \left( \frac{\nu}{\sqrt{2}} \right) \\ &= \sqrt{\frac{2}{\pi}} \frac{\rho_m}{M} \frac{d \ln \sigma^{-1}}{d \ln M} \nu \exp(-\nu^2/2)\end{aligned}$$

- High mass: exponential cut off above  $M_*$  where  $\sigma(M_*) = \delta_c$

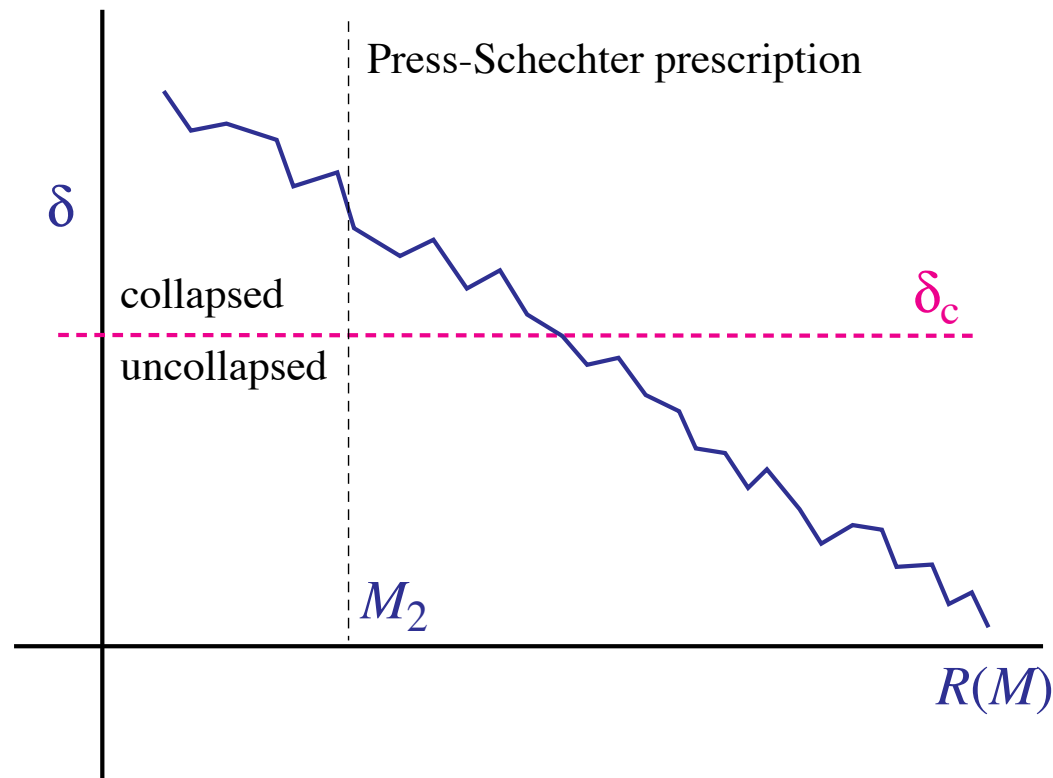
$$M_* \sim 10^{13} h^{-1} M_\odot \quad \text{today}$$

- Low mass divergence: (too many for the observations?)

$$\frac{dn}{d \ln M} \propto M^{-1}$$

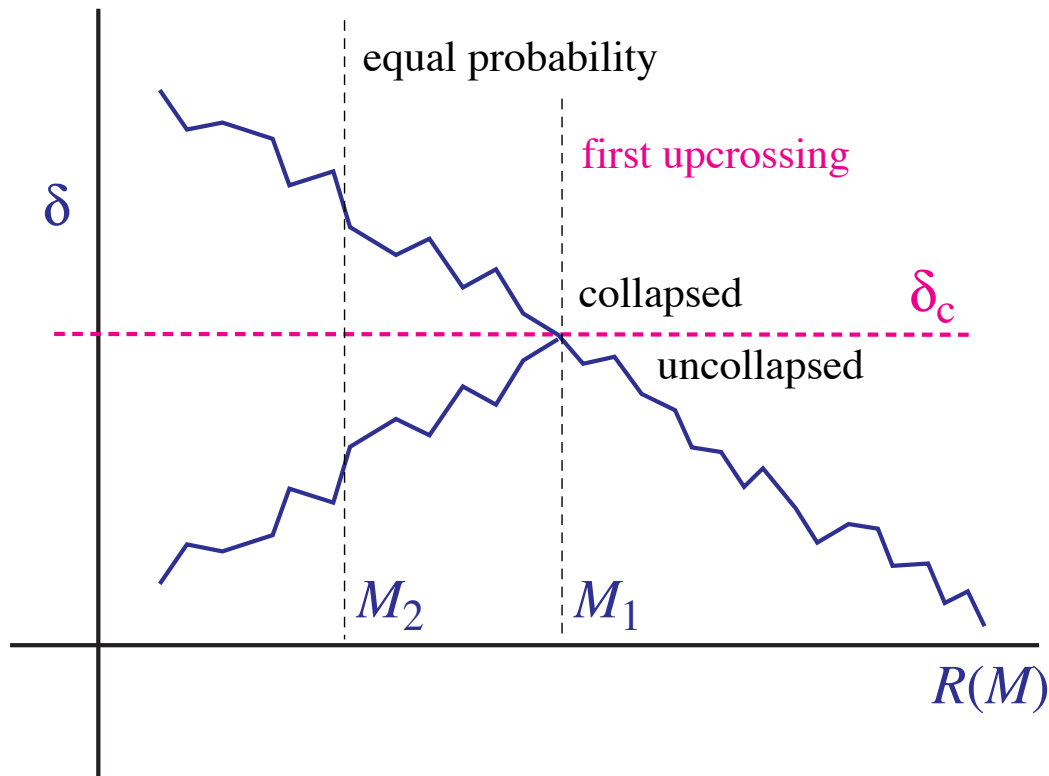
# Extended Press-Schechter Formalism

- A region that is **underdense** when smoothed on the scale  $M$  may be **overdense** on a scale of a larger  $M$
- If smoothing is a tophat in  $k$ -space, independence of  $k$ -modes implies fluctuation executes a **random walk**



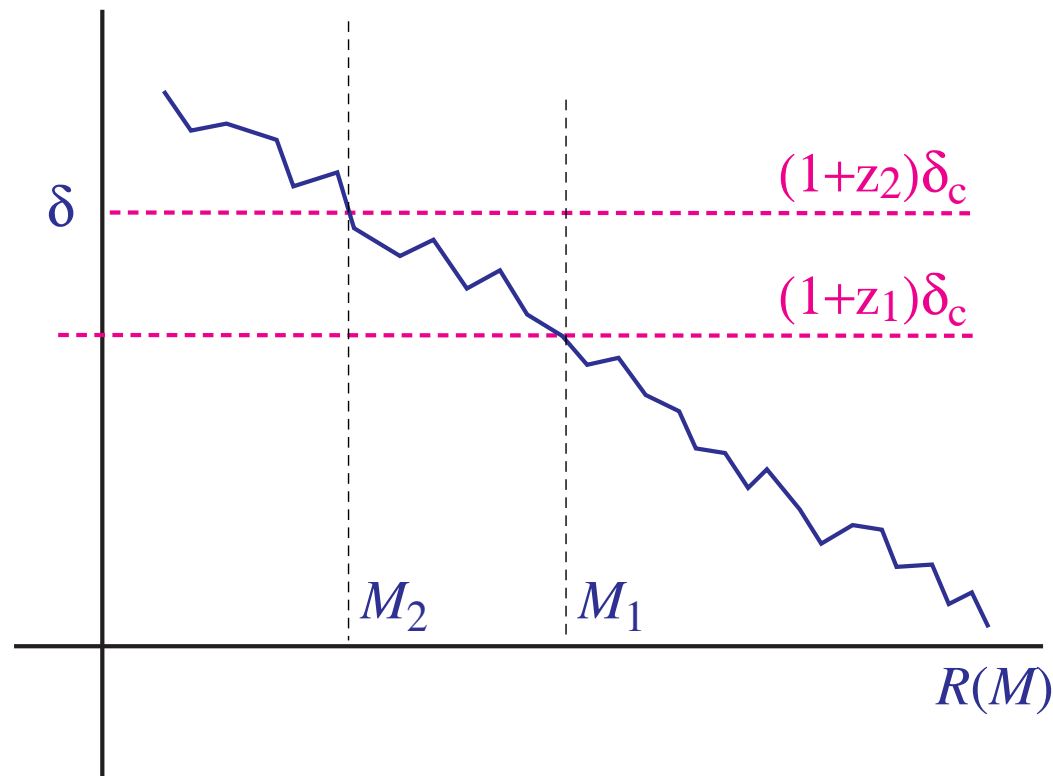
# Extended Press-Schechter Formalism

- For each trajectory that lies above threshold at  $M_2$ , there is an equivalent trajectory that is its mirror image reflected around  $\delta_c$
- Press-Schechter ignored this branch. It supplies the missing factor of 2



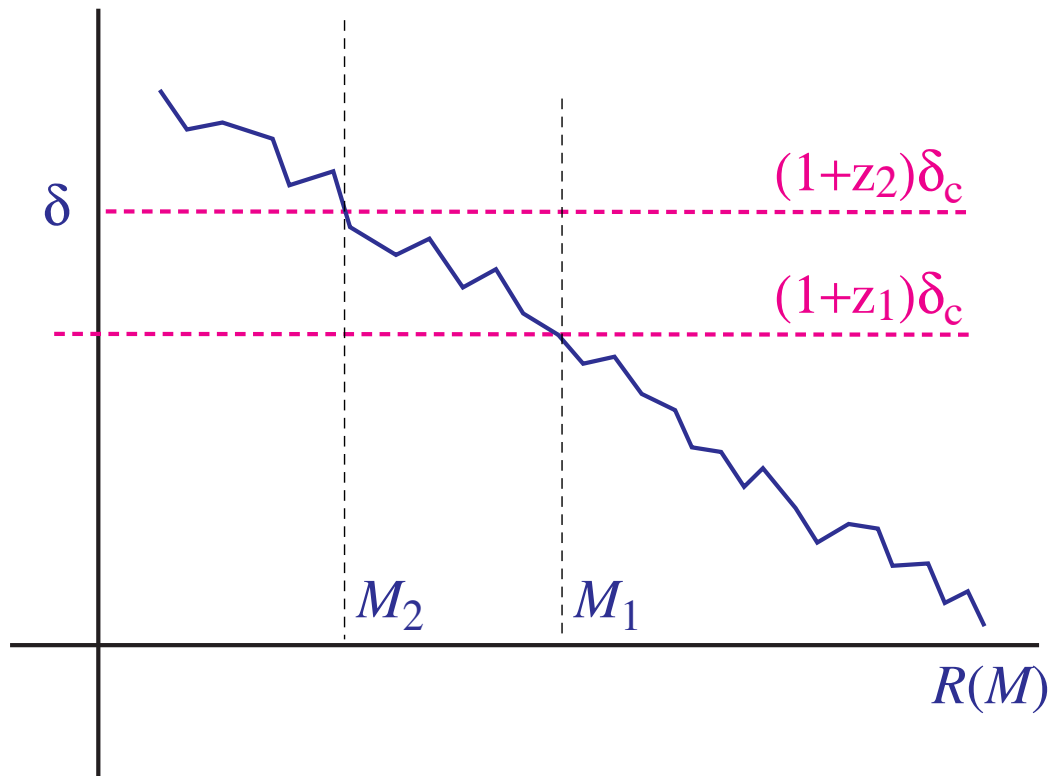
# Conditional Mass Function

- Extended Press-Schechter also gives the conditional mass function, useful for merger histories.
- Given a halo of mass  $M_1$  exists at  $z_1$ , what is the probability that it was part of a halo of mass  $M_2$  at  $z_2$



# Conditional Mass Function

- Same as before but with the origin translated.
- Conditional mass function is mass function with  $\delta_c$  and  $\sigma^2(M)$  shifted



# Magic “2” resolved?

- Spherical collapse is defined for a **real-space** not  $k$ -space smoothing. Random walk is only a **qualitative explanation**.
- Modern approach: think of spherical collapse as motivating a **fitting form** for the mass function

$$\nu \exp(-\nu^2/2) \rightarrow A[1 + (a\nu^2)^{-p}] \sqrt{a\nu^2} \exp(-a\nu^2/2)$$

**Sheth-Torman 1999**,  $a = 0.75$ ,  $p = 0.3$ . or a completely empirical fitting

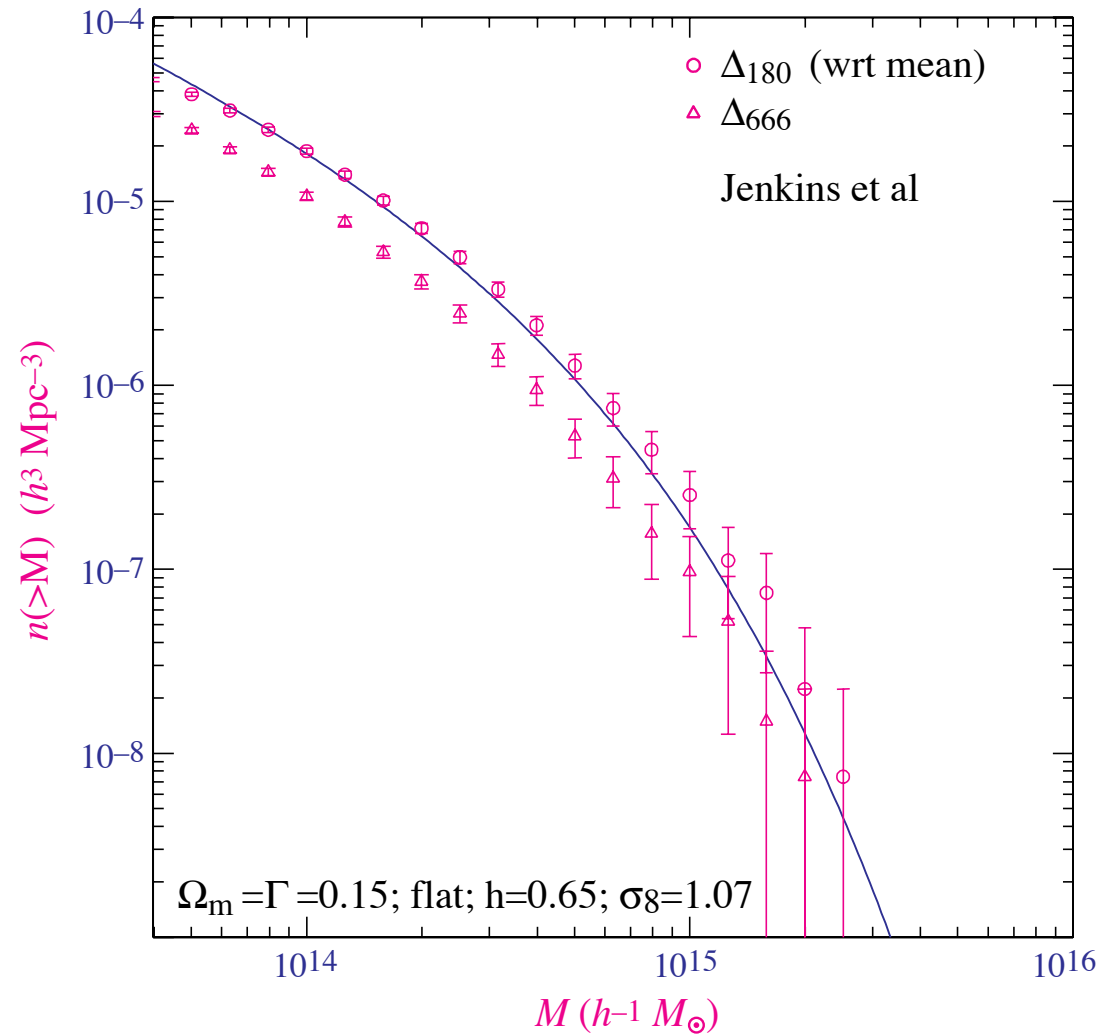
$$\frac{dn}{d \ln M} = 0.301 \frac{\rho_m}{M} \frac{d \ln \sigma^{-1}}{d \ln M} \exp[-|\ln \sigma^{-1} + 0.64|^{3.82}]$$

**Jenkins et al 2001**. Choice is tied up with the question: **what is the mass of a halo?**



# Numerical Mass Function

- Example of difference in mass definition (from Hu & Kravstov 2002)



# Halo Bias

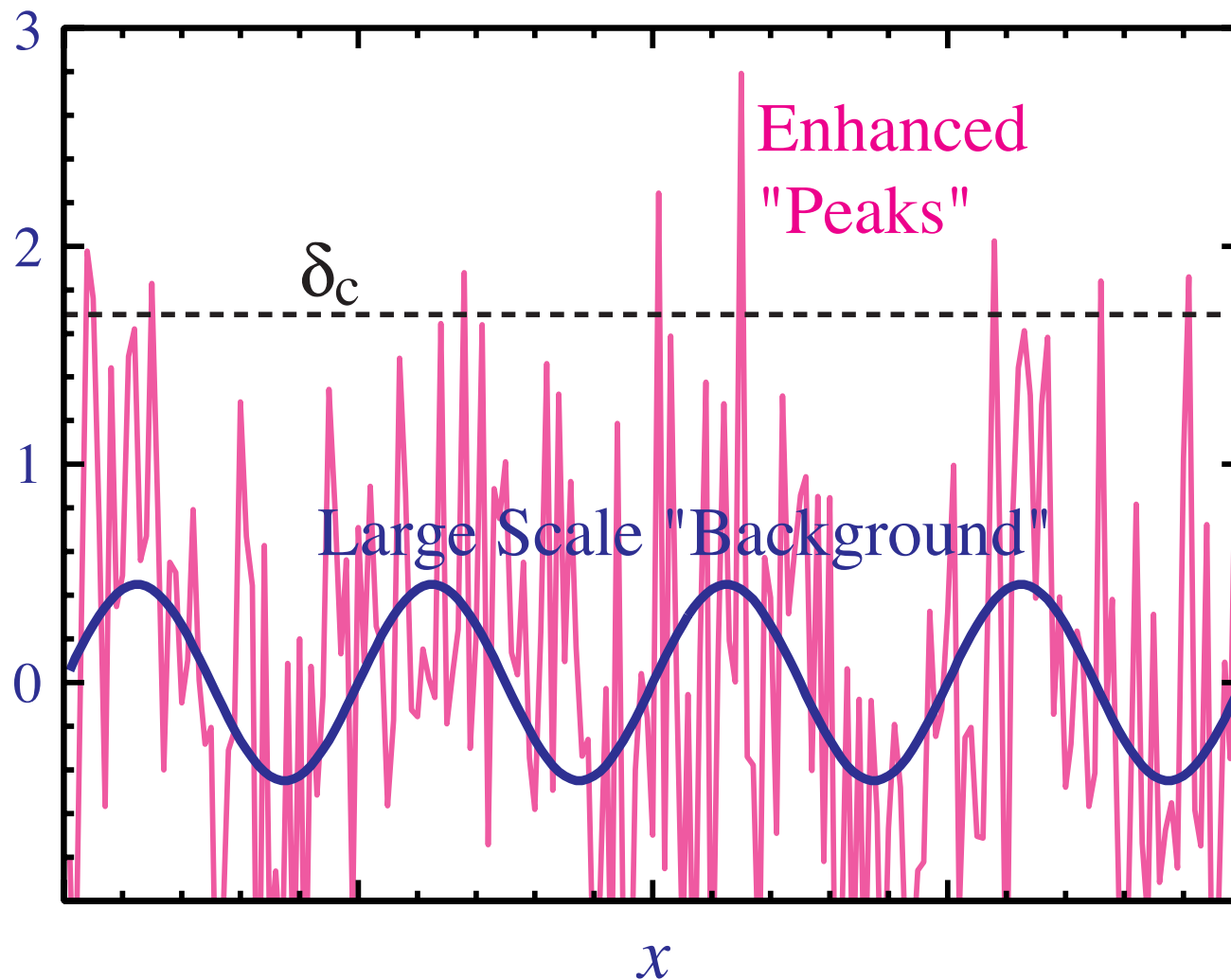
- If halos are formed without regard to the underlying density fluctuation and move under the **gravitational field** then their number density is an **unbiased tracer** of the dark matter density fluctuation

$$\left(\frac{\delta n}{n}\right)_{\text{halo}} = \left(\frac{\delta \rho}{\rho}\right)$$

- However **spherical collapse** says the probability of forming a halo depends on the **initial density field**
- **Large scale density** field acts as “background” enhancement of probability of forming a halo or “peak”
- **Peak-Background Split** (Efstathiou 1998; Cole & Kaiser 1989; Mo & White 1997)

# Peak-Background Split

- Schematic Picture:



# Perturbed Mass Function

- Density fluctuation split

$$\delta = \delta_b + \delta_p$$

- Lowers the threshold for collapse

$$\delta_{cp} = \delta_c - \delta_b$$

so that  $\nu = \delta_{cp}/\sigma$

- Taylor expand number density  $n_M \equiv dn/d \ln M$

$$n_M + \frac{dn_M}{d\nu} \frac{d\nu}{d\delta_b} \delta_b \dots = n_M \left[ 1 + \frac{(\nu^2 - 1)}{\sigma\nu} \delta_b \right]$$

if mass function is given by **Press-Schechter**

$$n_M \propto \nu \exp(-\nu^2/2)$$

# Halo Bias

- Halos are **biased tracers** of the “background” dark matter field with a bias  $b(M)$  that is given by spherical collapse and the form of the mass function
- Combine the enhancement with the original unbiased expectation

$$\frac{\delta n_M}{n_M} = b(M)\delta_b$$

- For Press-Schechter

$$b(M) = 1 + \frac{\nu^2 - 1}{\delta_c}$$

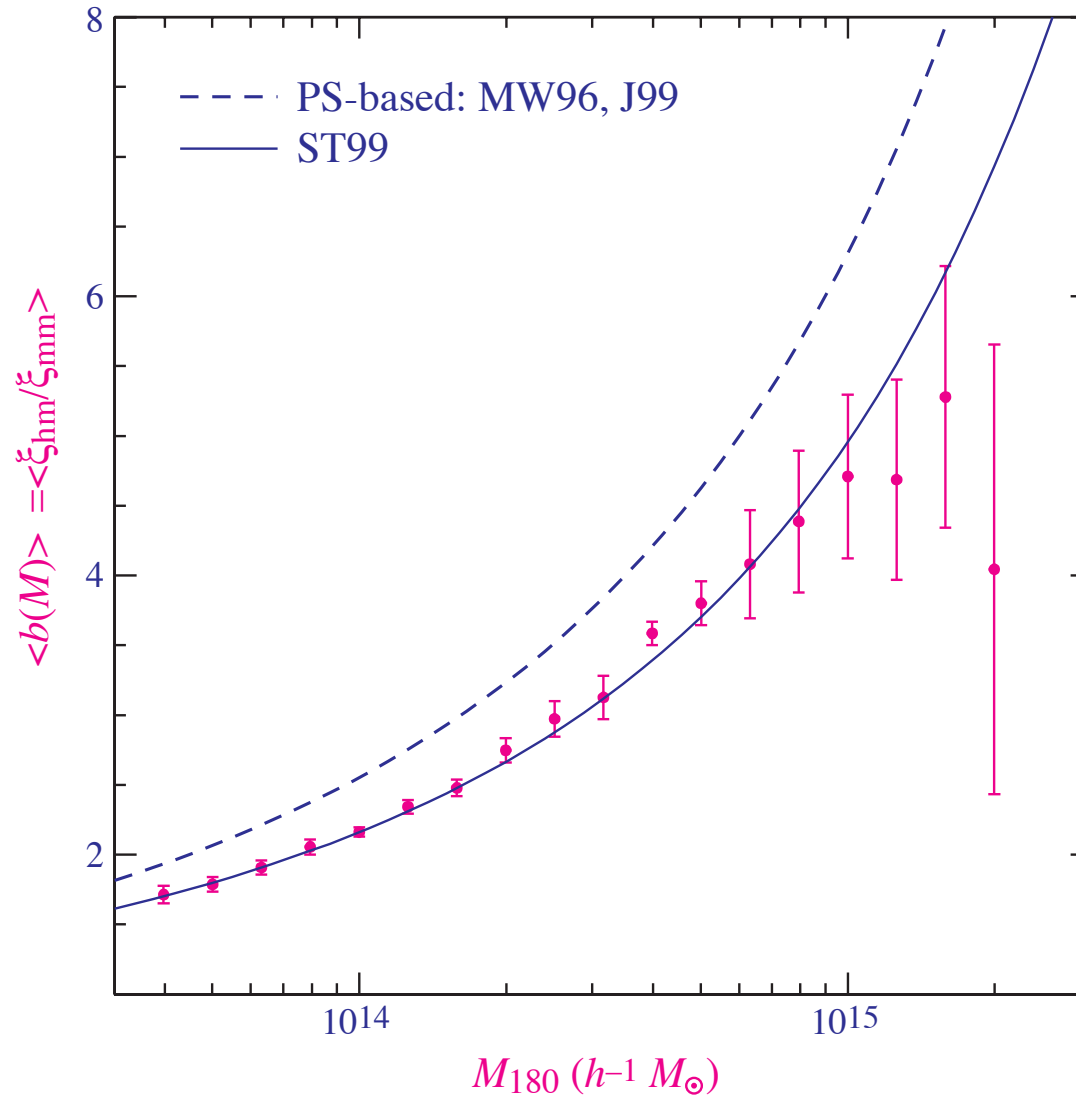
- Improved by the Sheth-Tormen mass function

$$b(M) = 1 + \frac{a\nu^2 - 1}{\delta_c} + \frac{2p}{\delta_c[1 + (a\nu^2)^p]}$$

with  $a = 0.75$  and  $p = 0.3$  to match simulations.

# Numerical Bias

- Example of halo bias from a simulation (from [Hu & Kravstov 2002](#))



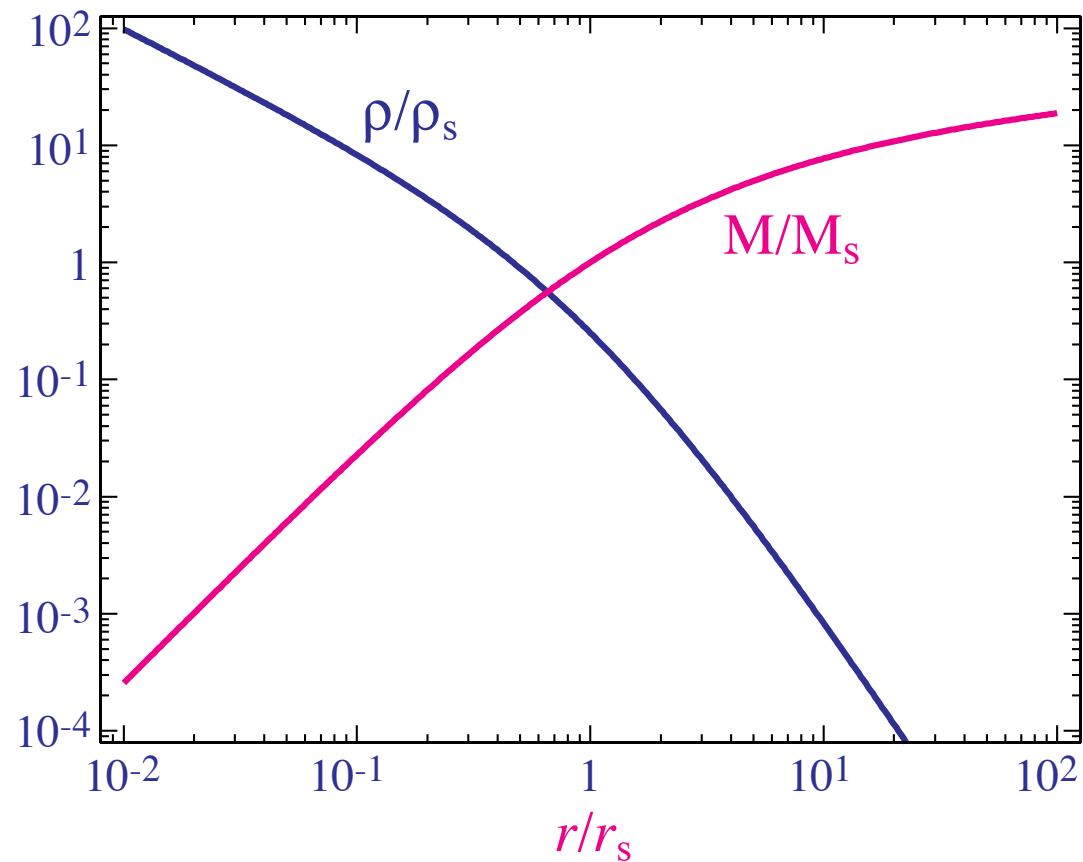
# What is a Halo?

- Mass function and halo bias depend on the definition of **mass of a halo**
- Agreement with simulations depend on how **halos are identified**
- Other **observables** (associated galaxies, *X*-ray, SZ) depend on the details of the density profile
- Fortunately, simulations have shown that halos take on a near **universal form** in their **density profile** at least on large scales.

# NFW Profile

- Density profile well-described by (Navarro, Frenk & White 1997)

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}$$





# Einasto Profile

- Current best simulations find that the inner slope runs rather than asymptotes to a cuspy constant
- This form is better fit by the Einasto profile (c.f. Sersic profile)

$$\ln \frac{\rho(r)}{\rho_s} = -\frac{2}{\alpha} \left[ \left( \frac{r}{r_s} \right)^\alpha - 1 \right]$$

- The local slope is given by

$$\frac{d \ln \rho}{d \ln r} = -2 \left( \frac{r}{r_s} \right)^\alpha$$

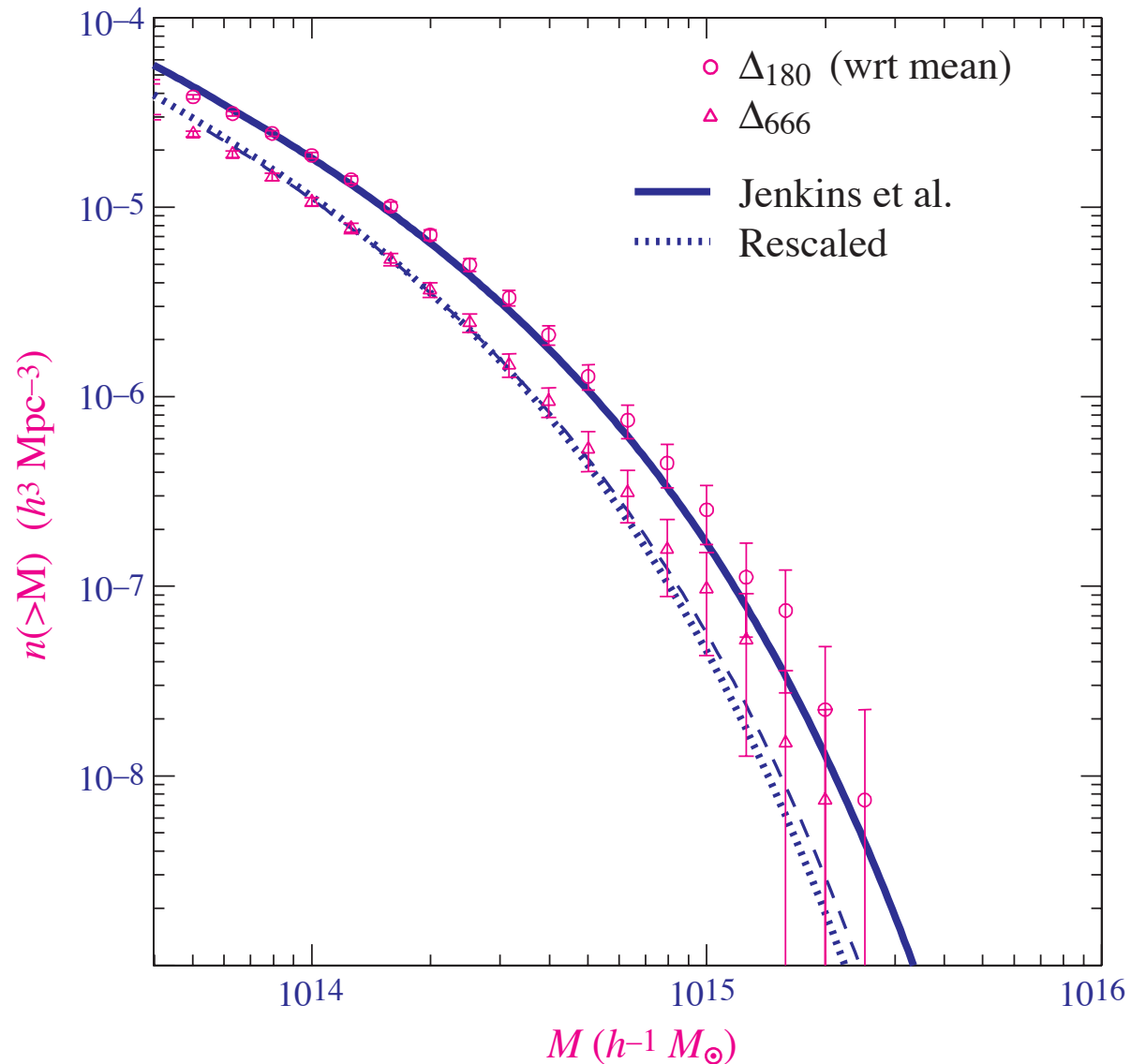
and continues to decrease as  $r/r_s \rightarrow 0$

# Whence Universal Profile?

- Recent investigations by Dalal, Lithwick, Kuhlen (2010) suggests that the universal halo profile arises generically from peaks in a Gaussian random field
- Outer  $r^{-3}$  profile predicted from slow accretion of material at low initial overdensity compared with peak
- Inner profile comes from adiabatic contraction (i.e. preserving adiabatic invariants during collapse) and depends on the initial density profile of peak
- Dynamical friction implies that the centroid of the initial density peak will settle to the center of the final halo

# Transforming the Masses

- NFW profile gives a way of transforming different mass definitions



# Lack of Concentration?

- NFW parameters may be recast into  $M_v$ , the mass of a halo out to the virial radius  $r_v$  where the overdensity wrt mean reaches  $\Delta_v = 180$ .

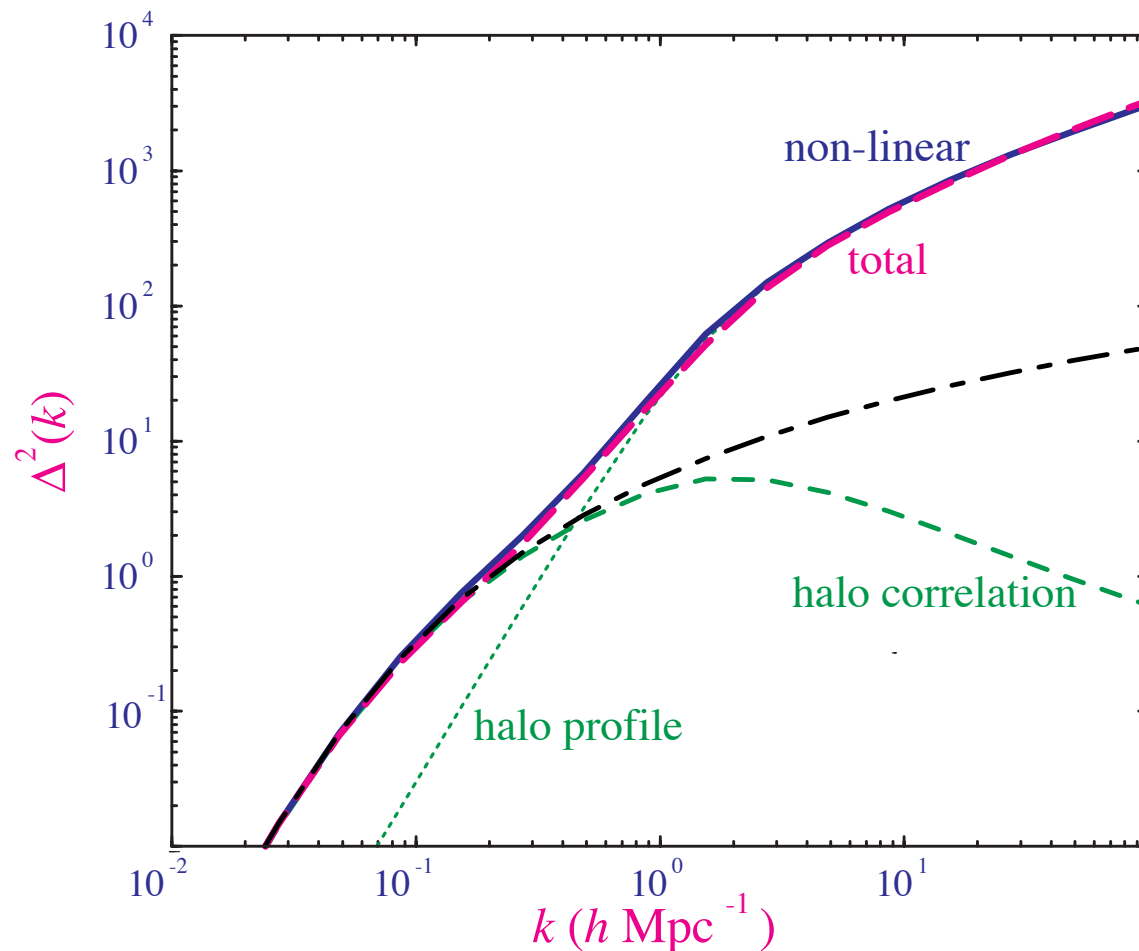
- Concentration parameter

$$c \equiv \frac{r_v}{r_s}$$

- CDM predicts  $c \sim 10$  for  $M_*$  halos. Too centrally concentrated for galactic rotation curves?
- Possible discrepancy has lead to the exploration of dark matter alternatives: warm ( $m \sim \text{keV}$ ) dark matter, self-interacting dark-matter, annihilating dark matter, ultra-light “fuzzy” dark matter, . . .

# The Halo Model

- NFW halos, of abundance  $n_M$  given by mass function, clustered according to the halo bias  $b(M)$  and the linear theory  $P(k)$
- Power spectrum example:



# Non-Linear Power Spectrum

- Non-linear power spectrum is composed of dark matter halos that are clustered according to the halo bias + the clustering due to the halo density profile

$$P_{\text{nl}}(k, z) = I_2^2(k, z)P(k, z) + I_1(k, z)$$

where

$$I_2(k, z) = \int d \ln M \left( \frac{M}{\rho_m(z=0)} \right) \frac{dn}{d \ln M} b(M) y(k, M)$$

$$I_1(k, z) = \int d \ln M \left( \frac{M}{\rho_m(z=0)} \right)^2 \frac{dn}{d \ln M} y^2(k, M)$$

and  $y$  is the Fourier transform of the halo profile with  $y(0, M) = 1$

$$y(k, M) = \frac{1}{M} \int_0^{r_h} dr 4\pi r^2 \rho(r, M) \frac{\sin(kr)}{kr}$$

# Galaxy Power Spectrum

- For galaxies, one defines a halo occupation distribution which determines the number of galaxies (satisfying a certain observational criteria) that can occupy a halo of mass  $M$
- Take a simple example of a mass selection on the galaxies, then  
 $N(M) = 0$  for  $M < M_{\text{th}}$  and above threshold  
 $N(M) = C + S(M)$  where  $C = 1$  accounts for the central galaxy and satellite galaxies follow a poisson distribution with mean  
 $S(M) \approx M/30M_{\text{th}}$

# Galaxy Power Spectrum

- Then assuming that satellites are distributed according to the mass profile

$$P_{\text{gal}}(k, z) = I_2^2(k, z)P(k, z) + I_1(k, z)$$

where

$$I_2(k, z) = \frac{1}{n_{\text{gal}}} \int d \ln M \frac{dn}{d \ln M} b(M) [C + y(k, M)S(M)]$$

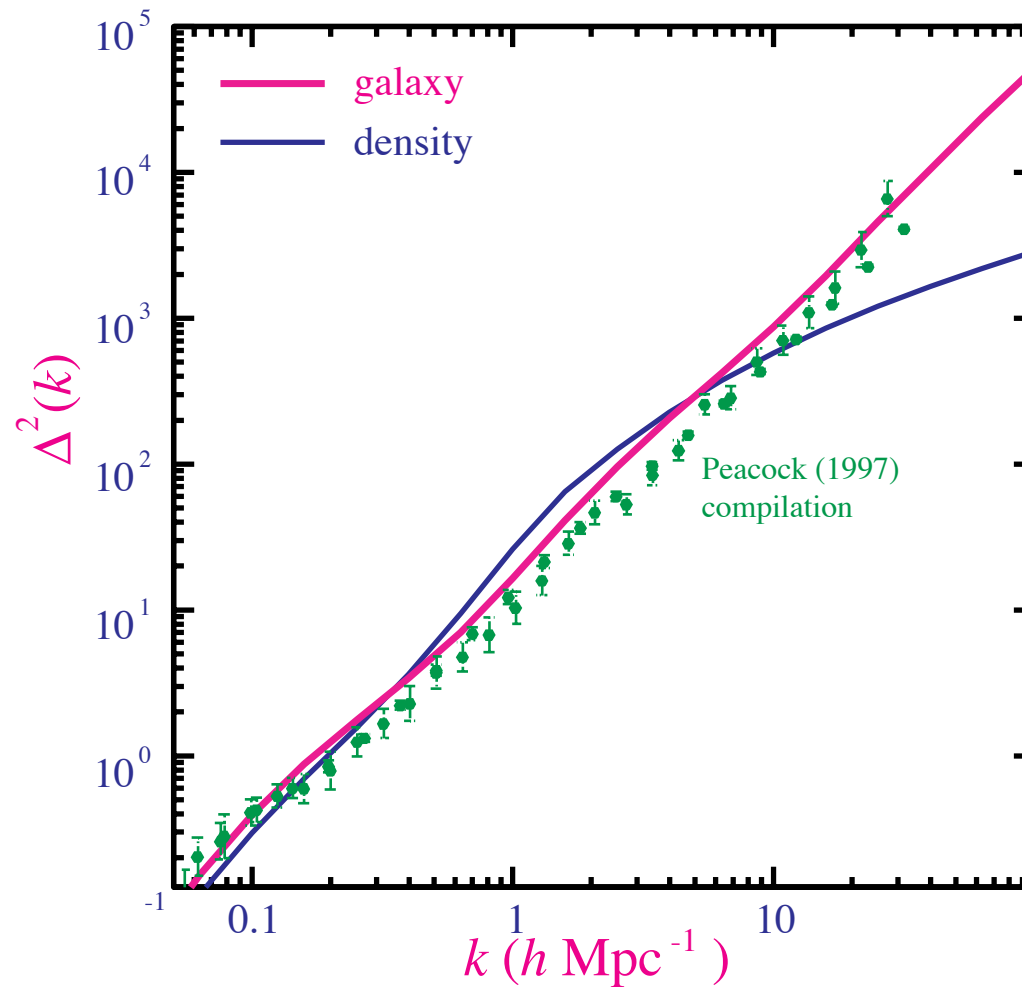
$$I_1(k, z) = \frac{1}{n_{\text{gal}}^2} \int d \ln M \frac{dn}{d \ln M} [S^2(M)y^2(k, M) + 2CS(M)y(k, M)]$$

- Break between the one and two halo regime first seen by SDSS



# Galaxy Power Spectrum

- Example (Seljak 2001)



- An explanation of the nearly power law galaxy spectrum

# Incredible, Extensible Halo Model

- An industry developed to build semi-analytic models for wide variety of cosmological observables based on the halo model
- Idea: associate an observable (galaxies, gas, ...) with dark matter halos
- Let the halo model describe the statistics of the observable
- The overextended halo model?

# Halo Temperature

- Motivate with **isothermal distribution**, correct from simulations

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$

- Express in terms of **virial mass**  $M_v$  enclosed at **virial radius**  $r_v$

$$M_v = \frac{4\pi}{3} r_v^3 \rho_m \Delta_v = \frac{2}{G} r_v \sigma^2$$

- Eliminate  $r_v$ , temperature  $T \propto \sigma^2$  velocity dispersion<sup>2</sup>
- Then  $T \propto M_v^{2/3} (\rho_m \Delta_v)^{1/3}$  or

$$\left( \frac{M_v}{10^{15} h^{-1} M_\odot} \right) = \left[ \frac{f}{(1+z)(\Omega_m \Delta_v)^{1/3}} \frac{T}{1\text{keV}} \right]^{3/2}$$

- Theory ( $X$ -ray weighted):  $f \sim 0.75$ ; observations  $f \sim 0.54$ .  
Difference is **crucial** in determining cosmology from **cluster counts!**

# Summary

- **Dark matter simulations** well-understood and can be modelled with dark matter **halos**
- Halo formation modelled by **spherical collapse**, two magic numbers  $\delta_c = 1.686$  and  $\Delta_v = 178$
- Halo abundance described by a **mass function** with **exponential** high mass cutoff – **rare clusters** extremely sensitive to power spectrum amplitude and **growth rate** → **dark energy**  
Possibly too many small halos or **sub-structure**?
- Halo clustering modelled with peak-background split leading to **halo bias**
- **Halo profile** described by NFW halos  
Possibly too high central **concentration**
- Associate an **observable** with a halo → a **halo model**