Astro 449

Linear Growth

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Metric Tensor

- Useful to think in a $3+1$ language since there are preferred spatial surfaces where the stress tensor is nearly homogeneous
- In general this is an Arnowitt-Deser-Misner (ADM) split
- Specialize to the case of a nearly FRW metric

$$g_{00} = -a^2, \qquad g_{ij} = a^2\gamma_{ij}\,.$$

where the "0" component is conformal time $\eta = dt/a$ and γ_{ij} is a spatial metric of constant curvature $K = H_0^2(\Omega_{\rm tot} - 1)$.

$$^{(3)}R = \frac{6K}{a^2}$$

Metric Tensor

- First define the slicing (lapse function A, shift function B^i)

$$
\begin{aligned}
g^{00} &= -a^{-2}(1-2A)\,, \\
g^{0i} &= -a^{-2}B^i\,,
\end{aligned}
$$

  A defines the lapse of proper time between 3-surfaces whereas B^i defines the threading or relationship between the 3-coordinates of the surfaces

- This absorbs 1+3=4 free variables in the metric, remaining 6 is in the spatial surfaces which we parameterize as

$$
g^{ij} = a^{-2}(\gamma^{ij} - 2H_L\gamma^{ij} - 2H_T^{ij})\,.
$$

  here (1) H_L a perturbation to the scale factor; (5) H_T^{ij} a trace-free distortion to spatial metric (which combined perturb the curvature)

Curvature Perturbation

- Curvature perturbation on the 3D slice

$$\delta[{}^{(3)}R] = -\frac{4}{a^2}\left(\nabla^2 + 3K\right) H_L + \frac{2}{a^2}\nabla_i\nabla_j H_T^{ij}$$

- Note that we will often loosely refer to H_L as the "curvature perturbation"
- We will see that many representations have $H_T = 0$
- It is easier to work with a dimensionless quantity
- First example of a 3-scalar - SVT decomposition

Matter Tensor

- Likewise expand the matter stress energy tensor around a homogeneous density ρ and pressure p:

$$
\begin{aligned}
T^0{}_0 &= -\rho - \delta\rho\,, \\
T^0{}_i &= (\rho + p)(v_i - B_i)\,, \\
T_0{}^i &= -(\rho + p)v^i\,, \\
T^i{}_j &= (p + \delta p)\delta^i{}_j + p\Pi^i{}_j\,,
\end{aligned}
$$

- (1) $\delta\rho$ a density perturbation; (3) v_i a vector velocity, (1) δp a pressure perturbation; (5) Π_{ij} an anisotropic stress perturbation
- So far this is fully general and applies to any type of matter or coordinate choice including non-linearities in the matter, e.g. scalar fields, cosmological defects, exotic dark energy.

Counting Variables

20	Variables (10 metric; 10 matter)
-10	Einstein equations
-4	Conservation equations
$+4$	Bianchi identities
-4	Gauge (coordinate choice 1 time, 3 space)
6	Free Variables

- Without loss of generality these can be taken to be the 6 components of the matter stress tensor
- For the background, specify $p(a)$ or equivalently $w(a) \equiv p(a)/\rho(a)$ the equation of state parameter.

Homogeneous Einstein Equations

- Einstein (Friedmann) equations:

$$\begin{aligned}\left(\frac{1}{a}\frac{da}{dt}\right)^2 &= -\frac{K}{a^2}+\frac{8\pi G}{3}\rho \quad [= \left(\frac{1}{a}\frac{\dot a}{a}\right)^2] \\ \frac{1}{a}\frac{d^2a}{dt^2} &= -\frac{4\pi G}{3}(\rho+3p) \quad [= \frac{1}{a^2}\frac{d}{d\eta}\frac{\dot a}{a}]\end{aligned}$$

  so that $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}$$

  overdots are conformal time but equally true with coordinate time

Homogeneous Einstein Equations

- Counting exercise:

20	Variables (10 metric; 10 matter)
-17	Homogeneity and Isotropy
-2	Einstein equations
-1	Conservation equations
$+1$	Bianchi identities
1	Free Variables

without loss of generality choose ratio of homogeneous & isotropic component of the stress tensor to the density $w(a) = p(a)/\rho(a)$.

Acceleration Implies Negative Pressure

- Role of stresses in the background cosmology
- Homogeneous Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ imply the two Friedmann equations (flat universe, or associating curvature $\rho_K = -3K/8\pi G a^2$)

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho$$

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p)$$

  so that the total equation of state $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}$$

  so that ρ must scale more slowly than a^{-2}

Scalar, Vector, Tensor

- In linear perturbation theory, perturbations may be separated by their transformation properties under 3D rotation and translation.
- The eigenfunctions of the Laplacian operator form a complete set

$$
\begin{aligned}
\nabla^2 Q^{(0)} &= -k^2 Q^{(0)} && \mathrm{S}\,, \\
\nabla^2 Q_i^{(\pm 1)} &= -k^2 Q_i^{(\pm 1)} && \mathrm{V}\,, \\
\nabla^2 Q_{ij}^{(\pm 2)} &= -k^2 Q_{ij}^{(\pm 2)} && \mathrm{T}\,,
\end{aligned}
$$

- Vector and tensor modes satisfy divergence-free and transverse-traceless conditions

$$\nabla^i Q_i^{(\pm 1)} = 0$$

$$\nabla^i Q_{ij}^{(\pm 2)} = 0$$

$$\gamma^{ij} Q_{ij}^{(\pm 2)} = 0$$

Vector and Tensor Quantities

- A scalar mode carries with it associated vector (curl-free) and tensor (longitudinal) quantities
- A vector mode carries and associated tensor (trace and divergence free) quantities
- A tensor mode has only a tensor (trace and divergence free)
- These are built from the mode basis out of covariant derivatives and the metric

$$\begin{aligned}
Q_i^{(0)} &= -k^{-1}\nabla_i Q^{(0)} \,, \\
Q_{ij}^{(0)} &= (k^{-2}\nabla_i\nabla_j + \frac{1}{3}\gamma_{ij})Q^{(0)} \,, \\
Q_{ij}^{(\pm 1)} &= -\frac{1}{2k}[\nabla_i Q_j^{(\pm 1)} + \nabla_j Q_i^{(\pm 1)}] \,,
\end{aligned}$$

Perturbation k-Modes

- For the kth eigenmode, the scalar components become

$$A(\mathbf{x}) = A(k)\,Q^{(0)}\,, \qquad H_L(\mathbf{x}) = H_L(k)\,Q^{(0)}\,,$$
$$\delta\rho(\mathbf{x}) = \delta\rho(k)\,Q^{(0)}\,, \qquad \delta p(\mathbf{x}) = \delta p(k)\,Q^{(0)}\,,$$

the vectors components become

$$B_i(\mathbf{x}) = \sum_{m=-1}^{1} B^{(m)}(k)\,Q_i^{(m)}\,, \qquad v_i(\mathbf{x}) = \sum_{m=-1}^{1} v^{(m)}(k)\,Q_i^{(m)}\,,$$

and the tensors components

$$H_{Tij}(\mathbf{x}) = \sum_{m=-2}^{2} H_T^{(m)}(k)\,Q_{ij}^{(m)}\,, \quad \Pi_{ij}(\mathbf{x}) = \sum_{m=-2}^{2} \Pi^{(m)}(k)\,Q_{ij}^{(m)}\,,$$

- Note that the curvature perturbation only involves scalars

$$\delta[^{(3)}R] = \frac{4}{a^2}(k^2 - 3K)(H_L^{(0)} + \frac{1}{3}H_T^{(0)})Q^{(0)}$$

Spatially Flat Case

- For a spatially flat background metric, harmonics are related to plane waves:

$$
\begin{aligned}
Q^{(0)} &= \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k}\cdot\mathbf{x})
\end{aligned}
$$

  where $\hat{\mathbf{e}}_3 \parallel \mathbf{k}$. Chosen as spin states, c.f. polarization.

- For vectors, the harmonic points in a direction orthogonal to $\mathbf{k}$ suitable for the vortical component of a vector

Fourier Conventions

- Suppress volume terms by making Fourier representation dimensionful

$$F(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}, \quad F(\mathbf{k}) = \int d^3x F(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$(2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') = \int d^3x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}$$

- Reality of field

$$F^*(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} F^*(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = \int \frac{d^3k}{(2\pi)^3} F^*(-\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$= F(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$F^*(-\mathbf{k}) = F(\mathbf{k})$$

Statistical Homogeneity and Isotropy

- Statistical homogeneity of two point correlation function

$$\langle F(\mathbf{x})F(\mathbf{x}')\rangle = \langle \int \frac{d^3k}{(2\pi)^3} F^*(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \int \frac{d^3k'}{(2\pi)^3} F(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}'}\rangle$$

$$= \langle F(\mathbf{x}+\mathbf{d})F(\mathbf{x}'+\mathbf{d})\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \langle F^*(\mathbf{k})F(\mathbf{k}')\rangle e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot\mathbf{x}'} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{d}}$$

- Requires 2pt is the Fourier transform of power spectrum

$$\langle F^*(\mathbf{k})F(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P_F(\mathbf{k})$$

$$\langle F(\mathbf{x})F(\mathbf{x}')\rangle = \int \frac{d^3k}{(2\pi)^3} P_F(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \xi(\mathbf{x}-\mathbf{x}')$$

- Statistical isotropy requires $P_F(\mathbf{k}) = P_F(k)$ and $\xi(\mathbf{x}-\mathbf{x}') = \xi(|\mathbf{x}-\mathbf{x}'|)$

N-point function and Gaussianity

- Generalize to N-point correlation function, e.g. 3pt

$$\langle F(\mathbf{x}_1)F(\mathbf{x}_2)F(\mathbf{x}_3)\rangle = \left[\prod_{i=1}^{3}\int \frac{d^3k_i}{(2\pi)^3} e^{-i\mathbf{k}_i\cdot\mathbf{x}_i}\right] \langle F(\mathbf{k}_1)F(\mathbf{k}_2)F(\mathbf{k}_3)\rangle$$

- Statistical homogeneity requires the $\mathbf{k}_i$ to sum to zero and isotropy independence of orientation

$$\langle F(\mathbf{k}_1)F(\mathbf{k}_2)F(\mathbf{k}_3)\rangle = (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B_F(k_1,k_2,k_3)$$

- Gaussian field: all N point correlation functions depend only on disconnected products of 2 point function or power spectrum, e.g. bispectrum is zero

Amplitude

- Variance

$$\sigma_F^2 \equiv \langle F(\mathbf{x})F(\mathbf{x})\rangle = \int \frac{d^3k}{(2\pi)^3} P_F(k)$$
$$= \int \frac{k^2 dk}{2\pi^2} \int \frac{d\Omega}{4\pi} P_F(k)$$
$$= \int d\ln k \frac{k^3}{2\pi^2} P_F(k)$$

- Define power per logarithmic interval

$$\Delta_F^2(k) \equiv \frac{k^3 P_F(k)}{2\pi^2}$$

- This quantity is dimensionless in all representations. Serves as a definition of the linear regime k's where $\Delta_F^2 \ll 1$

Linearity

- Fields related by a linear equation obey equation independent equations

$$F(\mathbf{x}) = AG(\mathbf{x}) + B \quad \to \quad F(\mathbf{k}) = AG(\mathbf{k}) \qquad (k > 0)$$

includes linear differential equation

$$F(\mathbf{x}) = A\nabla G(\mathbf{k}) + B$$

$$F(\mathbf{k}) = A\int d^3x e^{i\mathbf{k}\cdot\mathbf{x}}\nabla\int\frac{d^3k'}{(2\pi)^3}e^{-i\mathbf{k}'\cdot\mathbf{x}}G(\mathbf{k}')$$

$$= A\int\frac{d^3k'}{(2\pi)^3}\int d^3x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}(-i\mathbf{k}')G(\mathbf{k}') = A(-i\mathbf{k})G(\mathbf{k})$$

converts differential equations to algebraic relations

Convolution

- Convolution in real space often occurs – smoothing of field by finite resolution and normalization $\int d^3x W(\mathbf{x}) = 1$

$$F_W(\mathbf{x}) = \int d^3y W(\mathbf{x}-\mathbf{y}) F(\mathbf{y})$$

$$= \int d^3y \int \frac{d^3k}{(2\pi)^3} W(\mathbf{k}) e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \int \frac{d^3k'}{(2\pi)^3} F(\mathbf{k}') e^{-i\mathbf{k}'\cdot\mathbf{y}}$$

$$= \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} W(\mathbf{k}) F(\mathbf{k}') \int d^3y e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{y}}$$

$$= \int \frac{d^3k}{(2\pi)^3} W(\mathbf{k}) F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$F_W(\mathbf{k}) = W(\mathbf{k}) F(\mathbf{k})$$

- Smoothing acts as a low pass filter: if $W(\mathbf{x})$ is a broad function of width L, $W(\mathbf{k})$ suppressed for $k > 2\pi/L$

Convolution

- Filtered Variance

$$\langle F_W(\mathbf{x}) F_W(\mathbf{x}) \rangle = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \langle F^*(\mathbf{k}) F(\mathbf{k}') \rangle W^*(\mathbf{k}) W(\mathbf{k}')$$

$$= \int \frac{d^3k}{(2\pi)^3} P_F(k) |W(\mathbf{k})|^2$$

- Common filter is the spherical tophat:

$$W_R(\mathbf{x}) = V_R^{-1} \qquad x < R$$

$$W_R(\mathbf{x}) = 0 \qquad x > R$$

- Fourier transform

$$W_R(\mathbf{k}) = \frac{3}{y^3}(\sin y - y \cos y)\,, \qquad (y = kR)$$

Normalization

- Normalization is often quoted as the top hat rms of the density field

$$\sigma_R^2 = \int d\ln k \, \Delta_\delta^2(k) |W_R(k)|^2$$

  where observationally $\sigma_{8h^{-1}\mathrm{Mpc}} \equiv \sigma_8 \approx 1$

- Note that $\Delta_\delta^2(k)$ itself can be thought of as the variance of the field with a filter that has sharp high and low pass filters in k-space
- Convention is that σ_R is defined against the linear density field, not the true non-linear density field

Spatially Flat Case

- Tensor harmonics are the transverse traceless gauge representation
- Tensor amplitude related to the more traditional

$$h_+[(\mathbf{e}_1)_i(\mathbf{e}_1)_j - (\mathbf{e}_2)_i(\mathbf{e}_2)_j]\,, \qquad h_\times[(\mathbf{e}_1)_i(\mathbf{e}_2)_j + (\mathbf{e}_2)_i(\mathbf{e}_1)_j]$$

as

$$h_+ \pm i h_\times = -\sqrt{6} H_T^{(\mp 2)}$$

- $H_T^{(\pm 2)}$ proportional to the right and left circularly polarized amplitudes of gravitational waves with a normalization that is convenient to match the scalar and vector definitions

Covariant Scalar Equations

- DOF counting exercise

8	Variables (4 metric; 4 matter)
-4	Einstein equations
-2	Conservation equations
$+2$	Bianchi identities
-2	Gauge (coordinate choice 1 time, 1 space)
——	
2	Free Variables

without loss of generality choose scalar components of the stress tensor δp, Π .

Covariant Scalar Equations

- Einstein equations (suppressing 0) superscripts

$$(k^2 - 3K)[H_L + \frac{1}{3}H_T] - 3(\frac{\dot a}{a})^2 A + 3\frac{\dot a}{a}\dot H_L + \frac{\dot a}{a}kB =$$

$$= 4\pi G a^2 \delta\rho\,, \quad \text{00 Poisson Equation}$$

$$k^2(A + H_L + \frac{1}{3}H_T) + \left(\frac{d}{d\eta} + 2\frac{\dot a}{a}\right)(kB - \dot H_T)$$

$$= -8\pi G a^2 p \Pi\,, \quad ij \text{ Anisotropy Equation}$$

$$\frac{\dot a}{a}A - \dot H_L - \frac{1}{3}\dot H_T - \frac{K}{k^2}(kB - \dot H_T)$$

$$= 4\pi G a^2(\rho + p)(v - B)/k\,, \quad 0i \text{ Momentum Equation}$$

$$\left[2\frac{\ddot a}{a} - 2\left(\frac{\dot a}{a}\right)^2 + \frac{\dot a}{a}\frac{d}{d\eta} - \frac{k^2}{3}\right]A - \left[\frac{d}{d\eta} + \frac{\dot a}{a}\right](\dot H_L + \frac{1}{3}kB)$$

$$= 4\pi G a^2(\delta p + \frac{1}{3}\delta\rho)\,, \quad ii \text{ Acceleration Equation}$$

Covariant Scalar Equations

- Conservation equations: continuity and Navier Stokes

$$
\begin{aligned}
\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho + 3\frac{\dot a}{a}\delta p &= -(\rho+p)(kv + 3\dot H_L)\,, \\
\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right]\left[(\rho+p)\frac{(v-B)}{k}\right] &= \delta p - \frac{2}{3}(1 - 3\frac{K}{k^2})p\Pi + (\rho+p)A\,,
\end{aligned}
$$

- Equations are not independent since $\nabla_\mu G^{\mu\nu} = 0$ via the Bianchi identities.

- Related to the ability to choose a coordinate system or "gauge" to represent the perturbations.

Gauge

- Metric and matter fluctuations take on different values in different coordinate system
- No such thing as a "gauge invariant" density perturbation!
- General coordinate transformation:

$$\begin{aligned} \tilde{\eta} &= \eta + T \\ \tilde{x}^i &= x^i + L^i \end{aligned}$$

free to choose (T, L^i) to simplify equations or physics - corresponds to a choice of slicing and threading in ADM.

- Decompose these into scalar T, $L^{(0)}$ and vector harmonics $L^{(\pm 1)}$.

Gauge

- $g_{\mu\nu}$ and $T_{\mu\nu}$ transform as tensors, so components in different frames can be related

$$\begin{aligned}\tilde{g}_{\mu\nu}(\tilde{\eta},\tilde{x}^i) &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\eta, x^i) \\ &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\tilde{\eta} - TQ, \tilde{x}^i - LQ^i)\end{aligned}$$

- Fluctuations are compared at the same coordinate positions (not same space time positions) between the two gauges
- For example with a TQ perturbation, an event labeled with $\tilde{\eta}$ =const. and $\tilde{x}$ =const. represents a different time with respect to the underlying homogeneous and isotropic background

Gauge Transformation

- Scalar Metric:

$$
\begin{aligned}
\tilde{A} &= A - \dot{T} - \frac{\dot{a}}{a}T\,, \\
\tilde{B} &= B + \dot{L} + kT\,, \\
\tilde{H}_L &= H_L - \frac{k}{3}L - \frac{\dot{a}}{a}T\,, \\
\tilde{H}_T &= H_T + kL\,, \qquad \tilde{H}_L + \frac{1}{3}\tilde{H}_T = H_L + \frac{1}{3}H_T - \frac{\dot{a}}{a}T
\end{aligned}
$$

curvature perturbation depends on slicing not threading

- Scalar Matter (Jth component):

$$
\begin{aligned}
\delta\tilde{\rho}_J &= \delta\rho_J - \dot{\rho}_J T, \\
\delta\tilde{p}_J &= \delta p_J - \dot{p}_J T, \\
\tilde{v}_J &= v_J + \dot{L},
\end{aligned}
$$

density and pressure likewise depend on slicing only

Gauge Transformation

- Vector:

$$\begin{aligned}
\tilde{B}^{(\pm1)} &= B^{(\pm1)} + \dot{L}^{(\pm1)}, \\
\tilde{H}_T^{(\pm1)} &= H_T^{(\pm1)} + kL^{(\pm1)}, \\
\tilde{v}_J^{(\pm1)} &= v_J^{(\pm1)} + \dot{L}^{(\pm1)},
\end{aligned}$$

- Spatial vector has no background component hence no dependence on slicing at first order

  Tensor: no dependence on slicing or threading at first order

- Gauge transformations and covariant representation can be extended to higher orders
- A coordinate system is fully specified if there is an explicit prescription for (T, L^i) or for scalars (T, L)

Slicing

Common choices for slicing T: set something geometric to zero

- Proper time slicing $A = 0$: proper time between slices corresponds to coordinate time – T allows c/a freedom
- Comoving (velocity orthogonal) slicing: $v - B = 0$, matter 4 velocity is related to N^ν and orthogonal to slicing - T fixed
- Newtonian (shear free) slicing: $\dot{H}_T - kB = 0$, expansion rate is isotropic, shear free, T fixed
- Uniform expansion slicing: $-(\dot{a}/a)A + \dot{H}_L + kB/3 = 0$, perturbation to the volume expansion rate θ vanishes, T fixed
- Flat (constant curvature) slicing, $\delta^{(3)}R = 0$, $(H_L + H_T/3 = 0)$, T fixed
- Constant density slicing, $\delta\rho_I = 0$, T fixed

Threading

- Threading specifies the relationship between constant spatial coordinates between slices and is determined by L

  Typically involves a condition on v, B, H_T

  - Orthogonal threading $B = 0$, constant spatial coordinates orthogonal to slicing (zero shift), allows $\delta L = c$ translational freedom
  - Comoving threading $v = 0$, allows $\delta L = c$ translational freedom.
  - Isotropic threading $H_T = 0$, fully fixes L

Newtonian (Longitudinal) Gauge

- Newtonian (shear free slicing, isotropic threading):

$$
\begin{aligned}
\tilde{B} &= \tilde{H}_T = 0 \\
\Psi &\equiv \tilde{A} \quad \text{(Newtonian potential)} \\
\Phi &\equiv \tilde{H}_L \quad \text{(Newtonian curvature)} \\
L &= -H_T/k \\
T &= -B/k + \dot{H}_T/k^2
\end{aligned}
$$

Good: intuitive Newtonian like gravity; matter and metric algebraically related; commonly chosen for analytic CMB and lensing work

Bad: numerically unstable

Newtonian (Longitudinal) Gauge

- Newtonian (shear free) slicing, isotropic threading $B = H_T = 0$:

$$
\begin{aligned}
(k^2 - 3K)\Phi &= 4\pi G a^2 \left[\delta\rho + 3\frac{\dot{a}}{a}(\rho + p)v/k\right] \quad \text{Poisson + Momentum} \\
k^2(\Psi + \Phi) &= -8\pi G a^2 p \Pi \quad \text{Anisotropy}
\end{aligned}
$$

so $\Psi = -\Phi$ if anisotropic stress $\Pi = 0$ and

$$
\begin{aligned}
\left[\frac{d}{d\eta} + 3\frac{\dot{a}}{a}\right]\delta\rho + 3\frac{\dot{a}}{a}\delta p &= -(\rho + p)(kv + 3\dot{\Phi}), \\
\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a}\right](\rho + p)v &= k\delta p - \frac{2}{3}(1 - 3\frac{K}{k^2})p\, k\Pi + (\rho + p)\, k\Psi,
\end{aligned}
$$

- Newtonian competition between stress (pressure and viscosity) and potential gradients
- Note: Poisson source is the density perturbation on comoving slicing

Total Matter Gauge

- Total matter: (comoving slicing, isotropic threading)

$$
\begin{aligned}
\tilde{B} &= \tilde{v} \quad (T^0_i = 0) \\
H_T &= 0 \\
\xi &= \tilde{A} \\
\mathcal{R} &= \tilde{H}_L \quad \text{(comoving curvature)} \\
\Delta &= \tilde{\delta} \quad \text{(total density pert)} \\
T &= (v - B)/k \\
L &= -H_T/k
\end{aligned}
$$

Good: Algebraic relations between matter and metric; comoving curvature perturbation obeys conservation law

Bad: Non-intuitive threading involving v

Total Matter Gauge

- Euler equation becomes an algebraic relation between stress and potential

$$(\rho + p)\xi = -\delta p + \frac{2}{3}\left(1 - \frac{3K}{k^2}\right) p\Pi$$

- Einstein equation lacks momentum density source

$$\frac{\dot a}{a}\xi - \dot{\mathcal{R}} - \frac{K}{k^2} kv = 0$$

Combine: $\mathcal{R}$ is conserved if stress fluctuations negligible, e.g. above the horizon if $|K| \ll H^2$

$$\dot{\mathcal{R}} + Kv/k = \frac{\dot a}{a}\left[-\frac{\delta p}{\rho + p} + \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\frac{p}{\rho + p}\Pi \right] \to 0$$

"Gauge Invariant" Approach

- Gauge transformation rules allow variables which take on a geometric meaning in one choice of slicing and threading to be accessed from variables on another choice
- Functional form of the relationship between the variables is gauge invariant (*not* the variable values themselves! – i.e. equation is *covariant*)
- E.g. comoving curvature and density perturbations

$$
\begin{aligned}
\mathcal{R} &= H_L + \frac{1}{3}H_T - \frac{\dot{a}}{a}(v - B)/k \\
\Delta\rho &= \delta\rho + 3(\rho + p)\frac{\dot{a}}{a}(v - B)/k
\end{aligned}
$$

Newtonian-Total Matter Hybrid

- With the gauge in(*or co*)variant approach, express variables of one gauge in terms of those in another – allows a mixture in the equations of motion
- Example: Newtonian curvature and comoving density

  $$(k^2 - 3K)\Phi = 4\pi G a^2 \rho \Delta$$

  ordinary Poisson equation then implies Φ approximately constant if stresses negligible.
- Example: Exact Newtonian curvature above the horizon derived through comoving curvature conservation

  Gauge transformation

  $$\Phi = \mathcal{R} + \frac{\dot{a}}{a}\frac{v}{k}$$

Hybrid “Gauge Invariant” Approach

Einstein equation to eliminate velocity

$$\frac{\dot{a}}{a}\Psi - \dot{\Phi} = 4\pi G a^2(\rho + p)v/k$$

Friedmann equation with no spatial curvature

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}a^2\rho$$

With $\dot{\Phi} = 0$ and $\Psi \approx -\Phi$

$$\frac{\dot{a}}{a}\frac{v}{k} = -\frac{2}{3(1+w)}\Phi$$

Newtonian-Total Matter Hybrid

Combining gauge transformation with velocity relation

$$\Phi = \frac{3+3w}{5+3w}\mathcal{R}$$

Usage: calculate $\mathcal{R}$ from inflation determines Φ for any choice of matter content or causal evolution.

- Example: Scalar field ("quintessence" dark energy) equations in total matter gauge imply a sound speed $\delta p/\delta\rho = 1$ independent of potential $V(\phi)$. Solve in synchronous gauge.

Synchronous Gauge

- Synchronous: (proper time slicing, orthogonal threading)

$$
\begin{aligned}
\tilde{A} &= \tilde{B} = 0 \\
\eta_T &\equiv -\tilde{H}_L - \frac{1}{3}\tilde{H}_T \\
h_L &\equiv 6H_L \\
T &= a^{-1}\int d\eta a A + c_1 a^{-1} \\
L &= -\int d\eta (B + kT) + c_2
\end{aligned}
$$

Good: stable, the choice of numerical codes

Bad: residual gauge freedom in constants c_1, c_2 must be specified as an initial condition, intrinsically relativistic, threading conditions breaks down beyond linear regime if c_1 is fixed to CDM comoving.

Synchronous Gauge

- The Einstein equations give

$$
\begin{aligned}
\dot\eta_T - \frac{K}{2k^2}(\dot h_L + 6\dot\eta_T) &= 4\pi G a^2(\rho+p)\frac{v}{k}\,, \\
\ddot h_L + \frac{\dot a}{a}\dot h_L &= -8\pi G a^2(\delta\rho + 3\delta p)\,, \\
-(k^2-3K)\eta_T + \frac{1}{2}\frac{\dot a}{a}\dot h_L &= 4\pi G a^2 \delta\rho
\end{aligned}
$$

[choose (1 & 2) or (1 & 3)] while the conservation equations give

$$
\begin{aligned}
\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho_J + 3\frac{\dot a}{a}\delta p_J &= -(\rho_J + p_J)(k v_J + \frac{1}{2}\dot h_L)\,, \\
\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho_J + p_J)\frac{v_J}{k} &= \delta p_J - \frac{2}{3}(1 - 3\frac{K}{k^2})p_J \Pi_J\,.
\end{aligned}
$$

Synchronous Gauge

- Lack of a lapse A implies no gravitational forces in Navier-Stokes equation. Hence for stress free matter like cold dark matter, zero velocity initially implies zero velocity always.
- Choosing the momentum and acceleration Einstein equations is good since for CDM domination, curvature η_T is conserved and $\dot{h}_L$ is simple to solve for.
- Choosing the momentum and Poisson equations is good when the equation of state of the matter is complicated since δp is not involved. This is the choice of CAMB.

  Caution: since the curvature η_T appears and it has zero CDM source, subtle effects like dark energy perturbations are important everywhere

Spatially Flat Gauge

- Spatially Flat (flat slicing, isotropic threading):

$$\begin{aligned}
\tilde{H}_L &= \tilde{H}_T = 0 \\
L &= -H_T/k \\
\tilde{A}\,, \tilde{B} &= \quad \text{metric perturbations} \\
T &= \left(\frac{\dot{a}}{a}\right)^{-1}\left(H_L + \frac{1}{3}H_T\right)
\end{aligned}$$

Good: eliminates spatial metric in evolution equations; useful in inflationary calculations (Mukhanov et al)

Bad: non-intuitive slicing (no curvature!) and threading

- Caution: perturbation evolution is governed by the behavior of stress fluctuations and an isotropic stress fluctuation δp is gauge dependent.

Uniform Density Gauge

- Uniform density: (constant density slicing, isotropic threading)

$$
\begin{aligned}
H_T &= 0\,, \\
\zeta_I &\equiv H_L \\
B_I &\equiv B \\
A_I &\equiv A \\
T &= \frac{\delta\rho_I}{\dot\rho_I} \\
L &= -H_T/k
\end{aligned}
$$

Good: Curvature conserved involves only stress energy conservation; simplifies isocurvature treatment

Bad: non intuitive slicing (no density pert! problems beyond linear regime) and threading

Uniform Density Gauge

- Einstein equations with I as the total or dominant species

$$(k^2 - 3K)\zeta_I - 3\left(\frac{\dot a}{a}\right)^2 A_I + 3\frac{\dot a}{a}\dot\zeta_I + \frac{\dot a}{a} k B_I = 0\,,$$

$$\frac{\dot a}{a}A_I - \dot\zeta_I - \frac{K}{k}B_I = 4\pi G a^2(\rho + p)\frac{v - B_I}{k}\,,$$

- The conservation equations (if $J = I$ then $\delta\rho_J = 0$)

$$\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho_J + 3\frac{\dot a}{a}\delta p_J = -(\rho_J + p_J)(k v_J + 3\dot\zeta_I)\,,$$

$$\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho_J + p_J)\frac{v_J - B_I}{k} = \delta p_J - \frac{2}{3}(1 - 3\frac{K}{k^2})p_J\Pi_J + (\rho_J + p_J)A_I\,.$$

Uniform Density Gauge

- Conservation of curvature - single component I

$$\dot{\zeta}_I = -\frac{\dot{a}}{a}\frac{\delta p_I}{\rho_I + p_I} - \frac{1}{3}kv_I\,.$$

- Since $\delta\rho_I = 0$, δp_I is the non-adiabatic stress and curvature is constant as $k \to 0$ for internally adiabatic stresses $p_I(\rho_I)$.
- Note that this conservation law does not involve the Einstein equations at all: just local energy momentum conservation so it is valid for alternate theories of gravity
- Curvature on comoving slices $\mathcal{R}$ and ζ_I related by

$$\zeta_I = \mathcal{R} + \frac{1}{3}\frac{\rho_I\Delta_I}{(\rho_I + p_I)}\Big|_{\rm comoving}\,.$$

and coincide above the horizon for adiabatic fluctuations

Uniform Density Gauge

- Simple relationship to density fluctuations in the spatially flat gauge

$$\zeta_I = \frac{1}{3} \frac{\delta\tilde{\rho}_I}{(\rho_I + p_I)}\Big|_{\rm flat} .$$

- For each particle species $\delta\rho/(\rho + p) = \delta n/n$, the number density fluctuation
- Multiple ζ_J carry information about number density fluctuations between species
- ζ_J constant component by component outside horizon if each component is adiabatic $p_J(\rho_J)$.

Poisson Equation

- Naive expectation: $\Phi = -\Psi$ and

$$\nabla^2\Phi = -4\pi G a^2 \delta\rho$$

$$k^2\Phi = 4\pi G a^2 \rho\delta$$

where a^2 comes from physical $\to$ comoving and $\delta\rho$ since background density goes into scale factor evolution

- Einstein equations put in a relativistic correction (flat universe)

$$k^2\Phi = 4\pi G a^2 \rho[\delta + 3\frac{\dot{a}}{a}(1+w)v/k]$$

$$k^2(\Phi + \Psi) = -8\pi G a^2 p\pi$$

convenient to call combination

$$\Delta \equiv \delta + 3\frac{\dot{a}}{a}(1+w)v/k$$

Constancy of Potential & Growth Rate

- Given the Poisson equation relates a redshifting total density ρ and the comoving derivative factor a the density perturbation must grow as $\Delta \propto (a^2\rho)^{-1} \propto a^{1-3w}$ to maintain a constant potential.
- Density perturbations are stabilized by the expanding universe (expansion drag) and do not grow exponentially. Presents a new version of the horizon problem.
- Naive (Newtonian) argument: in the absence of stress perturbations the Euler equation takes the form $\dot{v} \sim k\Psi$
- Given an initial potential perturbation Ψ_i a velocity perturbation $v \sim (k\eta)\Psi_i$
- Given a velocity perturbation continuity grows a density fluctuation as $\dot{\Delta} \sim -kv$ or $\Delta = -(k\eta)^2\Psi_i$.

Constancy of Potential & Growth Rate

- The growing density perturbation is exactly that required to maintain the potential constant

$$\Psi \approx -\frac{4\pi G a^2 \rho}{k^2}\Delta \approx \frac{4\pi G a^2 \rho}{k^2}(k\eta)^2 \Psi_i$$

$\eta \propto a^{(1+3w)/2}$, $a^2\rho \propto a^{-(1+3w)}$

- Under gravity alone, the density fluctuations grow just fast enough to maintain constant potentials

- Stress fluctuations only decrease the rate of growth of the potential. Starting from an unperturbed $\Psi_i = 0$ universe, where do the fluctuations that form large scale structure come from

Bardeen Curvature

- A proper relativistic generalization involves the $(\dot{a}/a)v/k$ corrections, called the Bardeen (or comoving) curvature

$$\mathcal{R} \equiv \Phi - \frac{\dot{a}}{a} v/k \, .$$

- Geometric meaning: space curvature fluctuation on comoving (velocity-orthogonal-isotropic) time slicing
- Same time slicing gives Δ as the density perturbation

Bardeen Curvature

- Continuity equation becomes

$$\dot{\Delta} = -3\frac{\dot{a}}{a}\left(C_s^2 - w\right)\Delta - (1+w)(kv + 3\dot{\mathcal{R}})\,,$$

where the transformed sound speed

$$C_s^2 \equiv \frac{\Delta p}{\Delta \rho}$$
$$\Delta p \equiv \delta p - \dot{p}v/k$$

- Euler equation becomes

$$\dot{\mathcal{R}} = \frac{\dot{a}}{a}\xi$$
$$\xi = -\frac{C_s^2}{1+w}\Delta + \frac{2}{3}\frac{w}{1+w}\pi\,.$$

Bardeen Curvature

- So that the Bardeen curvature only changes in the presence of stress fluctuations – scales below the horizon
- Extremely useful result (proven in problem set) says that calculated $\mathcal{R}$ once and for all – e.g. during formation in an inflationary epoch
- Relationship to gravitational potential: (from Poisson & conservation equations)

$$\frac{\dot{a}}{a}\Psi - \dot{\Phi} = 4\pi G a^2 (\rho + p) v/k$$

so that if Φ constant and $\Psi = -\Phi$ then

$$-\left(\frac{\dot{a}}{a}\right)^2 \Phi = 4\pi G a^2 \rho (1+w) \frac{\dot{a}}{a} v/k$$

$$= \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 (1+w) \frac{\dot{a}}{a} v/k$$

Bardeen Curvature

- Relationship between the curvature Φ and v

$$\frac{\dot{a}}{a} v/k = -\frac{2}{3(1+w)}\Phi \quad \rightarrow \mathcal{R} = \left[1 + \frac{2}{3(1+w)}\right]\Phi$$

- Matter dominated $\Phi = 3\mathcal{R}/5$, radiation dominated $\Phi = 2\mathcal{R}/3$, Λ dominated $\Phi \rightarrow 0$.
- So: put these pieces together assuming dark energy is smooth

$$\begin{aligned}
\frac{k^3}{2\pi^2} P_\Delta(k) &= \left(\frac{k^2}{4\pi G a^2 \rho_m}\right)^2 \frac{k^3}{2\pi^2} P_\Phi(k) \\
&= \left(\frac{k^2}{4\pi G a^2 \rho_m}\right)^2 \frac{k^3}{2\pi^2} P_\Phi(k) \\
&= \frac{4}{9} \frac{a^2 k^4}{\Omega_m^2 H_0^4} \frac{k^3}{2\pi^2} P_\Phi(k)
\end{aligned}$$

Bardeen Curvature

- Assume initial curvature power spectrum

$$\frac{k^3}{2\pi^2}P_{\mathcal{R}}(k) = A_S \left(\frac{k}{k_{\rm norm}}\right)^{n_S-1}$$

  and a transfer function $T(k)$ that defines the subhorizon evolution which is influenced by pressure effects during radiation domination

- Finally normalize to the matter dominated expectation and take $\Phi = [3G(a)/5]\,\mathcal{R}$ where $G(a)$ is the modification to the growth rate of Φ due to the dark energy and curvature

$$\Phi(a,k) = \frac{3}{5}G(a)T(k)\mathcal{R}(0,k)$$

$$\frac{k^3}{2\pi^2}P_{\Delta}(k) = \frac{4}{25}A_S \left(\frac{G(a)a}{\Omega_m}\right)^2 \left(\frac{k}{H_0}\right)^4 \left(\frac{k}{k_{\rm norm}}\right)^{n_S-1} T^2(k)$$

Transfer Function

- Transfer function transfers the initial Newtonian curvature to its value today (linear response theory)

$$T(k) = \frac{\Phi(k, a = 1)}{\Phi(k, a_{\rm init})} \frac{\Phi(k_{\rm norm}, a_{\rm init})}{\Phi(k_{\rm norm}, a = 1)}$$

- Conservation of Bardeen curvature: Newtonian curvature is a constant when stress perturbations are negligible: above the horizon during radiation and dark energy domination, on all scales during matter domination
- When stress fluctuations dominate, perturbations are stabilized by the Jeans mechanism
- Hybrid Poisson equation: Newtonian curvature, comoving density perturbation $\Delta \equiv (\delta\rho/\rho)_{\rm com}$ implies Φ decays

$$(k^2 - 3K)\Phi = 4\pi G\rho\Delta \sim \eta^{-2}\Delta$$

Transfer Function

- Freezing of Δ stops at $\eta_{\rm eq}$

$$\Phi \sim (k\eta_{\rm eq})^{-2}\Delta_H \sim (k\eta_{\rm eq})^{-2}\Phi_{\rm init}$$

- Transfer function has a k^{-2} fall-off beyond $k_{\rm eq} \sim \eta_{\rm eq}^{-1}$
- Small correction since growth with a smooth radiation component is logarithmic not frozen
- Transfer function is a direct output of an Einstein-Boltzmann code

Fitting Function

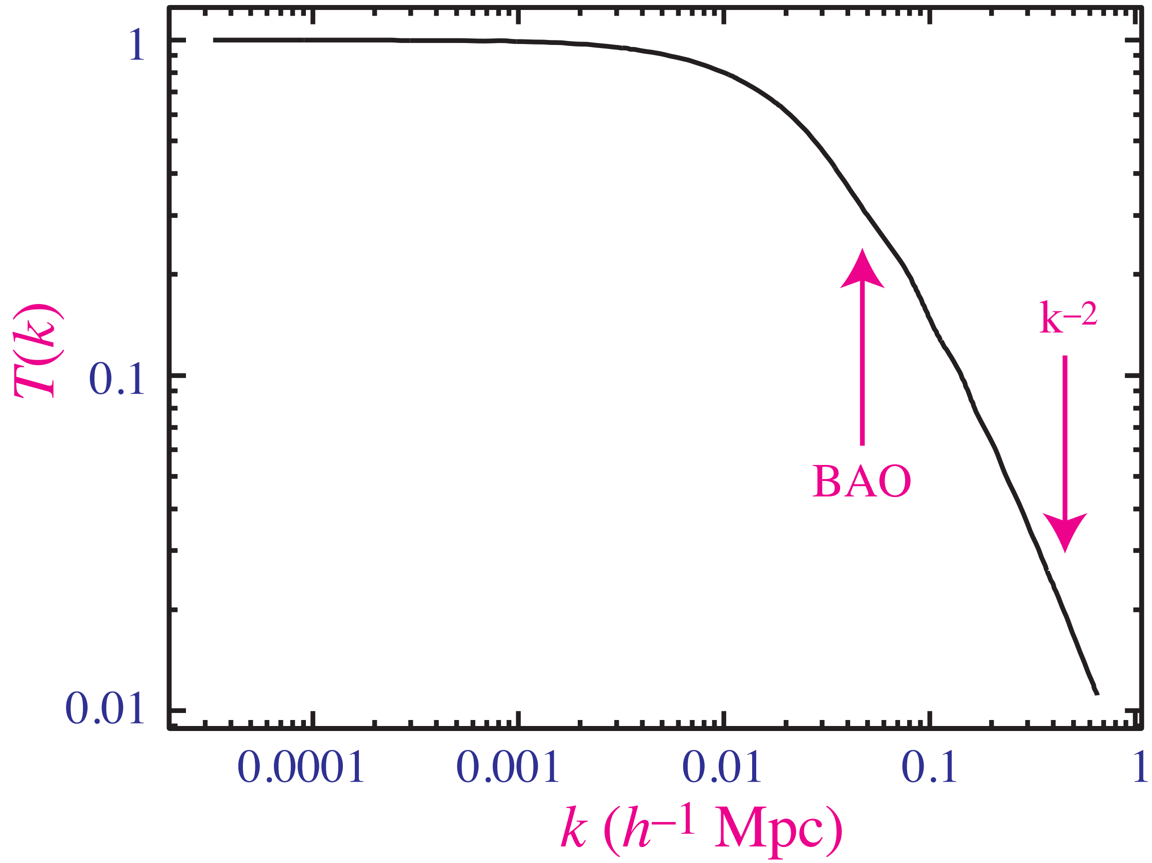
- Alternately accurate fitting formula exist, e.g. pure CDM form:

$$T(k(q)) = \frac{L(q)}{L(q) + C(q)q^2}$$
$$L(q) = \ln(e + 1.84q)$$
$$C(q) = 14.4 + \frac{325}{1 + 60.5q^{1.11}}$$
$$q = k/\Omega_m h^2 \mathrm{Mpc}^{-1} (T_{\mathrm{CMB}}/2.7K)^2$$

- In h Mpc^{-1}, the critical scale depends on $\Gamma \equiv \Omega_m h$ also known as the shape parameter

Transfer Function

- Numerical calculation



Dark Matter and the Transfer Function

- Baryons caught up in the acoustic oscillations of the CMB and impart acoustic wiggles to the transfer function. Density enhancements are produced kinematically through the continuity equation $\delta_b \sim (k\eta)v_b$ and hence are out of phase with CMB temperature peaks
- Dissipation of the acoustic oscillations eliminates both the CMB and baryon perturbations – known as Silk damping for the baryons. This suppression and the general fact that baryons are caught up with photons was one of the main arguments for CDM
- Neutrino dark matter suffers similar effects and hence cannot be the main component of dark matter in the universe

Massive Neutrinos

- Relativistic stresses of a light neutrino slow the growth of structure
- Neutrino species with cosmological abundance contribute to matter as $\Omega_\nu h^2 = \sum m_\nu / 94\text{eV}$, suppressing power as $\Delta P/P \approx -8\Omega_\nu/\Omega_m$
- Current data from 2dF galaxy survey and CMB indicate $\sum m_\nu < 0.9\text{eV}$ assuming a ΛCDM model with constant tilt based on the shape of the transfer function.

Growth Function

- Same physics applies to the dark energy dominated universe
- Under the dark energy sound horizon or Jeans scale, dark energy density frozen. Potential decays at the same rate for all scales

$$G(a) = \frac{\Phi(k_{\rm norm}, a)}{\Phi(k_{\rm norm}, a_{\rm init})} \qquad {}' \equiv \frac{d}{d\ln a}$$

- Continuity + Euler + Poisson

$$G'' + \left(1 - \frac{\rho''}{\rho'} + \frac{1}{2}\frac{\rho_c'}{\rho_c}\right) G' + \left(\frac{1}{2}\frac{\rho_c' + \rho'}{\rho_c} - \frac{\rho''}{\rho'}\right) G = 0$$

where ρ is the Jeans unstable matter and ρ_c is the critical density

Dark Energy Growth Suppression

- Pressure growth suppression: $\delta \equiv \delta\rho_m/\rho_m \propto aG$

$$\frac{d^2G}{d\ln a^2} + \left[\frac{5}{2} - \frac{3}{2}w(z)\Omega_{DE}(z)\right]\frac{dG}{d\ln a} + \frac{3}{2}[1 - w(z)]\Omega_{DE}(z)G = 0\,,$$

where $w \equiv p_{DE}/\rho_{DE}$ and $\Omega_{DE} \equiv \rho_{DE}/(\rho_m + \rho_{DE})$ with initial conditions $G = 1$, $dg/d\ln a = 0$

- As $\Omega_{DE} \to 0$ $G =$const. is a solution. The other solution is the decaying mode, elimated by initial conditions
- As $\Omega_{DE} \to 1$ $G \propto a^{-1}$ is a solution. Corresponds to a frozen density field.

Velocity field

- Continuity gives the velocity from the density field as

$$v = -\dot{\Delta}/k = -\frac{aH}{k}\frac{d\Delta}{d\ln a}$$
$$= -\frac{aH}{k}\Delta\frac{d\ln(ag)}{d\ln a}$$

- In a ΛCDM model or open model $d\ln(ag)/d\ln a \approx \Omega_m^{0.6}$
- Measuring both the density field and the velocity field (through distance determination and redshift) allows a measurement of Ω_m
- Practically one measures $\beta = \Omega_m^{0.6}/b$ where b is a bias factor for the tracer of the density field, i.e. with galaxy numbers $\delta n/n = b\Delta$
- Can also measure this factor from the redshift space power spectrum - the Kaiser effect where clustering in the radial direction is apparently enhanced by gravitational infall

Gravitational Lensing

- Gravitational potentials along the line of sight $\hat{\mathbf{n}}$ to some source at comoving distance D_s lens the images according to (flat universe)

$$\phi(\hat{\mathbf{n}}) = 2 \int dD \frac{D_s - D}{D D_s} \Phi(D\hat{\mathbf{n}}, \eta(D))$$

remapping image positions as

$$\hat{\mathbf{n}}^I = \hat{\mathbf{n}}^S + \nabla_{\hat{\mathbf{n}}} \phi(\hat{\mathbf{n}})$$

- Since absolute source position is unknown, use image distortion defined by the Jacobian matrix

$$\frac{\partial n_i^I}{\partial n_j^S} = \delta_{ij} + \psi_{ij}$$

Weak Lensing

- Small image distortions described by the convergence κ and shear components (γ_1, γ_2)

$$\psi_{ij} = \begin{pmatrix} \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & \kappa + \gamma_1 \end{pmatrix}$$

where $\nabla_{\hat{\mathbf{n}}} = D\nabla$ and

$$\psi_{ij} = 2 \int dD \frac{D(D_s - D)}{D_s} \nabla_i \nabla_j \Phi(D\hat{\mathbf{n}}, \eta(D))$$

- In particular, through the Poisson equation the convergence (measured from shear) is simply the projected mass

$$\kappa = \frac{3}{2} \Omega_m H_0^2 \int dD \frac{D(D_s - D)}{D_s} \frac{\Delta(D\hat{\mathbf{n}}, \eta(D))}{a}$$