

Set 6:

FRW Cosmology

# Notes vs Book

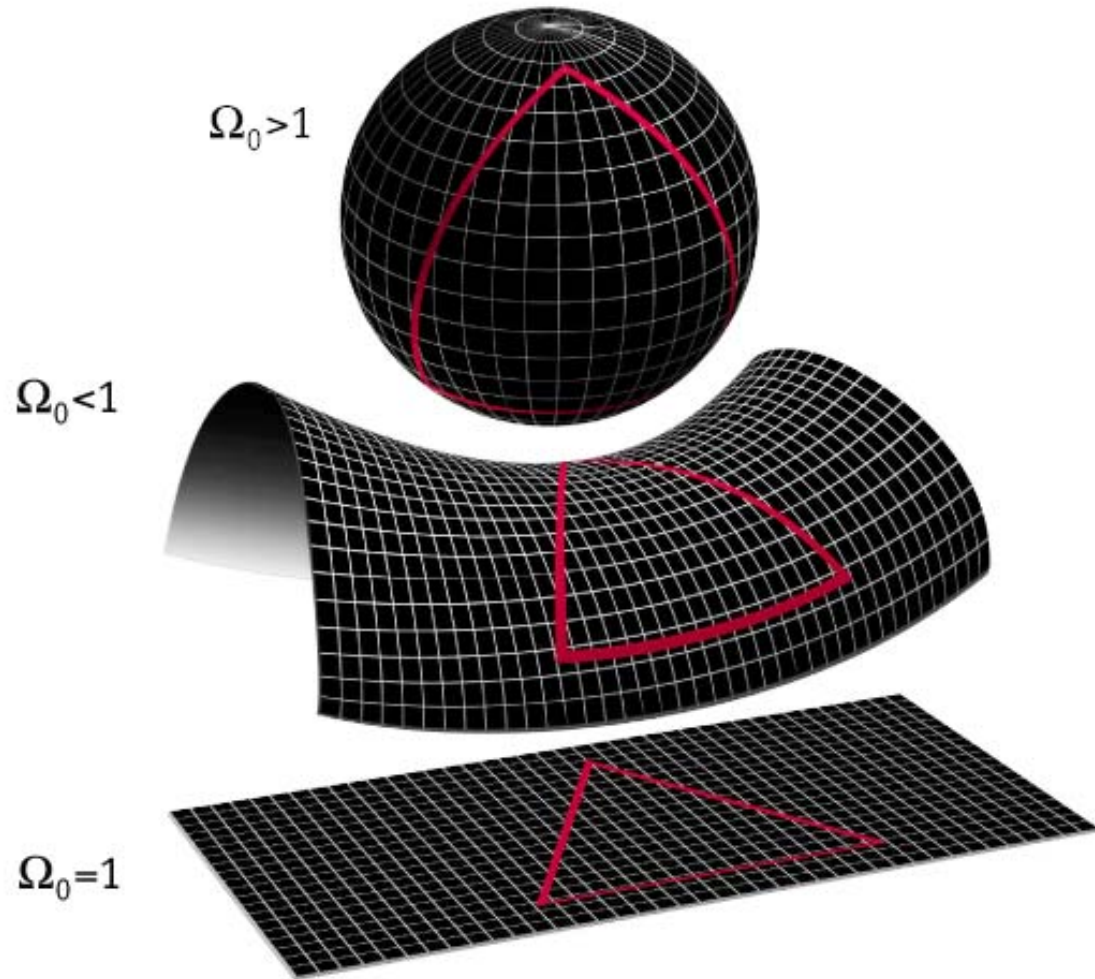
- Following notes uses a slightly different notation than the book
- Natural units  $c = 1, \hbar = 1$
- Scale factor  $a \rightarrow R$
- Comoving angular diameter distance  $D_A =$  book's coordinate distance  $r$
- Comoving distance  $D =$  book's proper distance  $D_A \rightarrow \omega$

# FRW Cosmology

- FRW cosmology = homogeneous and isotropic on large scales
- Universe observed to be nearly isotropic (e.g. CMB, radio point sources, galaxy surveys)
- Copernican principle: must be isotropic to all observers (all locations)
- Implies homogeneity; also galaxy redshift surveys (LCRS, 2dF, SDSS) have seen the “end of greatness”, large scale homogeneity directly
- FRW cosmology (homogeneity, isotropy & Einstein equations) generically implies the expansion of the universe, except for special unstable cases

# FRW Geometry

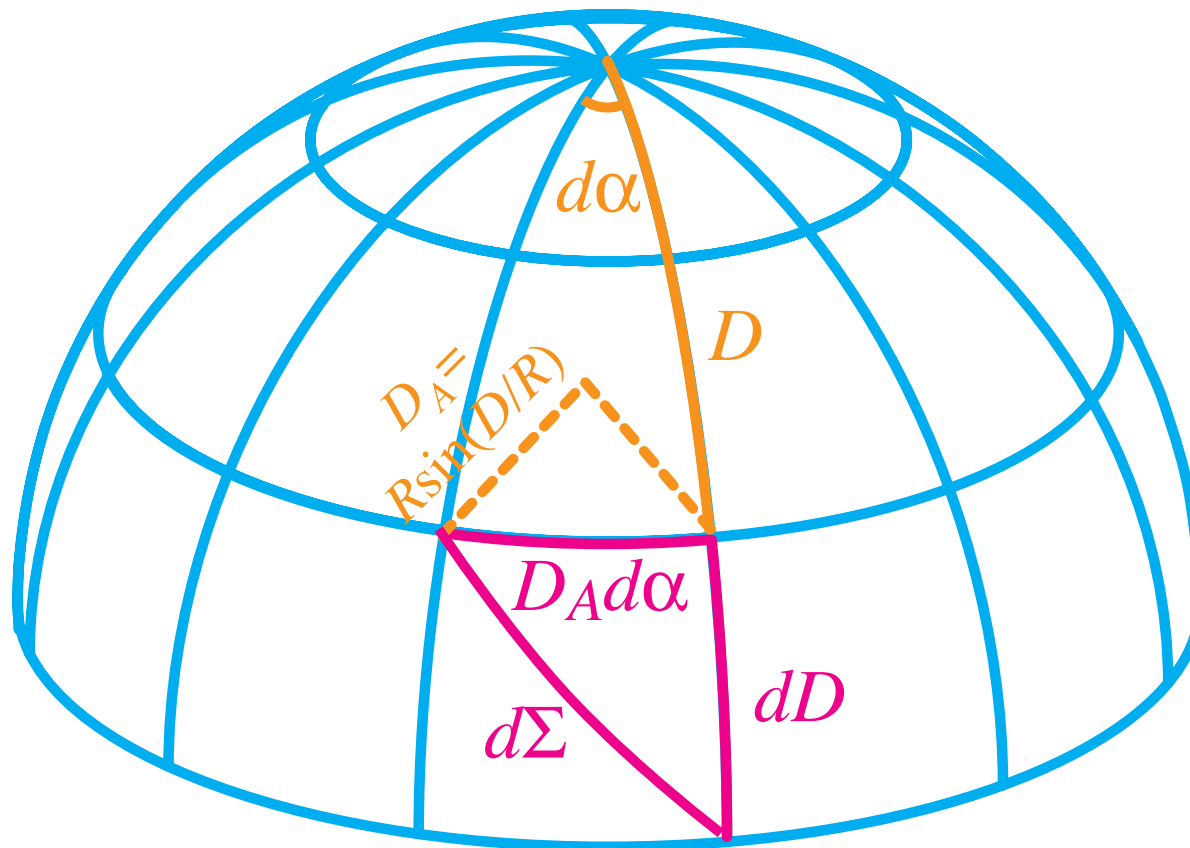
- Spatial geometry is that of a constant curvature
  - Positive: sphere
  - Negative: saddle
  - Flat: plane
- Metric tells us how to measure distances on this surface



MAP990006

# FRW Geometry

- Consider the closed geometry of a sphere of radius  $R$  and suppress one dimension: there are two important distances  $D$  - the distance along the arc, and  $D_A = R \sin(D/R)$  the distance from the center at a constant latitude



# Angular Diameter Distance

- Spatial distance: restore 3rd dimension with the usual spherical polar angles

$$\begin{aligned}d\Sigma^2 &= dD^2 + D_A^2 d\alpha^2 \\ &= dD^2 + D_A^2 (d\theta^2 + \sin^2 \theta d\phi^2)\end{aligned}$$

- $D_A$  is called the angular diameter distance since  $D_A d\alpha$  corresponds to the transverse separation or size as opposed to the Euclidean  $D d\alpha$ , i.e. is the apparent distance to an object through the gravitational lens of the background geometry
- In a positively curved geometry  $D_A < D$  and objects are further than they appear
- In a negatively curved universe  $R$  is imaginary and  $R \sin(D/R) = i|R| \sin(D/i|R|) = |R| \sinh(D/|R|)$  – and  $D_A > D$  objects are closer than they appear

# Angular Diameter Distance

- The line element is often also written using  $D_A$  as the coordinate distance

$$dD_A^2 = \left( \frac{dD_A}{dD} \right)^2 dD^2$$

$$\left( \frac{dD_A}{dD} \right)^2 = \cos^2(D/R) = 1 - \sin^2(D/R) = 1 - (D_A/R)^2$$

$$dD^2 = \frac{1}{1 - (D_A^2/R^2)} dD_A^2$$

and defining the curvature  $K = 1/R^2$  the line element becomes

$$d\Sigma^2 = \frac{1}{1 - D_A^2 K} dD_A^2 + D_A^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where  $K < 0$  for a negatively curved space

# Volume Element

- Metric also defines the volume element

$$\begin{aligned}dV &= (dD)(D_A d\theta)(D_A \sin \theta d\phi) \\ &= D_A^2 dD d\Omega\end{aligned}$$

- Most of classical cosmology boils down to these three quantities, (comoving) distance, (comoving) angular diameter distance, and volume element
- For example, distance to a high redshift supernova, angular size of the horizon at last scattering, number density of clusters...



# Comoving Coordinates

- Remaining degree of freedom (preserving homogeneity and isotropy) is an overall scale factor that relates the geometry (fixed by the radius of curvature  $R$ ) to physical coordinates – a function of time only

$$d\sigma^2 = a^2(t)d\Sigma^2$$

our conventions are that the scale factor today  $a(t_0) \equiv 1$

- Similarly physical distances are given by  $d(t) = a(t)D$ ,  
 $d_A(t) = a(t)D_A$ .
- Distances in capital case are *comoving* i.e. they comove with the expansion and do not change with time – simplest coordinates to work out geometrical effects

# Time and Conformal Time

- Proper time

$$\begin{aligned}d\tau^2 &= dt^2 - d\sigma^2 \\ &= dt^2 - a^2(t)d\Sigma^2 \\ &\equiv a^2(t) (d\eta^2 - d\Sigma^2)\end{aligned}$$

- Taking out the scale factor in the time coordinate  $d\eta = dt/a$  defines **conformal time** – useful in that photons travelling radially from observer then obey

$$\Delta D = \Delta\eta = \int \frac{dt}{a}$$

so that time and distance may be interchanged

# Horizon

- Distance travelled by a photon in the whole lifetime of the universe defines the **horizon**
- By  $d\tau = 0$ , the horizon is simply the conformal time elapsed

$$D_{\text{horizon}}(t) = \int_0^t \frac{dt'}{a} = \eta(t)$$

- Since the horizon always grows with time, there is always a point in time before which two observers separated by a distance  $D$  could not have been in causal contact
- Horizon problem: why is the universe homogeneous and isotropic on large scales, near the current horizon – problem deepens for objects seen at early times, e.g. CMB

# FRW Metric

- Proper time defines the metric  $g_{\mu\nu}$

$$d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu$$

signature follows Peacock's convention. Caveat reader: this is opposite to what I'm used to so I *will* occasionally mess up the sign

- Usually we will use comoving coordinates and conformal time as the “ $x$ ” ’s unless otherwise specified – metric for other choices are related by  $a(t)$
- We will generally skirt around real General Relativity but rudimentary knowledge will be useful

# Hubble Parameter

- Useful to define the expansion rate or Hubble parameter

$$H(t) \equiv \frac{1}{a} \frac{da}{dt}$$

since dynamics (Einstein equations) will give this directly as

$$H(a) \equiv H(t(a))$$

- Time becomes

$$t = \int dt = \int \frac{da}{aH(a)}$$

- Conformal time becomes

$$\eta = \int \frac{dt}{a} = \int \frac{da}{a^2 H(a)}$$

# Redshift

- Wavelength of light “stretches” with the scale factor, so that it is convenient to define a shift-to-the-red or redshift as the scale factor increases

$$\lambda(a) = a(t)\Lambda$$

$$\frac{\lambda(1)}{\lambda(a)} = \frac{1}{a} \equiv (1 + z)$$

$$\frac{\delta\lambda}{\lambda} = -\frac{\delta\nu}{\nu} = z$$

- Given known frequency of emission  $\nu(a)$ , redshift can be precisely measured (modulo Doppler shifts from peculiar velocities) – interpreting the redshift as a Doppler shift, objects recede in an expanding universe
- Given a measure of distance,  $D(z) \equiv D(z(a))$  can be measured

# Distance-Redshift Relation

- All distance redshift relations are based on the comoving distance  $D(z)$

$$D(a) = \int dD = \int_a^1 \frac{da'}{a'^2 H(a')} \\ (da = -(1+z)^{-2} dz = -a^2 dz)$$

$$D(z) = - \int_z^0 \frac{dz'}{H(z')} = \int_0^z \frac{dz'}{H(z')}$$

- Note limiting case is the Hubble law

$$\lim_{z \rightarrow 0} D(z) = z/H(z=0) \equiv z/H_0$$

- Hubble constant usually quoted as  $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$ , observationally  $h \sim 0.7$ ; in natural units  $H_0 = (2997.9)^{-1} h \text{ Mpc}^{-1}$  defines an inverse length scale

# Distance-Redshift Relation

- Example: object of known physical size  $\lambda = a(t)\Lambda$  (“standard ruler”) subtending an (observed) angle on the sky  $\alpha$

$$\begin{aligned}\alpha &= \frac{\Lambda}{D_A(z)} = \frac{\lambda}{aR \sin(D(z)/R)} \\ &= \frac{\lambda}{R \sin(D(z)/R)} (1+z) \equiv \frac{\lambda}{d_A(z)}, \quad d_A \equiv aD_A\end{aligned}$$

- Example: object of known luminosity  $L$  (“standard candle”) with a measured flux  $F$ . Comoving surface area  $4\pi D_A^2$ , frequency/energy  $(1+z)$ , time-dilation or arrival rate of photons (crests)  $(1+z)$ :

$$\begin{aligned}F &= \frac{L}{4\pi D_A^2} \frac{1}{(1+z)^2} \\ &\equiv \frac{L}{4\pi d_L^2} \quad (d_L = (1+z)D_A = (1+z)^2 d_A)\end{aligned}$$



# Distance-Redshift Relation

- The physical luminosity distance  $d_L$  is the distance that goes into the magnitude or distance modulus relation

$$m - M = 5 \log_{10}(d_L(z)/10\text{pc})$$

- If absolute magnitude is known then distance redshift measures Hubble constant at low  $z = z_0$ :  $d_L \approx z_0/H_0$
- If absolute calibration of standards unknown, measures evolution of Hubble parameter  $H(z)/H_0$ :

$$\frac{d_{A,L}(z)}{d_{A,L}(z_0)} = \frac{H_0}{z_0} d_{A,L}(z)$$

- Alternately, distances & curvature are measured in units of  $h^{-1}$  Mpc.

# Evolution of Scale Factor

- FRW cosmology is fully specified if the function  $a(t)$  is given
- General relativity relates the scale factor with the matter content of universe.
- Build the Einstein tensor  $G_{\mu\nu}$  out of the metric and use Einstein equation

$$G_{\mu\nu} = -8\pi GT_{\mu\nu}$$

$$G^0_0 = -\frac{3}{a^2} \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{1}{R^2} \right]$$
$$G^i_j = -\frac{1}{a^2} \left[ 2\frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 + \frac{1}{R^2} \right] \delta^i_j$$

# Einstein Equations

- Isotropy demands that the stress-energy tensor take the form

$$T^0_0 = \rho$$

$$T^i_j = -p\delta^i_j$$

where  $\rho$  is the energy density and  $p$  is the pressure

- So Einstein equations become

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{R^2} = \frac{8\pi G}{3}a^2\rho$$

$$2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{R^2} = -8\pi Ga^2p$$

$$\text{or } \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -\frac{4\pi G}{3}a^2(\rho + 3p)$$

# Friedman Equations

- More usual to see Einstein equations expressed in time not conformal time

$$\frac{\dot{a}}{a} = \frac{da}{d\eta} \frac{1}{a} = \frac{da}{dt} = aH(a)$$

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{d}{d\eta} \left(\frac{\dot{a}}{a}\right) = a \frac{d}{dt} \left(\frac{da}{dt}\right) = a \frac{d^2 a}{dt^2}$$

- Friedmann equations:

$$H^2(a) + \frac{1}{a^2 R^2} = \frac{8\pi G}{3} \rho$$
$$\frac{1}{a} \frac{d^2 a}{dt^2} = -\frac{4\pi G}{3} (\rho + 3p)$$

- Convenient fiction to describe curvature as an energy density component  $\rho_K = -3/(8\pi G a^2 R^2) \propto a^{-2}$

# Critical Density

- Friedmann equation for  $H$  then reads

$$H^2(a) = \frac{8\pi G}{3}(\rho + \rho_K) \equiv \frac{8\pi G}{3}\rho_c$$

defining a critical density today  $\rho_c$  in terms of the expansion rate

- In particular, its value today is given by the Hubble constant as

$$\rho_c(z = 0) = 3H_0^2/8\pi G = 1.8788 \times 10^{-29} h^2 \text{g cm}^{-3}$$

- Energy density today is given as a fraction of critical

$\Omega \equiv \rho/\rho_c|_{z=0}$ . Radius of curvature then given by

$$R^{-2} = H_0^2(\Omega - 1)$$

- If  $\Omega \approx 1$ ,  $\rho \approx \rho_c$ , then  $\rho_K \ll \rho_c$  or  $H_0 R \ll 1$ , universe is flat across the Hubble distance.  $\Omega < 1$  negatively curved;  $\Omega > 1$  positively curved

# Newtonian Interpretation

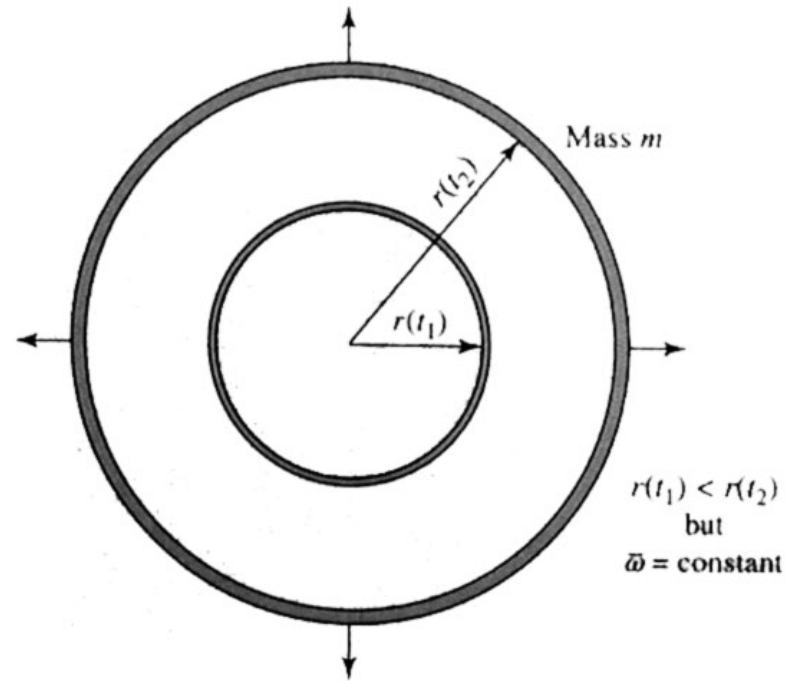
- Consider a test particle of mass  $m$  in expanding spherical region of radius  $r$  and total mass  $M$ . Energy conservation

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{const}$$

$$\frac{1}{2} \left( \frac{dr}{dt} \right)^2 - \frac{GM}{r} = \text{const}$$

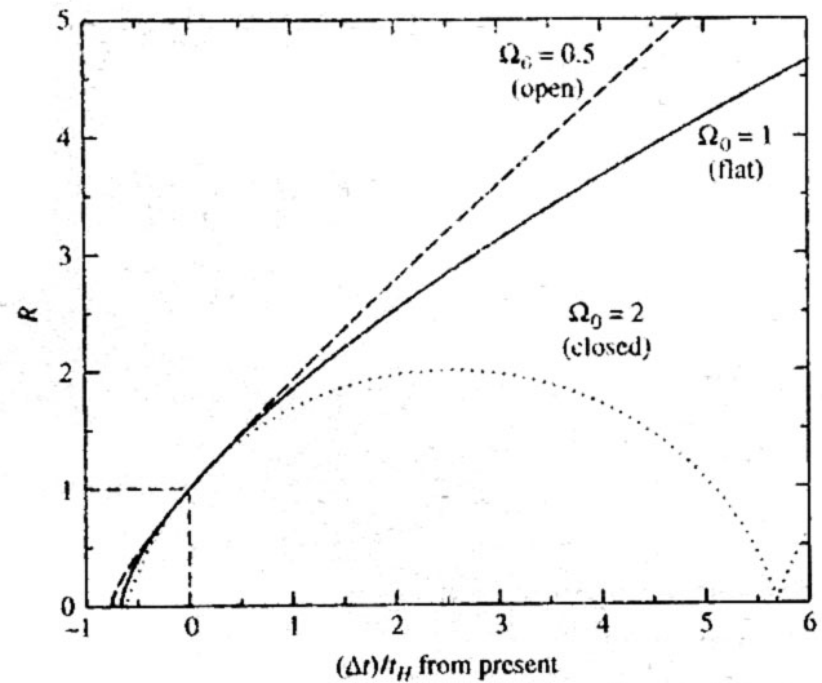
$$\frac{1}{2} \left( \frac{1}{r} \frac{dr}{dt} \right)^2 - \frac{GM}{r^3} = \frac{\text{const}}{r^2}$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{\text{const}}{a^2}$$



# Newtonian Interpretation

- Constant determines whether the system is bound and in the Friedmann equation is associated with curvature – not general since neglects pressure



# Conservation Law

- Second Friedmann equation, or acceleration equation, simply expresses energy conservation (why: stress energy is automatically conserved in GR via Bianchi identity)

$$d\rho V + p dV = 0$$

$$d\rho a^3 + p da^3 = 0$$

$$\dot{\rho} a^3 + 3 \frac{\dot{a}}{a} \rho a^3 + 3 \frac{\dot{a}}{a} p a^3 = 0$$

$$\frac{\dot{\rho}}{\rho} = -3(1 + w) \frac{\dot{a}}{a} \quad w \equiv p/\rho$$

- If  $w = \text{const.}$  then the energy density depends on the scale factor as  $\rho \propto a^{-3(1+w)}$ .



# Multicomponent Universe

- The total energy density can be composed of a sum of components with differing equations of state

$$\rho(a) = \sum_i \rho_i(a) = \sum_i \rho_i(a=1) a^{-3(1+w_i)}, \quad \Omega_i \equiv \rho_i/\rho_c|_{a=1}$$

- Important cases: nonrelativistic matter  $\rho_m = mn_m \propto a^{-3}$ ,  $w_m = 0$ ; relativistic radiation  $\rho_r = En_r = \nu n_r \propto a^{-4}$ ,  $w_r = 1/3$ ; “curvature”  $\rho_K \propto a^{-2}$ ,  $w_K = -1/3$ ; constant energy density or cosmological constant  $\rho_\Lambda \propto a^0$ ,  $w_\Lambda = -1$
- Or generally with  $w_c = p_c/\rho_c = (p + p_K)/(\rho + \rho_K)$

$$\rho_c(a) = \rho_c(a=1) e^{-\int d \ln a 3(1+w_c(a))}$$

$$H^2(a) = H_0^2 e^{-\int d \ln a 3(1+w_c(a))}$$

# Acceleration Equation

- Time derivative of (first) Friedman equation

$$\begin{aligned}2 \frac{1}{a} \frac{da}{dt} \left[ \frac{1}{a} \frac{d^2 a}{dt^2} - H^2(a) \right] &= \frac{8\pi G}{3} \frac{d\rho_c}{dt} \\ \left[ \frac{1}{a} \frac{d^2 a}{dt^2} - \frac{8\pi G}{3} \rho_c \right] &= \frac{4\pi G}{3} [-3(1 + w_c)\rho_c] \\ \frac{1}{a} \frac{d^2 a}{dt^2} &= -\frac{4\pi G}{3} [(1 + 3w_c)\rho_c] \\ &= -\frac{4\pi G}{3} (\rho + \rho_K + 3p + 3p_K) \\ &= -\frac{4\pi G}{3} (1 + 3w)\rho\end{aligned}$$

- Acceleration equation says that universe decelerates if  $w > -1/3$

# Expansion Required

- Friedmann equations “predict” the expansion of the universe. Non-expanding conditions  $da/dt = 0$  and  $d^2a/dt^2 = 0$  require

$$\rho = -\rho_K \quad \rho = -3p$$

i.e. a positive curvature and a total equation of state

$$w \equiv p/\rho = -1/3$$

- Since matter is known to exist, one can in principle achieve this with

$$\rho = \rho_m + \rho_\Lambda = -\rho_K = -3p = 3\rho_\Lambda$$
$$\rho_\Lambda = -\frac{1}{3}\rho_K \quad \rho_m = -\frac{2}{3}\rho_K$$

Einstein introduced  $\rho_\Lambda$  for exactly this reason – “biggest blunder”; but note that this balance is unstable:  $\rho_m$  can be perturbed but  $\rho_\Lambda$ , a true constant cannot

# Dark Energy

- Distance redshift relation depends on energy density components

$$\begin{aligned} H_0 D(z) &= \int \frac{da}{a^2} \frac{H_0}{H(a)} \\ &= \int \frac{da}{a^2} e^{\int d \ln a \frac{3}{2}(1+w_c(a))} \end{aligned}$$

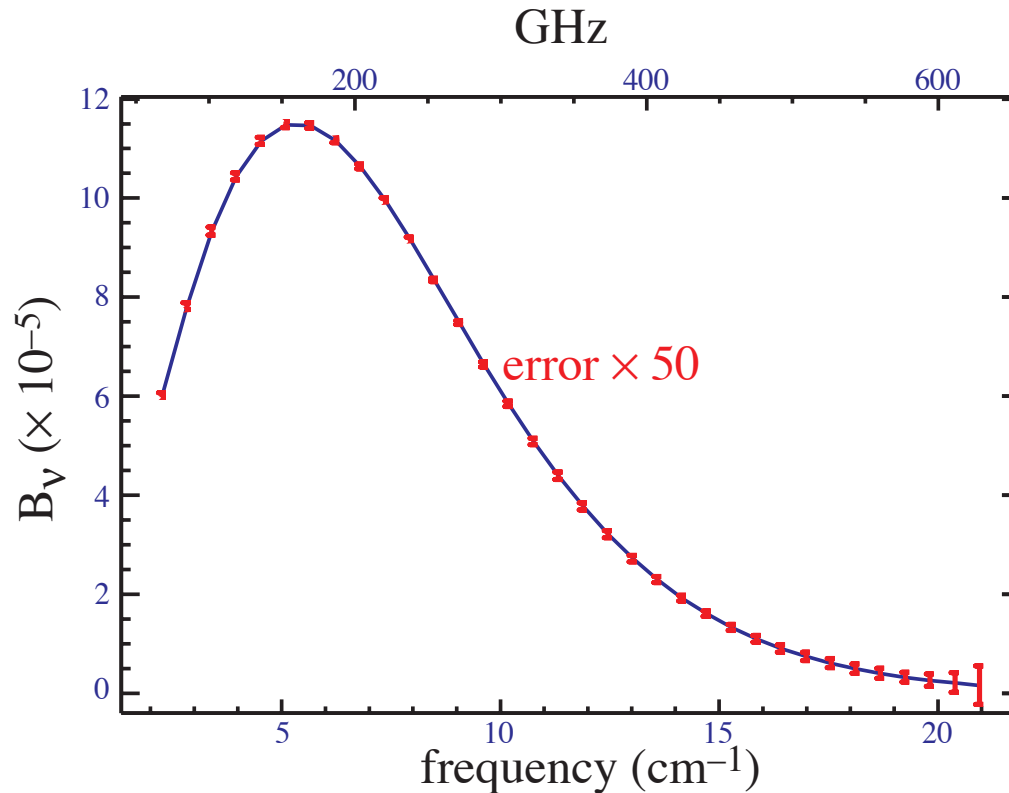
- Distant supernova Ia as standard candles imply that  $w_c < -1/3$  so that the expansion is accelerating
- Consistent with a cosmological constant that is  $\Omega_\Lambda = \rho_\Lambda / \rho_{\text{crit}} = 3/4$  of the total energy density
- Coincidence problem: different components of matter scale differently with  $a$ . Why are (at least) two components comparable today? – Anthropic?

# Cosmic Microwave Radiation

- Existence of a  $\sim 10\text{K}$  radiation background predicted by Gamow and Alpher in 1948 based on the formation of light elements in a hot big bang
- Peebles, Dicke, Wilkinson & Roll in 1965 independently predicted this background and proceeded to build instrument to detect it
- Penzias & Wilson 1965 found unexplained excess isotropic noise in a communications antennae and learning of the Peeble's calculation announced the discovery of the blackbody radiation
- Thermal radiation proves that the universe began in a hot dense state when matter and radiation was in equilibrium - ruling out a competing steady state theory

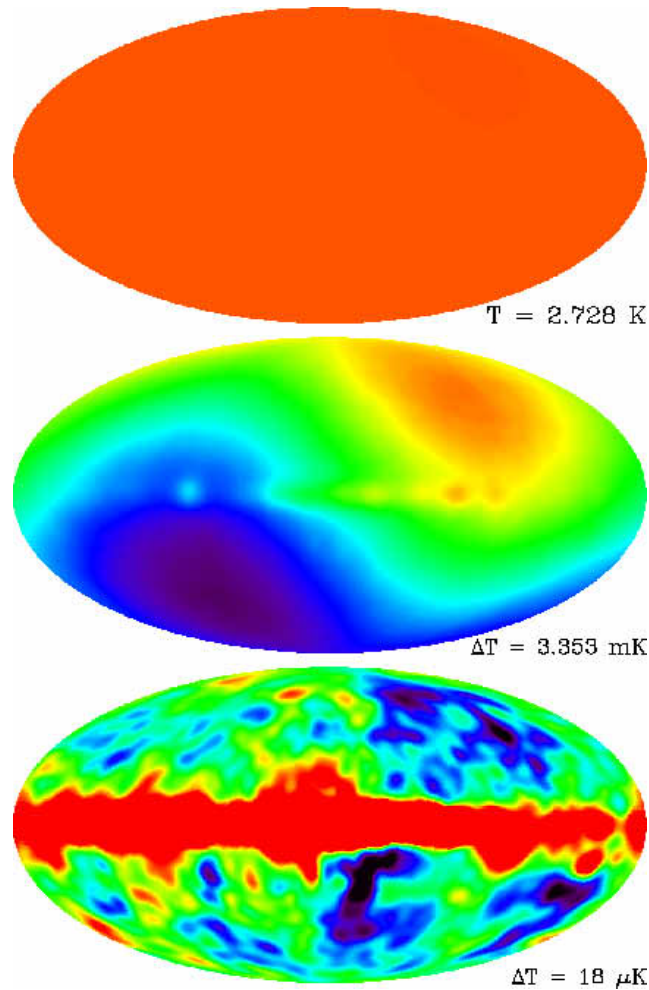
# Cosmic Microwave Radiation

- Modern measurement from COBE satellite of black body spectrum.  $T = 2.725\text{K}$  giving  $\Omega_\gamma h^2 = 2.471 \times 10^{-5}$



# Cosmic Microwave Radiation

- Radiation is isotropic to  $10^{-5}$  in temperature  $\rightarrow$  horizon problem



# Total Radiation

- Adding in neutrinos to the radiation gives the total radiation (next lecture set) content as  $\Omega_r h^2 = 4.153 \times 10^{-5}$
- Since radiation redshifts faster than matter by one factor of  $1 + z$  even this small radiation content will dominate the total energy density at sufficiently high redshift
- Epoch at which matter and radiation equal

$$1 + z_{\text{eq}} = \frac{\Omega_m h^2}{\Omega_r h^2}$$

$$1 + z_{\text{eq}} = 3130 \frac{\Omega_m h^2}{0.13}$$



# Dark Matter

- The mass to light ratio of objects increases up to the scale of galaxy clusters which are large enough samples of the universe that it approaches the universal value
- Since Zwicky in the 1930's non-luminous or dark matter has been known to dominate over luminous matter in stars (and hot gas) in clusters
- Like galactic dark matter based on balance of gravitational force against internal motions - here galaxy velocity dispersion and gas pressure
- Assuming that the object is somehow typical in its non-luminous to luminous density, these measures are converted to an overall dark matter density through a “mass-to-light ratio”

# Dark Matter

- From galaxy surveys the luminosity density in solar units is

$$\rho_L = 2 \pm 0.7 \times 10^8 h L_\odot \text{Mpc}^{-3}$$

( $h$ 's: distances in  $h^{-1}$  Mpc; luminosity inferred from flux  
 $L \propto Sd^2 \propto h^{-2}$ ; inverse volume  $\propto h^3$ )

- Critical density in solar units is  $\rho_c = 2.7754 \times 10^{11} h^2 M_\odot \text{Mpc}^{-3}$   
so that the critical mass-to-light ratio in solar units is

$$\left( \frac{M}{L} \right) \approx 1400h$$

- From the virial velocity of galaxies Zwicky got  $M/L \approx 300h$ .
- Evidence for dark matter in  $X$ -ray clusters also comes from direct hydrostatic equilibrium inference from the gas: balance radial pressure gradient with gravitational potential gradient

# Dark Matter

- Infinitesimal volume of area  $dA$  and thickness  $dr$  at radius  $r$  and interior mass  $M(r)$ : pressure difference supports the gas

$$[p_g(r) - p_g(r + dr)]dA = \frac{GmM}{r^2} = \frac{G\rho_g M}{r^2}dV$$
$$\frac{dp_g}{dr} = -\rho_g \frac{d\Phi}{dr}$$

with  $p_g = \rho_g T_g / m$  becomes

$$\frac{GM}{r} = -\frac{T_g}{m} \left( \frac{d \ln \rho_g}{d \ln r} + \frac{d \ln T_g}{d \ln r} \right)$$

- $\rho_g$  from X-ray luminosity;  $T_g$  sometimes taken as isothermal
- Gravitational lensing mass measurements also consistent with these results
- Infer  $\Omega_m \sim 0.3$

# Dark Matter

- Currently best measurement comes from applying hydrostatic equilibrium arguments for the CMB - photon pressure gradient vs dark matter gravitational potential gradient
- From WMAP 3year data  $\Omega_m h^2 = 0.13 \pm 0.01$
- With  $h = 0.72$ ,  $\Omega_m \sim 0.24$