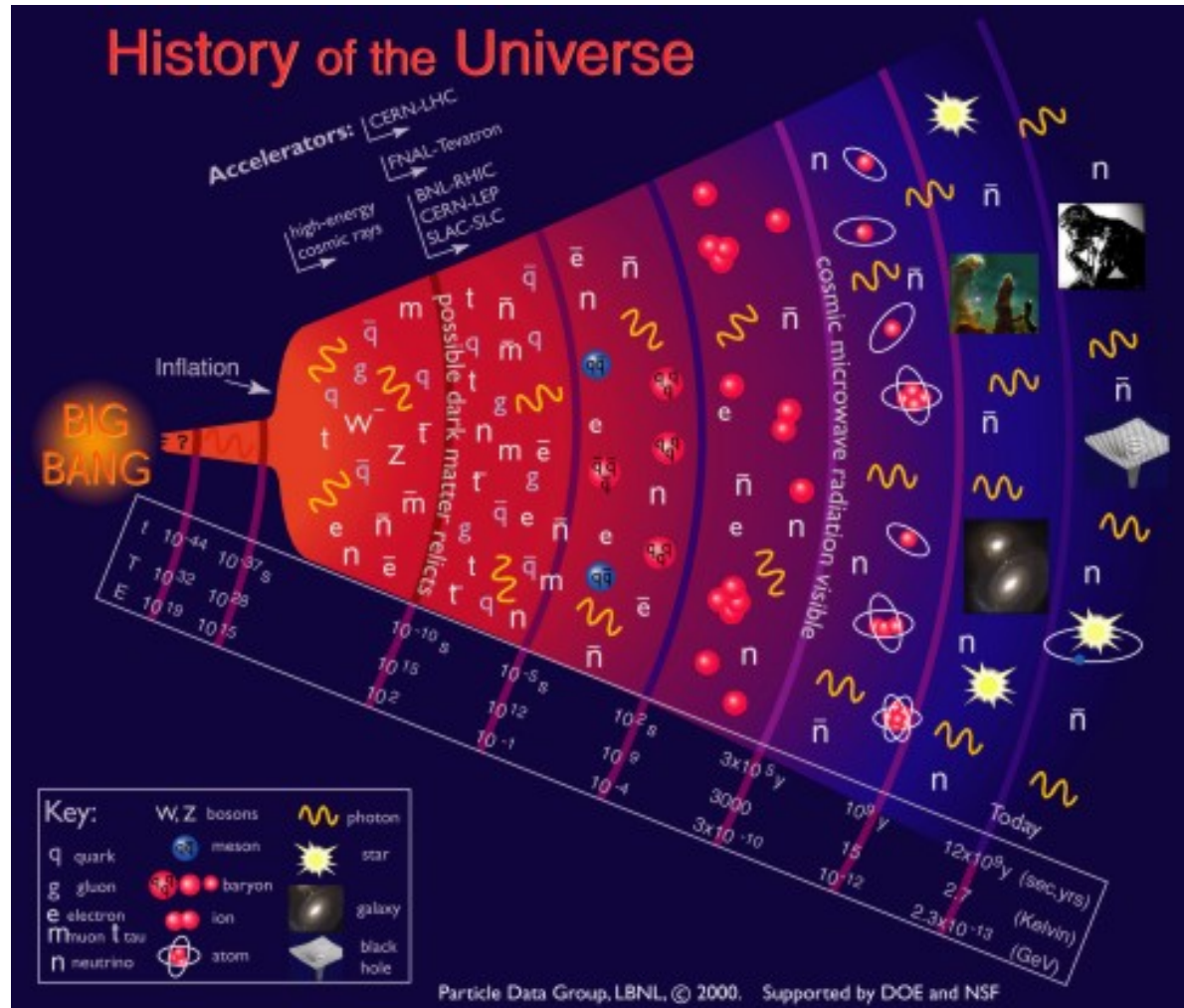


Set 7:

Thermal History

# Brief Thermal History



# Distribution Function

- The distribution function  $f$  gives the number of particles per unit phase space  $d^3x d^3q$  where  $q$  is the momentum (conventional to work in physical coordinates)
- Consider a box of volume  $V = L^3$ . Periodicity implies that the allowed momentum states are given by  $q_i = n_i 2\pi/L$  so that the density of states is

$$dN_s = g \frac{V}{(2\pi)^3} d^3q$$

where  $g$  is the degeneracy factor (spin/polarization states)

- The distribution function  $f(\mathbf{x}, \mathbf{q}, t)$  describes the particle occupancy of these states, i.e.

$$N = \int dN_s f = gV \int \frac{d^3q}{(2\pi)^3} f$$

# Bulk Properties

- Integrals over the distribution function define the bulk properties of the collection of particles
- Number density

$$n(\mathbf{x}, t) \equiv N/V = g \int \frac{d^3q}{(2\pi)^3} f$$

- Energy density

$$\rho(\mathbf{x}, t) = g \int \frac{d^3q}{(2\pi)^3} E(q) f$$

where  $E^2 = q^2 + m^2$

# Bulk Properties

- Pressure: particles bouncing off a surface of area  $A$  in a volume spanned by  $L_x$ : per momentum state

$$\begin{aligned} p_q &= \frac{F}{A} = \frac{N_{\text{part}}}{A} \frac{\Delta q}{\Delta t} \\ &\quad (\Delta q = 2|q_x|, \quad \Delta t = 2L_x/v_x) \\ &= \frac{N_{\text{part}}}{V} |q_x| |v_x| = f \frac{|q||v|}{3} = f \frac{q^2}{3E} \end{aligned}$$

so that summed over states

$$p(\mathbf{x}, t) = g \int \frac{d^3 q}{(2\pi)^3} \frac{|q|^2}{3E(q)} f$$

# Liouville Equation

- Liouville theorem states that the phase space distribution function is conserved along a trajectory in the absence of particle interactions

$$\frac{Df}{Dt} = \left[ \frac{\partial}{\partial t} + \frac{d\mathbf{q}}{dt} \frac{\partial}{\partial \mathbf{q}} + \frac{d\mathbf{x}}{dt} \frac{\partial}{\partial \mathbf{x}} \right] f = 0$$

subtlety in expanding universe is that the de Broglie wavelength of particles changes with the expansion so that

$$q \propto a^{-1}$$

- Homogeneous and isotropic limit

$$\frac{\partial f}{\partial t} + \frac{dq}{dt} \frac{\partial f}{\partial q} = \frac{\partial f}{\partial t} - H(a) \frac{\partial f}{\partial \ln q} = 0$$

# Energy Density Evolution

- Integrate Liouville equation over  $g \int d^3q/(2\pi)^3 E$  to form

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= H(a)g \int \frac{d^3q}{(2\pi)^3} E q \frac{\partial}{\partial q} f \\ &= H(a)g \int \frac{d\Omega}{(2\pi)^3} \int dq q^3 E \frac{\partial}{\partial q} f \\ &= -H(a)g \int \frac{d\Omega}{(2\pi)^3} \int dq \frac{d(q^3 E)}{dq} f \\ &= -H(a)g \int \frac{d\Omega}{(2\pi)^3} \int dq (3q^2 E + q^3 \frac{dE}{dq}) f \\ &\quad \left( \frac{dE}{dq} = \frac{d(q^2 + m^2)^{1/2}}{dq} = \frac{1}{2} \frac{2q}{E} = \frac{q}{E} \right) \\ &= -3H(a)g \int \frac{d^3q}{(2\pi)^3} \left( E + \frac{q^2}{3E} \right) f = -3H(a)(\rho + p)\end{aligned}$$

as derived previously from energy conservation

# Boltzmann Equation

- Boltzmann equation says that Liouville theorem must be modified to account for collisions

$$\frac{Df}{Dt} = C[f]$$

- If collisions are sufficiently rapid, distribution will tend to thermal equilibrium form



# Poor Man's Boltzmann Equation

- Non expanding medium

$$\frac{\partial f}{\partial t} = \Gamma (f - f_{\text{eq}})$$

where  $\Gamma$  is some rate for collisions

- Add in expansion in a homogeneous medium

$$\frac{\partial f}{\partial t} + \frac{dq}{dt} \frac{\partial f}{\partial q} = \Gamma (f - f_{\text{eq}})$$

$$(q \propto a^{-1} \rightarrow \frac{1}{q} \frac{dq}{dt} = -\frac{1}{a} \frac{da}{dt} = H)$$

$$\frac{\partial f}{\partial t} - H \frac{\partial f}{\partial \ln q} = \Gamma (f - f_{\text{eq}})$$

- So equilibrium will be maintained if collision rate exceeds expansion rate  $\Gamma > H$

# Thermal & Diffusive Equilibrium

- A gas in thermal & diffusive contact with a reservoir at temperature  $T$
- Probability of system being in state of energy  $E_i$  and number  $N_i$  (Gibbs Factor)

$$P(E_i, N_i) \propto \exp[-(E_i - \mu N_i)/T]$$

where  $\mu$  is the chemical potential (defines the free energy “cost” for adding a particle at fixed temperature and volume)

- Chemical potential appears when particles are conserved
- CMB photons can carry chemical potential if creation and annihilation processes inefficient, as they are after  $t \sim 1\text{yr}$ .

# Distribution Function

- Mean occupation of the state in thermal equilibrium

$$f \equiv \frac{\sum N_i P(E_i, N_i)}{\sum P(E_i, N_i)}$$

where the total energy is related to the particle energy as

$$E_i = N_i E \text{ (ignoring zero pt)}$$

- Density of (energy) states in phase space makes the net spatial density of particles

$$n = g \int \frac{d^3 p}{(2\pi)^3} f$$

where  $g$  is the number of spin states

# Fermi-Dirac Distribution

- For fermions, the occupancy can only be  $N_i = 0, 1$

$$\begin{aligned} f &= \frac{P(E, 1)}{P(0, 0) + P(E, 1)} \\ &= \frac{e^{-(E-\mu)/T}}{1 + e^{-(E-\mu)/T}} \\ &= \frac{1}{e^{(E-\mu)/T} + 1} \end{aligned}$$

- In the non-relativistic limit

$$E = (p^2 + m^2)^{1/2} \approx m + \frac{1}{2} \frac{p^2}{m}$$

and  $m \gg T$  so the distribution is Maxwell-Boltzmann

$$f = e^{-(m-\mu)/T} e^{-p^2/2mT} = e^{-(m-\mu)/T} e^{-mv^2/2T}$$

# Bose-Einstein Distribution

- For bosons each state can have multiple occupation,

$$f = \frac{\frac{d}{d\mu/T} \sum_{N=0}^{\infty} (e^{-(E-\mu)/T})^N}{\sum_{N=0}^{\infty} (e^{-(E-\mu)/T})^N} \quad \text{with} \quad \sum_{N=0}^{\infty} x^N = \frac{1}{1-x}$$
$$= \frac{1}{e^{(E-\mu)/T} - 1}$$

- Again, non relativistic distribution is Maxwell-Boltzmann

$$f = e^{-(m-\mu)/T} e^{-p^2/2mT} = e^{-(m-\mu)/T} e^{-mv^2/2T}$$

with a spatial number density

$$n = g e^{-(m-\mu)/T} \int \frac{d^3p}{(2\pi)^3} e^{-p^2/2mT}$$
$$= g e^{-(m-\mu)/T} \left( \frac{mT}{2\pi} \right)^{3/2}$$

# Ultra-Relativistic Bulk Properties

- Chemical potential  $\mu = 0$ ,  $\zeta(3) \approx 1.202$
- Number density

$$n_{\text{boson}} = gT^3 \frac{\zeta(3)}{\pi^2} \quad \zeta(n+1) \equiv \frac{1}{n!} \int_0^\infty dx \frac{x^n}{e^x - 1}$$
$$n_{\text{fermion}} = \frac{3}{4} gT^3 \frac{\zeta(3)}{\pi^2}$$

- Energy density

$$\rho_{\text{boson}} = gT^4 \frac{3}{\pi^2} \zeta(4) = gT^4 \frac{\pi^2}{30}$$
$$\rho_{\text{fermion}} = \frac{7}{8} gT^4 \frac{3}{\pi^2} \zeta(4) = \frac{7}{8} gT^4 \frac{\pi^2}{30}$$

- Pressure  $q^2/3E = E/3 \rightarrow p = \rho/3$ ,  $w_r = 1/3$

# Entropy Density

- Second law of thermodynamics

$$dS = \frac{1}{T}(d\rho(T)V + p(T)dV)$$

so that

$$\left. \frac{\partial S}{\partial V} \right|_T = \frac{1}{T}[\rho(T) + p(T)]$$
$$\left. \frac{\partial S}{\partial T} \right|_V = \frac{V}{T} \frac{d\rho}{dT}$$

- Since  $S(V, T) \propto V$  is extensive

$$S = \frac{V}{T}[\rho(T) + p(T)] \quad \sigma = S/V = \frac{1}{T}[\rho(T) + p(T)]$$

# Entropy Density

- Integrability condition  $dS/dV dT = dS/dT dV$  relates the evolution of entropy density

$$\begin{aligned}\frac{d\sigma}{dT} &= \frac{1}{T} \frac{d\rho}{dT} \\ \frac{d\sigma}{dt} &= \frac{1}{T} \frac{d\rho}{dt} = \frac{1}{T} [-3(\rho + p)] \frac{d \ln a}{dt} \\ \frac{d \ln \sigma}{dt} &= -3 \frac{d \ln a}{dt} \quad \sigma \propto a^{-3}\end{aligned}$$

comoving entropy density is conserved in thermal equilibrium

- For ultra relativistic bosons  $\sigma_{\text{boson}} = 3.602 n_{\text{boson}}$ ; for fermions factor of 7/8 from energy density.

$$g_* = \sum_{\text{bosons}} g_b + \frac{7}{8} \sum g_f$$



# Neutrino Freezeout

- Neutrino equilibrium maintained by weak interactions, e.g.  
 $e^+ + e^- \leftrightarrow \nu + \bar{\nu}$
- Weak interaction cross section  $T_{10} = T/10^{10} K \sim T/1\text{MeV}$

$$\sigma_w \sim G_F^2 E_\nu^2 \approx 4 \times 10^{-44} T_{10}^2 \text{cm}^2$$

- Rate  $\Gamma = n_\nu \sigma_w = H$  at  $T_{10} \sim 3$  or  $t \sim 0.2\text{s}$
- After neutrino freezeout, electrons and positrons annihilate dumping their entropy into the photons
- Before  $g_*$ :  $\gamma, e^+, e^- = 2 + \frac{7}{8}(2 + 2) = \frac{11}{2}$
- After  $g_*$ :  $\gamma = 2$ ; so conservation of entropy gives

$$g_* T^3 \Big|_{\text{initial}} = g_* T^3 \Big|_{\text{final}} \quad T_\nu = \left( \frac{4}{11} \right)^{1/3} T_\gamma$$

# Relic Neutrinos

- Relic number density (zero chemical potential; now required by oscillations & BBN)

$$n_\nu = n_\gamma \frac{3}{4} \frac{4}{11} = 112 \text{cm}^{-3}$$

- Relic energy density assuming one species with finite  $m_\nu$ :

$$\rho_\nu = m_\nu n_\nu$$

$$\rho_\nu = 112 \frac{m_\nu}{\text{eV}} \text{eV cm}^{-3} \quad \rho_c = 1.05 \times 10^4 h^2 \text{eV cm}^{-3}$$

$$\Omega_\nu h^2 = \frac{m_\nu}{93.7 \text{eV}}$$

- Candidate for dark matter? an eV mass neutrino goes non relativistic around  $z \sim 1000$  and retains a substantial velocity dispersion  $\sigma_\nu$ .

# Hot Dark Matter

- Momenta for a nonrelativistic species redshifts like temperature for a relativistic one, so average momentum is still given by

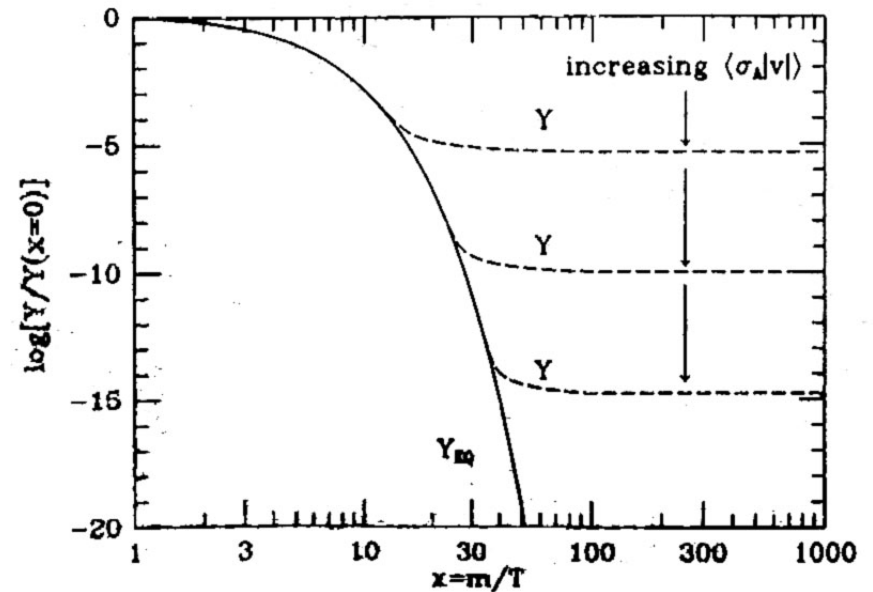
$$\langle q \rangle = 3T_\nu = m\sigma_\nu$$

$$\begin{aligned}\sigma_\nu &= 3 \left( \frac{m_\nu}{1\text{eV}} \right)^{-1} \left( \frac{T_\nu}{1\text{eV}} \right) = 3 \left( \frac{m_\nu}{1\text{eV}} \right)^{-1} \left( \frac{T_\nu}{10^4\text{K}} \right) \\ &= 6 \times 10^{-4} \left( \frac{m_\nu}{1\text{eV}} \right)^{-1} = 200\text{km/s} \left( \frac{m_\nu}{1\text{eV}} \right)^{-1}\end{aligned}$$

on order the rotation velocity of galactic halos and higher at higher redshift - small objects can't form: top down structure formation – not observed – must not constitute the bulk of the dark matter

# Cold Dark Matter

- Problem with neutrinos is they decouple while relativistic and hence have a comparable number density to photons - for a reasonable energy density, the mass must be small
- The equilibrium distribution for a non-relativistic species declines exponentially beyond the mass threshold



$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

# Cold Dark Matter

- Freezeout when annihilation rate equal expansion rate  $\Gamma \propto \sigma_A$ , increasing annihilation cross section decreases abundance
- Appropriate candidates supplied by supersymmetry
- Alternate solution: keep light particle but not created in thermal equilibrium, axion dark matter

# Big Bang Nucleosynthesis

- Most of light element synthesis can be understood through nuclear statistical equilibrium and reaction rates
- Equilibrium abundance of species with mass number  $A$  and charge  $Z$  ( $Z$  protons and  $A - Z$  neutrons)

$$n_A = g_A \left( \frac{m_A T}{2\pi} \right)^{3/2} e^{(\mu_A - m_A)/T}$$

- In chemical equilibrium with protons and neutrons

$$\mu_A = Z\mu_p + (A - Z)\mu_n$$

$$n_A = g_A \left( \frac{m_A T}{2\pi} \right)^{3/2} e^{-m_A/T} e^{(Z\mu_p + (A - Z)\mu_n)/T}$$

# Big Bang Nucleosynthesis

- Eliminate chemical potentials with  $n_p, n_n$

$$e^{\mu_p/T} = \frac{n_p}{g_p} \left( \frac{2\pi}{m_p T} \right)^{3/2} e^{m_p/T}$$

$$e^{\mu_n/T} = \frac{n_n}{g_n} \left( \frac{2\pi}{m_n T} \right)^{3/2} e^{m_n/T}$$

$$n_A = g_A g_p^{-Z} g_n^{Z-A} \left( \frac{m_A T}{2\pi} \right)^{3/2} \left( \frac{2\pi}{m_p T} \right)^{3Z/2} \left( \frac{2\pi}{m_n T} \right)^{3(A-Z)/2} \\ \times e^{-m_A/T} e^{(Z\mu_p + (A-Z)\mu_n)/T} n_p^Z n_n^{A-Z}$$

$$(g_p = g_n = 2; m_p \approx m_n = m_b = m_A/A)$$

$$(B_A = Zm_p + (A - Z)m_n - m_A)$$

$$= g_A 2^{-A} \left( \frac{2\pi}{m_b T} \right)^{3(A-1)/2} A^{3/2} n_p^Z n_n^{A-Z} e^{B_A/T}$$

# Big Bang Nucleosynthesis

- Convenient to define abundance fraction

$$\begin{aligned}
 X_A &\equiv A \frac{n_A}{n_b} = Ag_A 2^{-A} \left( \frac{2\pi}{m_b T} \right)^{3(A-1)/2} A^{3/2} n_p^Z n_n^{A-Z} n_b^{-1} e^{B_A/T} \\
 &= Ag_A 2^{-A} \left( \frac{2\pi n_b^{2/3}}{m_b T} \right)^{3(A-1)/2} A^{3/2} e^{B_A/T} X_p^Z X_n^{A-Z} \\
 &\quad \left( n_\gamma = \frac{2}{\pi^2} T^3 \zeta(3) \quad \eta_{b\gamma} \equiv n_b/n_\gamma \right) \\
 &= A^{5/2} g_A 2^{-A} \left[ \left( \frac{2\pi T}{m_b} \right)^{3/2} \frac{2\zeta(3)\eta_{b\gamma}}{\pi^2} \right]^{A-1} e^{B_A/T} X_p^Z X_n^{A-Z}
 \end{aligned}$$

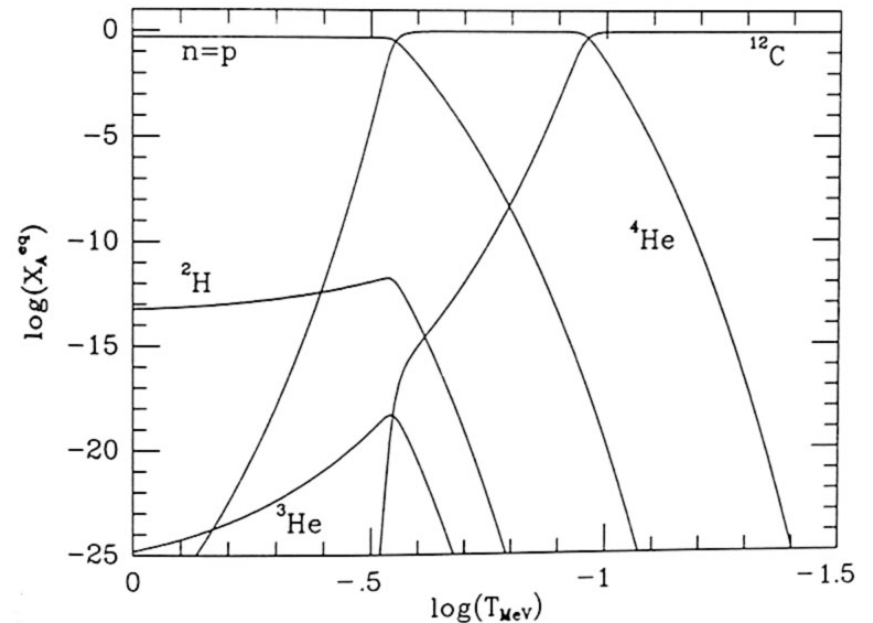


# Deuterium

- Deuterium  $A = 2, Z = 1, g_2 = 3, B_2 = 2.225 \text{ MeV}$

$$X_2 = \frac{3}{\pi^2} \left( \frac{4\pi T}{m_b} \right)^{3/2} \eta_{b\gamma} \zeta(3) e^{B_2/T} X_p X_n$$

- Deuterium  
“bottleneck” is mainly  
due to the low baryon-photon  
number of the universe  
 $\eta_{b\gamma} \sim 10^{-9}$ , secondarily due  
to the low binding energy  $B_2$



# Deuterium

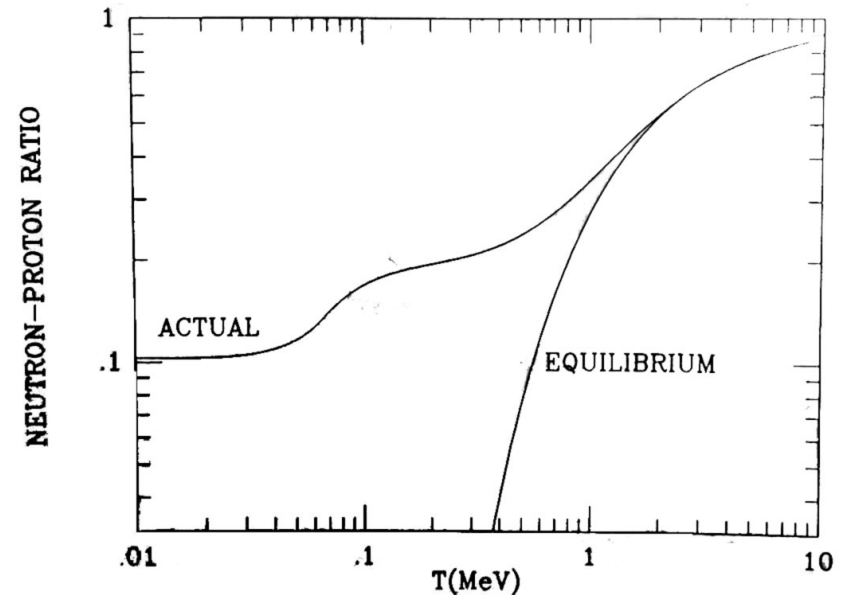
- $X_2/X_p X_n \approx \mathcal{O}(1)$  at  $T \approx 100\text{keV}$  or  $10^9\text{ K}$ , much lower than the binding energy  $B_2$
- Most of the deuterium formed then goes through to helium via  
 $D + D \rightarrow {}^3\text{He} + p, {}^3\text{He} + D \rightarrow {}^4\text{He} + n$
- Deuterium freezes out as number abundance becomes too small to maintain reactions  $n_D = \text{const.}$  The deuterium freezeout fraction  $n_D/n_b \propto \eta_{b\gamma}^{-1} \propto (\Omega_b h^2)^{-1}$  and so is fairly sensitive to the baryon density.
- Observations of the ratio in quasar absorption systems give  $\Omega_b h^2 \approx 0.02$

# Helium

- Essentially all neutrons around during nucleosynthesis end up in Helium
- In equilibrium, the neutron-to-proton ratio is determined by the mass difference

$$Q = m_n - m_p = 1.293 \text{ MeV}$$

$$\frac{n_n}{n_p} = \exp[-Q/T]$$



# Helium

- Equilibrium is maintained through weak interactions, e.g.  
 $n \leftrightarrow p + e^- + \bar{\nu}, \nu + n \leftrightarrow p + e^-, e^+ + n \leftrightarrow p + \bar{\nu}$  with rate

$$\frac{\Gamma}{H} \approx \frac{T}{0.8\text{MeV}}$$

- Freezeout fraction

$$\frac{n_n}{n_p} = \exp[-1.293/0.8] \approx 0.2$$

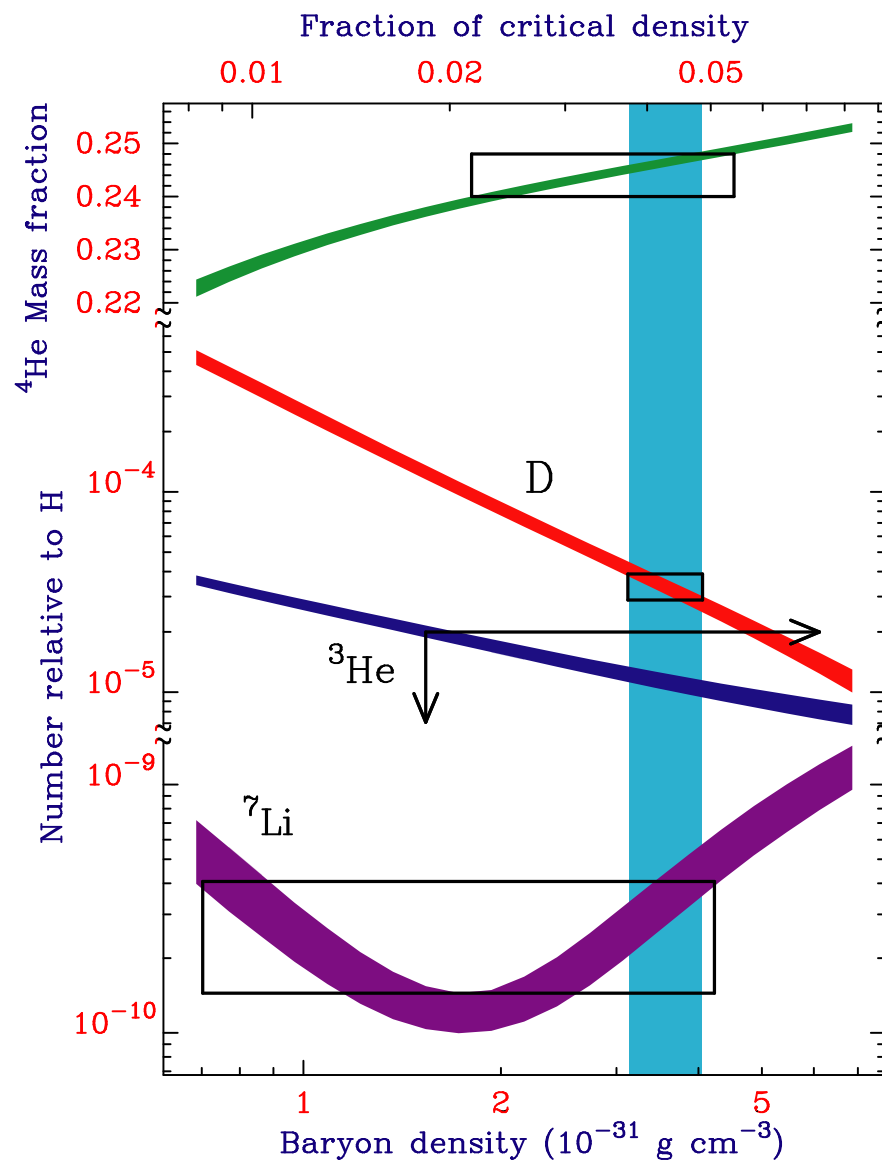
- Finite lifetime of neutrons brings this to  $\sim 1/7$  by  $10^9\text{K}$
- Helium mass fraction

$$\begin{aligned} Y_{\text{He}} &= \frac{4n_{\text{He}}}{n_b} = \frac{4(n_n/2)}{n_n + n_p} \\ &= \frac{2n_n/n_p}{1 + n_n/n_p} \approx \frac{2/7}{8/7} \approx \frac{1}{4} \end{aligned}$$

# Helium

- Depends mainly on the expansion rate during BBN - measure number of relativistic species
- Traces of  ${}^7\text{Li}$  as well. Measured abundances in reasonable agreement with deuterium measure  $\Omega_b h^2 = 0.02$

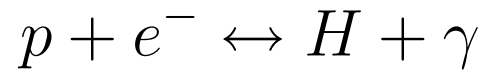
# Light Elements



Burles, Nollett, Turner (1999)

# Recombination

- Maxwell-Boltzmann distribution determines the equilibrium distribution for reactions, e.g. big-bang nucleosynthesis, recombination:



$$\frac{n_p n_e}{n_H} \approx e^{-B/T} \left( \frac{m_e T}{2\pi} \right)^{3/2} e^{(\mu_p + \mu_e - \mu_H)/T}$$

where  $B = m_p + m_e - m_H = 13.6\text{eV}$  is the binding energy,  $g_p = g_e = \frac{1}{2}g_H = 2$ , and  $\mu_p + \mu_e = \mu_H$  in equilibrium

- Define ionization fraction

$$n_p = n_e = x_e n_b$$

$$n_H = n_{\text{tot}} - n_b = (1 - x_e) n_b$$

# Recombination

- Saha Equation

$$\begin{aligned}\frac{n_e n_p}{n_H n_b} &= \frac{x_e^2}{1 - x_e} \\ &= \frac{1}{n_b} \left( \frac{m_e T}{2\pi} \right)^{3/2} e^{-B/T}\end{aligned}$$

- Naive guess of  $T_* = B$  wrong due to the low baryon-photon ratio  
–  $T_* \approx 0.3\text{eV}$  so recombination at  $z_* \approx 1000$
- But the photon-baryon ratio is very low

$$\eta_{b\gamma} \equiv n_b/n_\gamma \approx 3 \times 10^{-8} \Omega_b h^2$$



# Recombination

- Eliminate in favor of  $\eta_{b\gamma}$  and  $B/T$  through

$$n_\gamma = 0.244T^3, \quad \frac{m_e}{B} = 3.76 \times 10^4$$

- Big coefficient

$$\frac{x_e^2}{1 - x_e} = 3.16 \times 10^{15} \left( \frac{B}{T} \right)^{3/2} e^{-B/T}$$

$$T = 1/3\text{eV} \rightarrow x_e = 0.7, \quad T = 0.3\text{eV} \rightarrow x_e = 0.2$$

- Further delayed by inability to maintain equilibrium since net is through  $2\gamma$  process and redshifting out of line

# Recombination

