Astro 321

Lecture Notes Set 3

Wayne Hu

Inhomogeneous Fields

- Like homogeneous cosmology, a full description of the matter fields is given through their phase space distribution function

$$f(\mathbf{x}, \mathbf{q}, t)$$

where the momentum dependence $\mathbf{q}$ describes the bulk motion of the particles

- Energy density and pressure are functions of position

$$\rho(\mathbf{x}, t) = g \int \frac{d^3 q}{(2\pi)^3} f(\mathbf{x}, \mathbf{q}, t) E$$
$$p(\mathbf{x}, t) = g \int \frac{d^3 q}{(2\pi)^3} f(\mathbf{x}, \mathbf{q}, t) \frac{|\mathbf{q}|^2}{3E}$$

and can be considered as low order moments of the distribution function

Inhomogeneous Boltzmann Equation

- Evolution of density inhomogeneities is governed by the Boltzmann equation. Switch over to comoving representation: η, comoving $\mathbf{x}$, retain physical momentum $\mathbf{q}$
- For non-interacting species, Liouville equation

$$\dot{f} + \dot{\mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{q}} + \dot{\mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0$$

- Momentum $\mathbf{q} = q\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a directional unit vector and in a flat universe $\dot{\mathbf{q}} = \dot{q}\hat{\mathbf{n}}$
- Particle velocity $\dot{\mathbf{x}} = \mathbf{q}/E$

$$\dot{f} + \dot{q} \cdot \frac{\partial f}{\partial q} + \frac{\mathbf{q}}{E} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0$$

Energy Equation

- Integrate Boltzmann equation over

$$g \int \frac{d^3 q}{(2\pi)^3} E \ldots$$

- Time term $\partial / \partial \eta \to \dot{\rho}$
- Momentum term: perturbation to scale factor $a(\mathbf{x}) = a(1 + \Phi)$

$$\dot{a}(\mathbf{x}) = \dot{a}(1 + \Phi) + a\dot{\Phi}$$

$$\frac{\dot{a}(\mathbf{x})}{a(\mathbf{x})} \approx \frac{\dot{a}}{a} + \frac{\dot{\Phi}}{1 + \Phi} \approx \frac{\dot{a}}{a} + \dot{\Phi}$$

$$\dot{q} = -\left(\frac{\dot{a}}{a} + \dot{\Phi}\right) q - (\nabla \Psi \cdot \hat{\mathbf{n}}) E$$

Energy Equation

Non-relativistic: gravitational force $\dot{q} = F = -m\nabla\Psi \cdot \hat{\mathbf{n}}$

Relativistic: gravitational redshift $\dot{q} = \dot{E} = \dot{x}\nabla E = (\mathbf{q}/E) \cdot q\nabla\Psi$ $= -E\nabla\Psi \cdot \hat{\mathbf{n}}$. Both: de Broglie redshift

- Combined momentum terms (since f is to leading order isotropic)

$$g \int \frac{d^3q}{(2\pi)^3} \dot{q} E \frac{\partial f}{\partial q} \approx 3\left[\frac{\dot{a}}{a} + \dot{\Phi}\right](\rho + p)$$

- Position term: define average momentum as momentum density

$$\nabla \cdot g \int \frac{d^3q}{(2\pi)^3} \mathbf{q} f \equiv \nabla \cdot (\rho + p)\mathbf{v}$$

- Linearized energy/continuity equation

$$\dot{\rho} = -3\left[\frac{\dot{a}}{a} + \dot{\Phi}\right](\rho + p) - \nabla \cdot (\rho + p)\mathbf{v}$$

Momentum Equation

- Closure requires an evolution equation for momentum
- Integrate Boltzmann equation over

$$g \int \frac{d^3q}{(2\pi)^3} \mathbf{q} \ldots$$

- Time term

$$\frac{\partial}{\partial \eta} \to \frac{\partial}{\partial \eta}[(\rho + p)\mathbf{v}]$$

- Momentum term: de Broglie redshift

$$\begin{aligned} -[\frac{\dot{a}}{a} + \dot{\Phi}]g \int \frac{d^3q}{(2\pi)^3} \mathbf{q} \frac{\partial f}{\partial q} &= 4[\frac{\dot{a}}{a} + \dot{\Phi}]g \int \frac{d^3q}{(2\pi)^3} \mathbf{q} f \\ &= 4\left[\frac{\dot{a}}{a} + \dot{\Phi}\right](\rho + p)\mathbf{v} \approx 4\frac{\dot{a}}{a}(\rho + p)\mathbf{v} \end{aligned}$$

Momentum Equation

- Momentum term: gravitational potential jth component

$$-\partial_i \Psi \cdot g \int \frac{d^3q}{(2\pi)^3} q E n_j n^i \frac{\partial f}{\partial q} \approx \partial_j \Psi (\rho + p)$$

where angle averaged $\langle n^i n_j \rangle = \frac{1}{3}\delta^i{}_j$ and used relation from homogeneous energy equation

- Spatial term: recall stress tensor divided into isotropic and anisotropic pieces

$$g \int \frac{d^3q}{(2\pi)^3} \frac{q^i q_j}{E} f \equiv p\delta^i{}_j + \pi^i{}_j$$

- Combined momentum terms

$$\frac{\partial}{\partial \eta}[(\rho + p)v^i] = -4\frac{\dot{a}}{a}(\rho + p)v^i - \partial^i p - \partial^j \pi^i{}_j - (\rho + p)\partial^i \Psi$$

Boltzmann Hierarchy

- Momentum equation is Navier-Stokes equation. Unless stress tensor is specified, equation is not closed
- In general, the time derivative of a low order moment of the Boltzmann equation is given by the spatial gradient of higher order moments (here anisotropic stress)
- Microphysics closes the Boltzmann equation. Energy and momentum equations simply reflect conservation of the stress energy tensor and is valid for *any* component of matter – even things like cosmological defects.

Density Fluctuation

- Given homogeneity on the large scale, it is useful to define the density fluctuation

$$\delta(\mathbf{x}, t) = \frac{\rho(\mathbf{x}, t) - \langle \rho(\mathbf{x}, t) \rangle}{\langle \rho(\mathbf{x}, t) \rangle}$$

  where $\delta \ll 1$ on large scales

- Evolution of the density fluctuation is given by Boltzmann equation as a partial differential equation
- For small fluctuations, evolution equations are linear and decouple in harmonic space
- In a spatially flat cosmology plane waves form a complete and orthogonal set of harmonics. In general, eigenfunctions of the Laplace operator form a complete set

$$\nabla^2 Q = -k^2 Q \quad \rightarrow \quad Q = e^{i\mathbf{k}\cdot\mathbf{x}}$$

Fourier Conventions

- Often required to relate harmonics in a finite (e.g. survey) volume to infinite volume
- Periodicity: assume a 1D field $F(x)$ periodic in finite volume of length L

$$F(x+L) = F(x)$$
$$= \sum_n F(k_n) e^{-ik_n x - ik_n L}$$
$$= \sum_n F(k_n) e^{-ik_n x} = F(x) \quad \text{if } k_n = \frac{2\pi}{L} n$$

- Reality:

$$F^*(x) = \sum_n F^*(k_n) e^{ik_n x} = F(x) = \sum_n F(k_n) e^{-ik_n x}$$

$$F^*(k_n) = F(-k_n)$$

Fourier Conventions

- Band limited: function has no high frequency structure, e.g. because smoothed

$$k_n < k_{\rm max} \equiv \frac{2\pi}{L}\frac{N}{2}$$

$$F(x) = \sum_{n=-N/2}^{N/2} F(k_n) e^{-ik_n x}$$

- Sampling theorem: sampling at a rate $\Delta = L/N$ is sufficient to reconstruct field exactly. Inverse relation

$$F(k_n) = \frac{1}{N}\sum_{m=0}^{N-1} F(x_m) e^{ik_n x_m}\,, \quad x_m = m\Delta$$

Fourier Conventions

- δ (Kronecker) function

$$F(k_n) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n'=-N/2}^{N/2} F(k_{n'})e^{-i(k_{n'}-k_n)x_m}$$

if $n' = n$ then

$$\frac{1}{N}\sum_{m=0}^{N-1} e^{-i(k_{n'}-k_n)x_m} = \frac{1}{N}\sum_{m=0}^{N-1} 1 = 1$$

if $n' \neq n$ then

$$e^{-i(k_{n'}-k_n)x_m} = \cos(k_n - k_{n'})x_m + i\sin(k_n - k_{n'})x_m$$

Fourier Conventions

$$\sum_{m=0}^{N-1} \cos[(k_n - k_{n'})\frac{2\pi m}{N}] = \frac{\sin[(N-\frac{1}{2})(n-n')\frac{2\pi}{N}]}{2\sin[(n-n')\frac{\pi}{N}]} + \frac{1}{2}$$

$$(N(n-n')2\pi/N = 2\pi(n-n')\,\text{where}\,n-n'\,\text{integer})$$

$$= -\frac{\sin[(n-n')\frac{\pi}{N}]}{2\sin[(n-n')\frac{\pi}{N}]} + \frac{1}{2} = 0$$

$$\sum_{m=0}^{N-1} \sin[(k_n - k_{n'})\frac{2\pi m}{N}] = \frac{\sin[(n-n')\pi]\ \sin[\frac{N-1}{N}(n-n')\pi]}{\sin[(n-n')\frac{\pi}{N}]} = 0$$

$$\rightarrow \frac{1}{N}\sum_{m=0}^{N-1} e^{-i(k_{n'}-k_n)x_m} = \delta_{nn'}$$

Fourier Conventions

- Two point correlation

$$\langle F(x)F(x')\rangle = \sum_{nn'} \langle F^*(k_n)F(k_{n'})\rangle e^{ik_n x - ik_{n'}x'}$$

- Translational invariance

$$\langle F(x+d)F(x'+d)\rangle = \langle F(x)F(x')\rangle$$

$$\sum_{nn'} \langle F^*(k_n)F(k_{n'})\rangle e^{ik_n x - ik_{n'}x'} e^{i(k_n - k_{n'})d} = \sum_{nn'} \langle F^*(k_n)F(k_{n'})\rangle e^{ik_n x - ik_{n'}x'}$$

$$\langle F^*(k_n)F(k_{n'})\rangle = \delta_{nn'} P_F(k_n)$$

two point statistical properties are given by the power spectrum P_F and correlation function depends only on separation

$$\langle F(x)F(x')\rangle = \xi(x - x')$$

Fourier Conventions

- Continuous conventions: let $L \to \infty$, density of k_n states gets high

$$\sum_n \to \int dn$$

- Forward and inverse transform

$$F(x) = \sum_{n=-N/2}^{N/2} F(k_n) e^{-ik_n x} = \int_{-N/2}^{N/2} dn F(k_n) e^{-ik_n x},$$

$$= L \int_{-k_{\max}}^{k_{\max}} \frac{dk}{2\pi} F(k) e^{-ikx} \qquad (dk_n = dn \frac{2\pi}{L})$$

$$F(k) = \frac{1}{N} \sum_{m=0}^{N-1} F(x_m) e^{ikx_m} = \frac{1}{L} \int dx F(x) e^{ikx} \qquad dx_m = \frac{L}{N} dm$$

Fourier Conventions

- The (Dirac) δ function

$$\delta_{nn'} = \frac{1}{N}\sum_{m=0}^{N-1} e^{-i(k_{n'}-k_n)x_m} = \frac{1}{L}\int dx e^{-i(k_{n'}-k_n)x_m}$$

$$F(k_n) = \sum_{n'} F(k_{n'})\delta_{nn'} = \int dn' F(k_{n'})\delta_{nn'} = \frac{L}{2\pi}\int dk'_{n'} F(k_{n'})\delta_{nn'}$$

- Define the δ function as

$$\int dk' F(k')\delta(k-k') = F(k)$$

$$\text{then} \quad \delta(k-k') = \frac{L}{2\pi}\delta_{nn'} = \frac{1}{2\pi}\int dx e^{i(k-k')x}$$

$$\langle F^*(k)F(k')\rangle = \frac{2\pi}{L}\delta(k-k')P_F(k)$$

Fourier Conventions

- 3D Fields

$$F(\mathbf{x}) = V \int \frac{d^3k}{(2\pi)^3} F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$F(\mathbf{k}) = \frac{1}{V} \int d^3x F(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$(2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') = \int d^3x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}$$

$$\langle F^*(\mathbf{k}) F(\mathbf{k}') \rangle = \frac{(2\pi)^3}{V} \delta(\mathbf{k} - \mathbf{k}') P_F(\mathbf{k})$$

- Statistical isotropy: $P_F(\mathbf{k}) = P_F(k)$

Fourier Conventions

- Suppress volume terms by making Fourier representation dimensionful $\tilde{F}(\mathbf{k}) \equiv V F(\mathbf{k})$, $\tilde{P}_F = V P_F$

$$F(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \tilde{F}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$\tilde{F}(\mathbf{k}) = \int d^3x F(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\begin{aligned}
\langle \tilde{F}^*(\mathbf{k}) \tilde{F}(\mathbf{k}') \rangle &= (2\pi)^3 V \delta(\mathbf{k} - \mathbf{k}') P_F(\mathbf{k}) \\
&= (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \tilde{P}_F(\mathbf{k})
\end{aligned}$$

- Hereafter, suppress $\sim$, power spectra have dimensions of volume
- So: what does it mean to have a large fluctuation in power?

Fourier Conventions

- Variance

$$\sigma_F^2 \equiv \langle F(\mathbf{x})F(\mathbf{x})\rangle = \int \frac{d^3k}{(2\pi)^3} P_F(k)$$
$$= \int \frac{k^2 dk}{2\pi^2} \int \frac{d\Omega}{4\pi} P_F(k)$$
$$= \int d\ln k \frac{k^3}{2\pi^2} P_F(k)$$

- Define power per logarithmic interval

$$\Delta_F^2(k) \equiv \frac{k^3 P_F(k)}{2\pi^2}$$

- This quantity is dimensionless in all representations. Serves as a definition of the linear regime k's where $\Delta_\delta^2 \ll 1$

Linearity

- Fields related by a linear equation obey equation independent equations

$$F(\mathbf{x}) = AG(\mathbf{x}) + B \quad \rightarrow \quad F(\mathbf{k}) = AG(\mathbf{k}) \qquad (k > 0)$$

includes linear differential equation

$$F(\mathbf{x}) = A\nabla G(\mathbf{k}) + B$$

$$F(\mathbf{k}) = A\int d^3x e^{i\mathbf{k}\cdot\mathbf{x}}\nabla \int \frac{d^3k'}{(2\pi)^3} e^{-i\mathbf{k}'\cdot\mathbf{x}} G(\mathbf{k}')$$

$$= A\int \frac{d^3k'}{(2\pi)^3} \int d^3x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}(-i\mathbf{k}')G(\mathbf{k}') = A(-i\mathbf{k})G(\mathbf{k})$$

converts differential equations to algebraic relations

Convolution

- Convolution in real space often occurs – smoothing of field by finite resolution

$$F_W(\mathbf{x}) = \int d^3y W(\mathbf{x}-\mathbf{y})F(\mathbf{y})$$

$$= \int d^3y \int \frac{d^3k}{(2\pi)^3} W(\mathbf{k}) e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \int \frac{d^3k'}{(2\pi)^3} F(\mathbf{k}') e^{-i\mathbf{k}'\cdot\mathbf{y}}$$

$$= \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} W(\mathbf{k}) F(\mathbf{k}') \int d^3y e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{y}}$$

$$= \int \frac{d^3k}{(2\pi)^3} W(\mathbf{k}) F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$F_W(\mathbf{k}) = W(\mathbf{k})F(\mathbf{k})$$

- Smoothing acts as a low pass filter: if $W(\mathbf{x})$ is a broad function of width L, $W(\mathbf{k})$ suppressed for $k > 2\pi/L$

Convolution

- Filtered Variance

$$\langle F_W(\mathbf{x}) F_W(\mathbf{x}) \rangle = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \langle F^*(\mathbf{k}) F(\mathbf{k}') \rangle W^*(\mathbf{k}) W(\mathbf{k}')$$
$$= \int \frac{d^3k}{(2\pi)^3} P_F(k) |W(\mathbf{k})|^2$$

- Common filter is the spherical tophat:

$$W_R(\mathbf{x}) = 1 \qquad x < R$$
$$W_R(\mathbf{x}) = 0 \qquad x > R$$

- Fourier transform

$$W_R(\mathbf{k}) = \frac{3}{y^3}(\sin y - y \cos y) \,, \qquad (y = kR)$$

Normalization

- Normalization is often quoted as the top hat rms of the density field

$$\sigma_R^2 = \int d\ln k\, \Delta_\delta^2(k)|W_R(k)|^2$$

where observationally $\sigma_{8h^{-1}\mathrm{Mpc}} \equiv \sigma_8 \approx 1$

- Note that $\Delta_\delta^2(k)$ itself can be thought of as the variance of the field with a filter that has sharp high and low pass filters in k-space

Linear Perturbation Theory

- Energy (continuity) and momentum (Navier-Stokes) equations are linearized and hence Fourier modes obey

$$\frac{\partial}{\partial\eta}[(\rho+p)v^i] = -4\frac{\dot{a}}{a}(\rho+p)v^i + +ikp + ik^j\pi^i{}_j + ik^i(\rho+p)\Psi$$

- If the source of perturbations is from the (scalar) gravitational potential, directional dependence of velocity and anisotropic stress follows the direction of the plane wave, so define scalar velocity and anisotropic stress as

$$\mathbf{v}(\mathbf{k}) = i\hat{\mathbf{k}}v$$

$$\pi^i{}_j(\mathbf{k}) = \left(-\hat{k}^i\hat{k}_j + \frac{1}{3}\delta^i{}_j\right)p\pi$$

Linear Perturbation Theory

- Navier-Stokes equation

$$\frac{\partial}{\partial \eta}[(\rho + p)v] = -4\frac{\dot{a}}{a}(\rho + p)v^i + kp - \frac{2}{3}kp\pi + (\rho + p)k\Psi$$

$$(w = p/\rho\,, \quad c_s^2 = \delta p/\delta\rho\,, \quad \dot{\rho}/\rho = -3(1+w)\dot{a}/a)$$

$$\dot{v} = -(1-3w)\frac{\dot{a}}{a}v - \frac{\dot{w}}{1+w}v + \frac{kc_s^2}{1+w}\delta - \frac{2}{3}\frac{w}{1+w}k\pi + k\Psi$$

- Continuity Equation

$$\dot{\rho} = -3\left[\frac{\dot{a}}{a} + \dot{\Phi}\right](\rho + p) + i\mathbf{k}\cdot(\rho + p)\mathbf{v}$$

$$\dot{\rho} = -3\left[\frac{\dot{a}}{a} + \dot{\Phi}\right](\rho + p) - k(\rho + p)v$$

$$\dot{\delta} = -3\frac{\dot{a}}{a}(c_s^2 - w)\delta - (1+w)(kv + 3\dot{\Phi})$$

Poisson Equation

- Naive expectation: $\Phi = -\Psi$ and

$$\nabla^2 \Phi = -4\pi G a^2 \delta\rho$$

$$k^2 \Phi = 4\pi G a^2 \rho\delta$$

where a^2 comes from physical $\rightarrow$ comoving and $\delta\rho$ since background density goes into scale factor evolution

- Einstein equations put in a relativistic correction (flat universe)

$$k^2 \Phi = 4\pi G a^2 \rho[\delta + 3\frac{\dot{a}}{a}(1+w)v/k]$$

$$k^2(\Phi + \Psi) = -8\pi G a^2 p\pi$$

- convenient to call combination

$$\Delta \equiv \delta + 3\frac{\dot{a}}{a}(1+w)v/k$$

Constancy of Potential & Growth Rate

- Given the Poisson equation relates a redshifting total density ρ and the comoving derivative factor a the density perturbation must grow as $\Delta \propto (a^2\rho)^{-1} \propto a^{1-3w}$ to maintain a constant potential.
- Density perturbations are stabilized by the expanding universe (expansion drag) and do not grow exponentially. Presents a new version of the horizon problem.
- Naive (Newtonian) argument: in the absence of stress perturbations the Euler equation takes the form $\dot{v} \sim k\Psi$
- Given an initial potential perturbation Ψ_i a velocity perturbation $v \sim (k\eta)\Psi_i$
- Given a velocity perturbation continuity grows a density fluctuation as $\dot{\Delta} \sim -kv$ or $\Delta = -(k\eta)^2\Psi_i$.

Constancy of Potential & Growth Rate

- The growing density perturbation is exactly that required to maintain the potential constant

$$\Psi \approx -\frac{4\pi G a^2 \rho}{k^2}\Delta \approx \frac{4\pi G a^2 \rho}{k^2}(k\eta)^2\Psi_i$$

$\eta \propto a^{(1+3w)/2}$, $a^2\rho \propto a^{-(1+3w)}$

- Under gravity alone, the density fluctuations grow just fast enough to maintain constant potentials
- Stress fluctuations only decrease the rate of growth of the potential. Starting from an unperturbed $\Psi_i = 0$ universe, where do the fluctuations that form large scale structure come from

Bardeen Curvature

- A proper relativistic generalization involves the $(\dot{a}/a)v/k$ corrections, called the Bardeen (or comoving) curvature

$$\zeta \equiv \Phi - \frac{\dot{a}}{a} v/k \, .$$

- Continuity equation becomes

$$\dot{\Delta} = -3\frac{\dot{a}}{a}\left(C_s^2 - w\right)\Delta - (1+w)(kv + 3\dot{\zeta}) \, ,$$

where the transformed sound speed

$$C_s^2 \equiv \frac{\Delta p}{\Delta \rho}$$

$$\Delta p \equiv \delta p - \dot{p} v/k$$

Bardeen Curvature

- Euler equation becomes

$$\dot{\zeta} = \frac{\dot{a}}{a}\xi$$

$$\xi = -\frac{C_s^2}{1+w}\Delta + \frac{2}{3}\frac{w}{1+w}\pi \,.$$

so that the Bardeen curvature only changes in the presence of stress fluctuations – scales below the horizon

- Extremely useful result (proven in problem set) says that calculated ζ once and for all – e.g. during formation in an inflationary epoch

Bardeen Curvature

- Relationship to gravitational potential: (from Poisson & conservation equations)

$$\frac{\dot a}{a}\Psi - \dot\Phi = 4\pi G a^2(\rho + p)v/k$$

so that if Φ constant and $\Psi = -\Phi$ then

$$-\left(\frac{\dot a}{a}\right)^2 \Phi = 4\pi G a^2 \rho(1+w)\frac{\dot a}{a} v/k$$

$$= \frac{3}{2}\left(\frac{\dot a}{a}\right)^2 (1+w)\frac{\dot a}{a} v/k$$

$$\frac{\dot a}{a} v/k = -\frac{2}{3(1+w)}\Phi \quad \rightarrow \zeta = 1 + \frac{2}{3(1+w)}\Phi$$

- Matter dominated $\Phi = 3\zeta/5$, radiation dominated $\Phi = 2\zeta/3$, Λ dominated $\Phi \rightarrow 0$.