Astro 321

Lecture Notes Set 4

Wayne Hu

Horizon Problem

- The horizon in a decelerating universe scales as $\eta \propto a^{(1+3w)/2}$, $w > -1/3$. For example in a matter dominated universe

$$\eta \propto a^{1/2}$$

- CMB decoupled at $a_* = 10^{-3}$ so subtends an angle on the sky

$$\frac{\eta_*}{\eta_0} = a_*^{1/2} \approx 0.03 \approx 2^\circ$$

- So why is the CMB sky isotropic to 10^{-5} in temperature if it is composed of $\sim 10^4$ causally disconnected regions
- If smooth by fiat, why are there 10^{-5} fluctuations correlated on superhorizon scales

Flatness & Relic Problems

- Flatness problem: why is the radius of curvature larger than the observable universe. (Before the CMB determinations, why is it at least comparable to observable universe $|\Omega_K| \lesssim \Omega_m$)
- Also phrased as a coincidence problem: since $\rho_K \propto a^{-2}$ and $\rho_m \propto a^{-3}$, why would they be comparable today – modern version is dark energy coincidence $\rho_\Lambda =$ const.
- Relic problem – why don't relics like monopoles dominate the energy density
- Inflation is a theory that solves all three problems at once and also supplies a source for density perturbations

Accelerating Expansion

- In a matter or radiation dominated universe, the horizon grows as a power law in a so that there is no way to establish causal contact on a scale longer than the inverse Hubble length ($1/aH$, comoving coordinates) at any given time: general for a decelerating universe

$$\eta = \int d\ln a \frac{1}{aH(a)}$$

- $H^2 \propto \rho \propto a^{-3(1+w)}$, $aH \propto a^{-(1+3w)/2}$, critical value of $w = -1/3$, the division between acceleration and deceleration
- In an accelerating universe, the Hubble length shrinks in comoving coordinates and so the horizon gets its contribution at the earliest times, e.g. in a cosmological constant universe, the horizon saturates to a constant value

Causal Contact

- Note confusion in nomenclature: the true horizon always grows meaning that one always sees more and more of the universe. The Hubble length decreases: the difference in conformal time, the distance a photon can travel between two epochs denoted by the scale factor decreases. Regions that were in causal contact, leave causal contact.
- Horizon problem solved if the universe was in an acceleration phase up to η_i and the conformal time since then is shorter than the total conformal age

$$\eta_0 \gg \eta_0 - \eta_i$$

$$\text{total distance} \gg \text{distance traveled since inflation}$$

$$\text{apparent horizon}$$

Flatness & Relic

- Comoving radius of curvature is constant and can even be small compared to the full horizon $R \ll \eta_0$ yet still $\eta_0 \gg R \gg \eta_0 - \eta_i$
- In physical coordinates, the rapid expansion of the universe makes the current observable universe much smaller than the curvature scale
- Likewise, the number density of relics formed before the accelerating (or inflationary) epoch is diluted to make them rare in the current observable volume
- Common to reference time to the end of inflation $\tilde{\eta} \equiv \eta - \eta_i$. Here conformal time is negative during inflation and its value (as a difference in conformal time) reflects the comoving Hubble length - defines leaving the horizon as $k|\tilde{\eta}| = 1$

Exponential Expansion

- If the accelerating component has equation of state $w = -1$, $\rho =$ const., $H = H_i$ const. so that $a \propto \exp(Ht)$

$$
\begin{aligned}
\tilde{\eta} &= -\int_a^{a_i} d\ln a \frac{1}{aH} = \frac{1}{aH_i}\Big|_a^{a_i} \\
&\approx -\frac{1}{aH_i} \qquad (a_i \gg a)
\end{aligned}
$$

- In particular, the current horizon scale $H_0\tilde{\eta}_0 \approx 1$ exited the horizon during inflation at

$$
\begin{aligned}
\tilde{\eta}_0 \approx H_0^{-1} &= \frac{1}{a_H H_i} \\
a_H &= \frac{H_0}{H_i}
\end{aligned}
$$

Sufficient Inflation

- Current horizon scale must have exited the horizon during inflation so that the start of inflation could not be after a_H. How long before the end of inflation must it have began?

$$\frac{a_H}{a_i} = \frac{H_0}{H_i a_i}$$

$$\frac{H_0}{H_i} = \sqrt{\frac{\rho_c}{\rho_i}}, \qquad a_i = \frac{T_{\rm CMB}}{T_i}$$

- $\rho_c^{1/4} = 3 \times 10^{-12}$ GeV, $T_{\rm CMB} = 3 \times 10^{-13}$ GeV

$$\frac{a_H}{a_i} = 3 \times 10^{-29} \left(\frac{\rho_i^{1/4}}{10^{14} {\rm GeV}} \right)^{-2} \left(\frac{T_i}{10^{10} {\rm GeV}} \right)$$

$$\ln \frac{a_i}{a_H} = 65 + 2 \ln \left(\frac{\rho_i^{1/4}}{10^{14} {\rm GeV}} \right) - \ln \left(\frac{T_i}{10^{10} {\rm GeV}} \right)$$

Perturbation Generation

- Horizon scale $\tilde{\eta}$ during inflation acts like an even horizon - things leaving causal contact
- Particle creation similar to Hawking radiation from a black hole with hubble length replacing the BH horizon

$$T_{\rm H} \approx H_i$$

- Because H_i remains roughly constant during inflation the result is a scale invariant spectrum of fluctuations due to zero-point fluctuations becoming classical
- Fluctuations in the field driving inflation (inflaton) carry the energy density of the universe and so their zero point fluctuations are net energy density or curvature fluctuations
- Any other light field (gravitational waves, etc...) will also carry scale invariant perturbations but are iso-curvature fluctuations

Slow Roll Inflation

- Single minimally coupled scalar field rolling slowly in a nearly flat potential
- Scalar field equation of motion $V' \equiv dV/d\phi$

$$\nabla_\mu \nabla^\mu \phi + V'(\phi) = 0$$

so that in the background $\phi = \phi_0$ and

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2 V' = 0$$

$$\frac{d^2\phi_0}{dt^2} + 3H\frac{d\phi_0}{dt} + V' = 0$$

- Simply the continuity equation with the associations

$$\rho_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + V \qquad p_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 - V$$

Slow Roll Parameters

- Net energy is dominated by potential energy and so acts like a cosmological constant $w \to -1$
- First slow roll parameter

$$\epsilon = \frac{3}{2}(1+w) = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2$$

- Second slow roll parameter $d^2\phi_0/dt^2 \approx 0$, or $\ddot{\phi}_0 \approx (\dot{a}/a)\dot{\phi}_0$

$$\delta = \frac{\ddot{\phi}_0}{\dot{\phi}_0}\left(\frac{\dot{a}}{a}\right)^{-1} - 1 = \epsilon - \frac{1}{8\pi G}\frac{V''}{V}$$

- Slow roll condition $\epsilon, \delta \ll 1$ corresponds to a very flat potential

Perturbations

- Linearize perturbation $\phi = \phi_0 + \phi_1$ then

$$\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + k^2\phi_1 + a^2V''\phi_1 = 0$$

  in slow roll inflation V'' term negligible

- Implicitly assume that the spatial metric fluctuations (curvature ζ) vanishes, otherwise covariant derivatives pick these up – formal justification is work in that frame and transform back.

- Curvature represents a warping of the scale factor $a \to (1+\zeta)a$ or $\delta a/a = \zeta$

$$\zeta = \frac{\delta a}{a} = \frac{\dot{a}}{a}\delta\eta = \frac{\dot{a}}{a}\frac{\phi_1}{\dot{\phi}_0}$$

  a change in the field value ϕ_1 defines a change in the epoch that inflation ends, imprinting a curvature fluctuation

Slow-Roll Evolution

- Rewrite in $u \equiv a\phi$ to remove expansion damping

$$\ddot{u} + [k^2 - 2\left(\frac{\dot{a}}{a}\right)^2]u = 0$$

- or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\rm end}$$

$$\tilde{\eta} = \int_{a_{\rm end}}^{a} \frac{da}{Ha^2} \approx -\frac{1}{aH}$$

- Compact, slow-roll equation:

$$\ddot{u} + [k^2 - \frac{2}{\tilde{\eta}^2}]u = 0$$

Slow Roll Limit

- Slow roll equation has the exact solution:

$$u = A(k \pm \frac{i}{\tilde{\eta}})e^{\mp ik\tilde{\eta}}$$

- For $|k\tilde{\eta}| \gg 1$ (early times, inside Hubble length) behaves as free oscillator

$$\lim_{|k\tilde{\eta}| \to \infty} u = Ake^{\mp ik\tilde{\eta}}$$

- Normalization A will be set by origin in quantum fluctuations of free field

Slow Roll Limit

- For $|k\tilde{\eta}| \ll 1$ (late times, $\gg$ Hubble length) fluctuation freezes in

$$\lim_{|k\tilde{\eta}|\to 0} u = \pm\frac{i}{\tilde{\eta}} A = \pm iHaA$$

$$\phi_1 = \pm iHA$$

$$\zeta = \mp iHA \left(\frac{\dot{a}}{a}\right) \frac{1}{\dot{\phi}_0}$$

- Slow roll replacement

$$\left(\frac{\dot{a}}{a}\right)^2 \frac{1}{\dot{\phi}_0^2} = \frac{8\pi G a^2 V}{3} \frac{3}{2a^2 V\epsilon} = \frac{4\pi G}{\epsilon} = \frac{4\pi}{\epsilon m_{\rm pl}^2}$$

- Bardeen curvature power spectrum

$$\Delta_\zeta^2 \equiv \frac{k^3|\zeta|^2}{2\pi^2} = \frac{2k^3}{\pi} \frac{H^2}{\epsilon m_{\rm pl}^2} A^2$$

Quantum Fluctuations

- Simple harmonic oscillator $\ll$ Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k,\eta)\hat{a} + u^*(k,\eta)\hat{a}^\dagger$$

where $u(k,\eta)$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \qquad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\eta] = i$

$$u(k,\eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

Quantum Fluctuations

- Zero point fluctuations of ground state

$$\begin{aligned}\langle u^2\rangle &= \langle 0|u^\dagger u|0\rangle \\ &= \langle 0|(u^*\hat{a}^\dagger + u\hat{a})(u\hat{a} + u^*\hat{a}^\dagger)|0\rangle \\ &= \langle 0|\hat{a}\hat{a}^\dagger|0\rangle|u(k,\tilde{\eta})|^2 \\ &= \langle 0|[\hat{a},\hat{a}^\dagger] + \hat{a}^\dagger\hat{a}|0\rangle|u(k,\tilde{\eta})|^2 \\ &= |u(k,\tilde{\eta})|^2 = \frac{1}{2k}\end{aligned}$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in. Slow roll equation
- So $A = (2k^3)^{-1/2}$ and curvature power spectrum

$$\Delta_\zeta^2 \equiv \frac{1}{\pi}\frac{H^2}{\epsilon m_{\rm pl}^2}$$

Tilt

- Curvature power spectrum is scale invariant to the extent that H is constant
- Scalar spectral index

$$\frac{d\ln \Delta_\zeta^2}{d\ln k} \equiv n_S - 1$$
$$= 2\frac{d\ln H}{d\ln k} - \frac{d\ln \epsilon}{d\ln k}$$

- Evaluate at horizon crossing where fluctuation freezes

$$\frac{d\ln H}{d\ln k}\Big|_{-k\tilde\eta=1} = \frac{k}{H}\frac{dH}{d\tilde\eta}\Big|_{-k\tilde\eta=1}\frac{d\tilde\eta}{dk}\Big|_{-k\tilde\eta=1}$$
$$= \frac{k}{H}(-aH^2\epsilon)\Big|_{-k\tilde\eta=1}\frac{1}{k^2} = -\epsilon$$

where $aH = -1/\tilde\eta = k$

Tilt

- Evolution of ϵ

$$\frac{d\ln\epsilon}{d\ln k} = -\frac{d\ln\epsilon}{d\ln\tilde{\eta}} = -2(\delta+\epsilon)\frac{\dot{a}}{a}\tilde{\eta} = 2(\delta+\epsilon)$$

- Tilt in the slow-roll approximation

$$n_S = 1 - 4\epsilon - 2\delta$$

Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation $(K = 0)$, same as scalar field

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0\,.$$

- Acquires quantum fluctuations in same manner as ϕ. Lagrangian sets the normalization

$$\phi_1 \rightarrow H_T^{(\pm 2)}\sqrt{\frac{3}{16\pi G}}$$

- Scale-invariant gravitational wave amplitude (each component: NB more traditional notation $H_T^{(\pm 2)} = (h_+ \pm i h_\times)/\sqrt{6}$)

$$\Delta_H^2 = \frac{16\pi G}{3 \cdot 2\pi^2}\frac{H^2}{2} = \frac{4}{3\pi}\frac{H^2}{m_{\rm pl}^2}$$

Gravitational Waves

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation
- Tensor tilt:

$$\frac{d \ln \Delta_H^2}{d \ln k} \equiv n_T = 2 \frac{d \ln H}{d \ln k} = -2\epsilon$$

- Consistency relation between tensor-scalar ratio and tensor tilt

$$\frac{\Delta_H^2}{\Delta_\zeta^2} = \frac{4}{3}\epsilon = -\frac{2}{3} n_T$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparision of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself