Astro 321

Lecture Notes Set 7

Wayne Hu

Closed Universe

- Friedman equation in a closed universe

$$\frac{1}{a}\frac{da}{dt} = H_0 \left(\Omega_m a^{-3} + (1-\Omega_m) a^{-2}\right)^{1/2}$$

- Parametric solution in terms of a development angle $\theta = H_0 \eta (\Omega_m - 1)^{1/2}$, scaled conformal time η

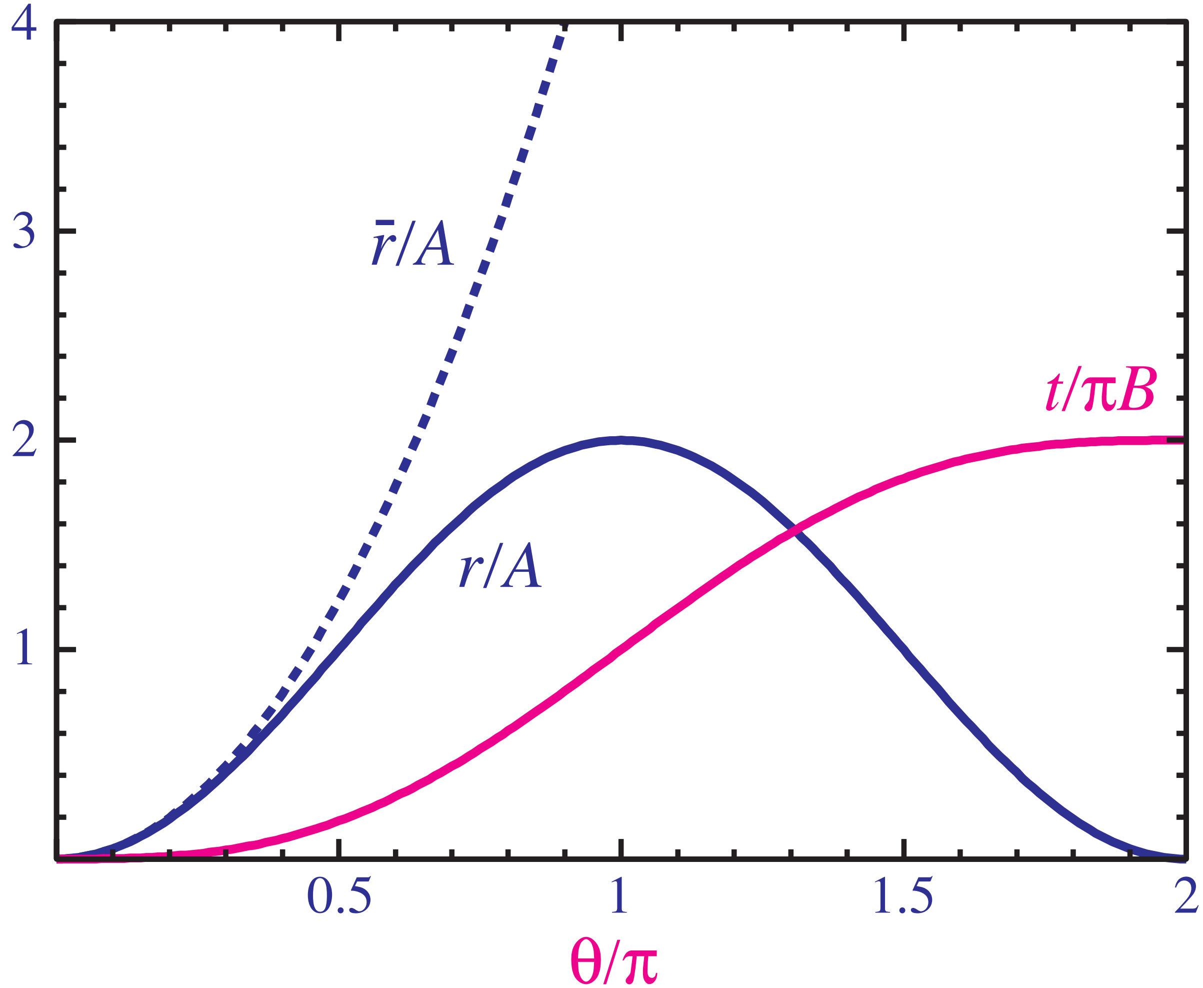
$$\begin{aligned} r(\theta) &= A(1 - \cos\theta) \\ t(\theta) &= B(\theta - \sin\theta) \end{aligned}$$

where $A = r_0 \Omega_m / 2(\Omega_m - 1)$, $B = H_0^{-1} \Omega_m / 2(\Omega_m - 1)^{3/2}$.

- Turn around at $\theta = \pi$, $r = 2A$, $t = B\pi$.
- Collapse at $\theta = 2\pi$, $r \to 0$, $t = 2\pi B$

Spherical Collapse

- Parametric Solution:



Correspondence

- Eliminate cosmological correspondence in A and B in terms of enclosed mass M

$$M = \frac{4\pi}{3} r_0^3 \Omega_m \rho_c = \frac{4\pi}{3} r_0^3 \Omega_m \frac{3H_0^2}{8\pi G}$$

- Related as $A^3 = GMB^2$, and to initial perturbation δ_i at a_i - require an explicit $r(t) \to r(a)$

$$\begin{aligned}
\lim_{\theta \to 0} r(\theta) &= A\left(\frac{1}{2}\theta^2 - \frac{1}{24}\theta^4\right) \\
\lim_{\theta \to 0} t(\theta) &= B\left(\frac{1}{6}\theta^3 - \frac{1}{120}\theta^5\right)
\end{aligned}$$

- Leading Order: $r = A\theta^2/2$, $t = B\theta^3/6$

$$r = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}$$

Next Order

- Leading order is unperturbed matter dominated expansion
  $r \propto a \propto t^{2/3}$
- Iterate r and t solutions

$$\lim_{\theta \to 0} t(\theta) = \frac{\theta^3}{6} B \left[1 - \frac{1}{20} \left(\frac{6t}{B} \right)^{2/3} \right]$$

$$\theta \approx \left(\frac{6t}{B} \right)^{1/3} \left[1 + \frac{1}{60} \left(\frac{6t}{B} \right)^{2/3} \right]$$

Next Order

- Substitute back into $r(\theta) \longrightarrow r(t)$

$$
\begin{aligned}
r(\theta) &= A\frac{\theta^2}{2}\left(1 - \frac{\theta^2}{12}\right) \\
&= \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left[1 - \frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right] \\
&= \frac{1}{2}(6t)^{2/3}(GM)^{1/3}\left[1 - \frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right]
\end{aligned}
$$

Density Correspondence

- Density

$$\begin{aligned}\rho_m &= \frac{M}{\frac{4}{3}\pi r^3} \\ &= \frac{1}{6\pi t^2 G}\left[1 + \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}\right]\end{aligned}$$

- Density perturbation

$$\delta \equiv \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m} \approx \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}$$

Density Correspondence

- Time $\longrightarrow$ scale factor

$$
\begin{aligned}
t &= \frac{2}{3H_0\Omega_m^{1/2}} a^{3/2} \\
\delta &= \frac{3}{20} a \left(4/B H_0 \Omega_m^{1/2}\right)^{2/3}
\end{aligned}
$$

- A and B constants $\longrightarrow$ initial cond.

$$
\begin{aligned}
B &= \frac{1}{2H_0\Omega_m^{1/2}} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right)^{3/2} \\
A &= \frac{3}{10}\frac{r_i}{\delta_i}
\end{aligned}
$$

Spherical Collapse Relations

- Scale factor $a \propto t^{2/3}$

$$a = \left(\frac{3}{4}\right)^{2/3} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right) (\theta - \sin\theta)^{2/3}$$

- At collapse $\theta = 2\pi$

$$a_{\rm col} = \left(\frac{3}{4}\right)^{2/3} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right) (2\pi)^{2/3} \approx 1.686\frac{a_i}{\delta_i}$$

- Perturbation collapses when linear theory predicts $\delta_c \equiv 1.686$

Virialization

- A real density perturbation is neither spherical nor homogeneous
- Shell crossing if δ_i doesn't monotonically decrease
- Collapse does not proceed to a point but reaches virial equilibrium

$$
\begin{aligned}
U &= -2K \\
E &= U + K = U(r_{\rm max}) = \frac{1}{2} U(r_{\rm vir}) \\
r_{\rm vir} &= \frac{1}{2} r_{\rm max}
\end{aligned}
$$

since $U \propto r^{-1}$. Thus $\theta_{\rm vir} = \frac{3}{2}\pi$

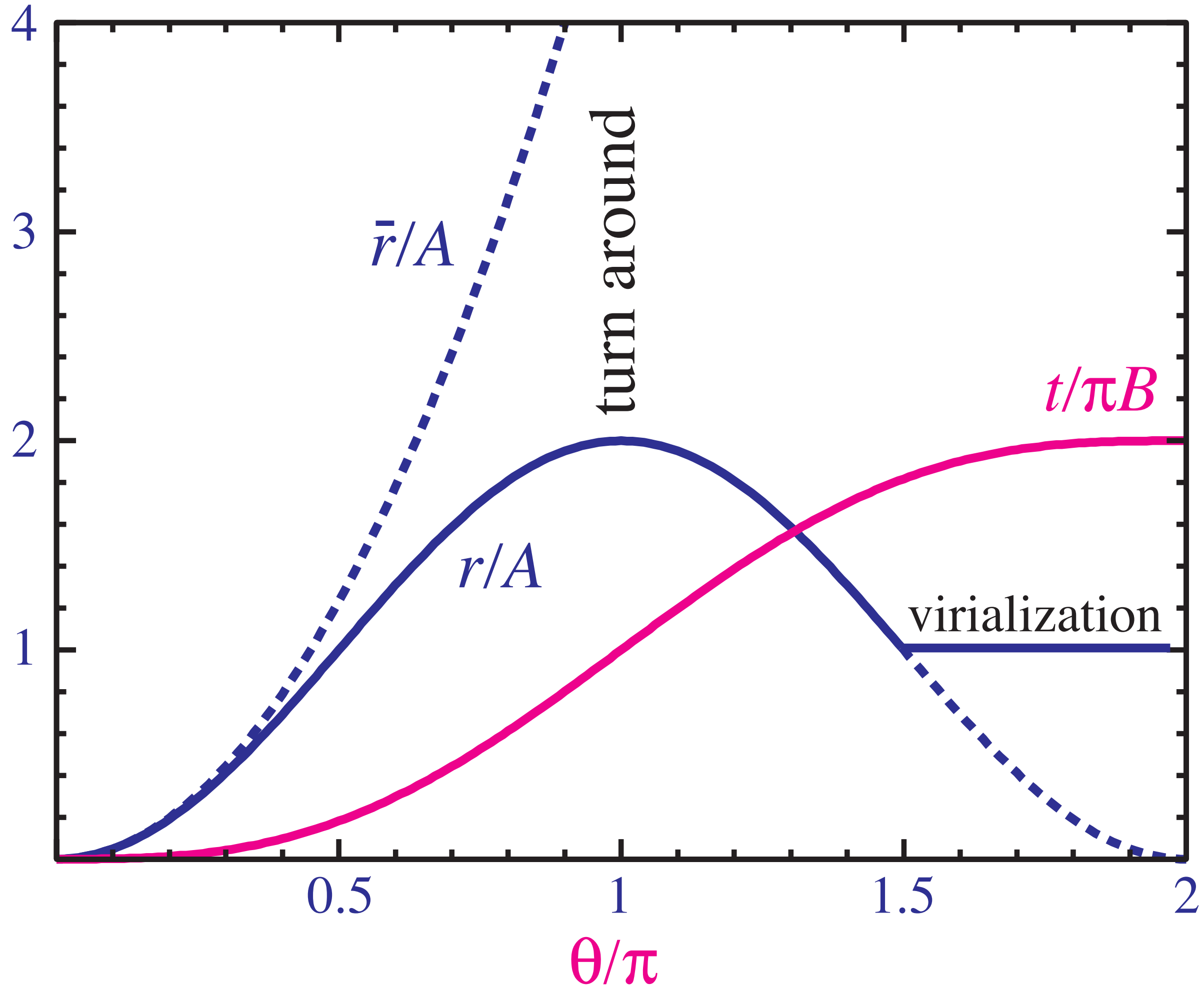
- Overdensity at virialization

$$
\frac{\rho_m(\theta = 3\pi/2)}{\bar{\rho}_m(\theta = 2\pi)} = 18\pi^2 \approx 178
$$

- Threshold $\Delta_v = 178$ often used to define a collapsed object

Virialization

- Schematic Picture:



The Mass Function

- Spherical collapse predicts the end state as virialized halos given an initial density perturbation
- Initial density perturbation is a Gaussian random field
- Compare the variance in the linear density field to threshold $\delta_c = 1.686$ to determine collapse fraction
- Combine to form the mass function, the number density of halos in a range dM around M.
- Halo density defined entirely by linear theory
- Fudge the result to get the right answer compared with simulations (a la Press-Schechter)!

Press-Schechter Formalism

- Smooth linear density density field on mass scale M with tophat

$$R = \left(\frac{3M}{4\pi}\right)^{1/3}$$

- Result is a Gaussian random field with variance $\sigma^2(M)$
- Fluctuations above the threshold δ_c correspond to collapsed regions. The fraction in halos $> M$ becomes

$$\frac{1}{\sqrt{2\pi}\sigma(M)}\int_{\delta_c}^{\infty} d\delta \exp\left(-\frac{\delta^2}{2\sigma^2(M)}\right) = \frac{1}{2}\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$

  where $\nu \equiv \delta_c/\sigma(M)$
- Problem: even as $\sigma(M) \to \infty$, $\nu \to 0$, collapse fraction $\to 1/2$ – only overdense regions participate in spherical collapse.
- Multiply by an ad hoc factor of 2!

Press-Schechter Mass Function

- Differentiate in M to find fraction in range dM and multiply by ρ_m/M the number density of halos if all of the mass were composed of such halos $\rightarrow$ differential number density of halos

$$\frac{dn}{d\ln M} = \frac{\rho_m}{M}\frac{d}{d\ln M}\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$
$$= \sqrt{\frac{2}{\pi}}\frac{\rho_m}{M}\frac{d\ln\sigma^{-1}}{d\ln M}\nu\exp(-\nu^2/2)$$

- High mass: exponential cut off above M_* where $\sigma(M_*) = \delta_c$

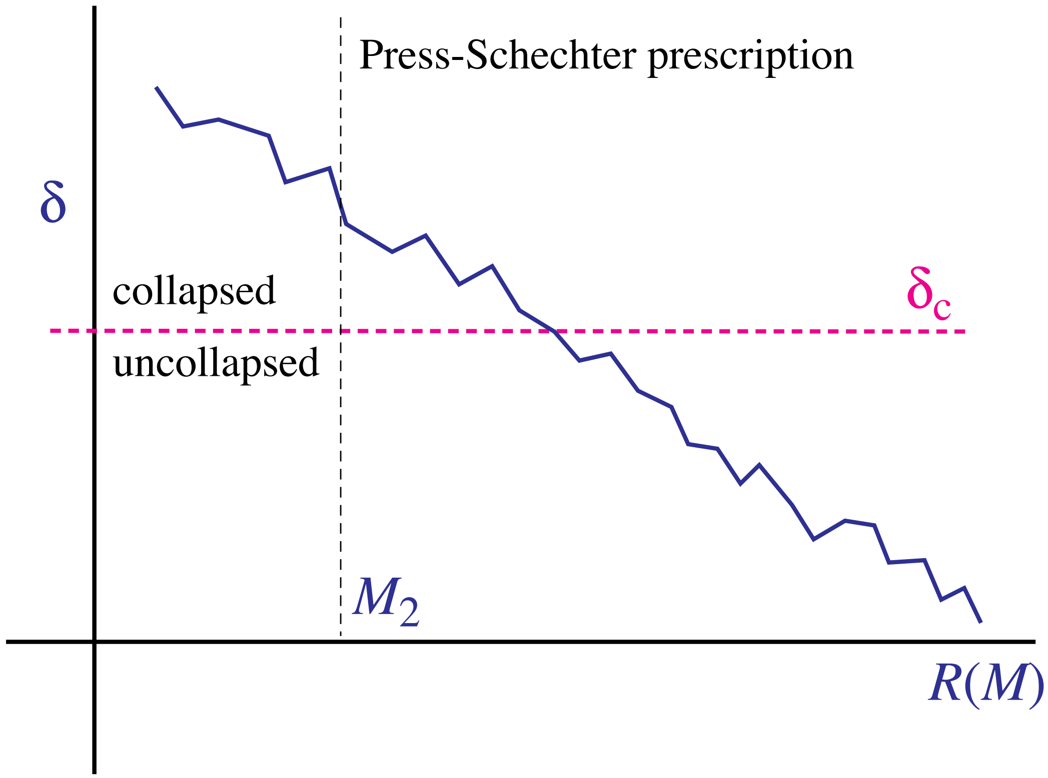
$$M_* \sim 10^{13}h^{-1}M_\odot \quad \text{today}$$

- Low mass divergence: (too many for the observations?)

$$\frac{dn}{d\ln M} \propto\sim M^{-1}$$

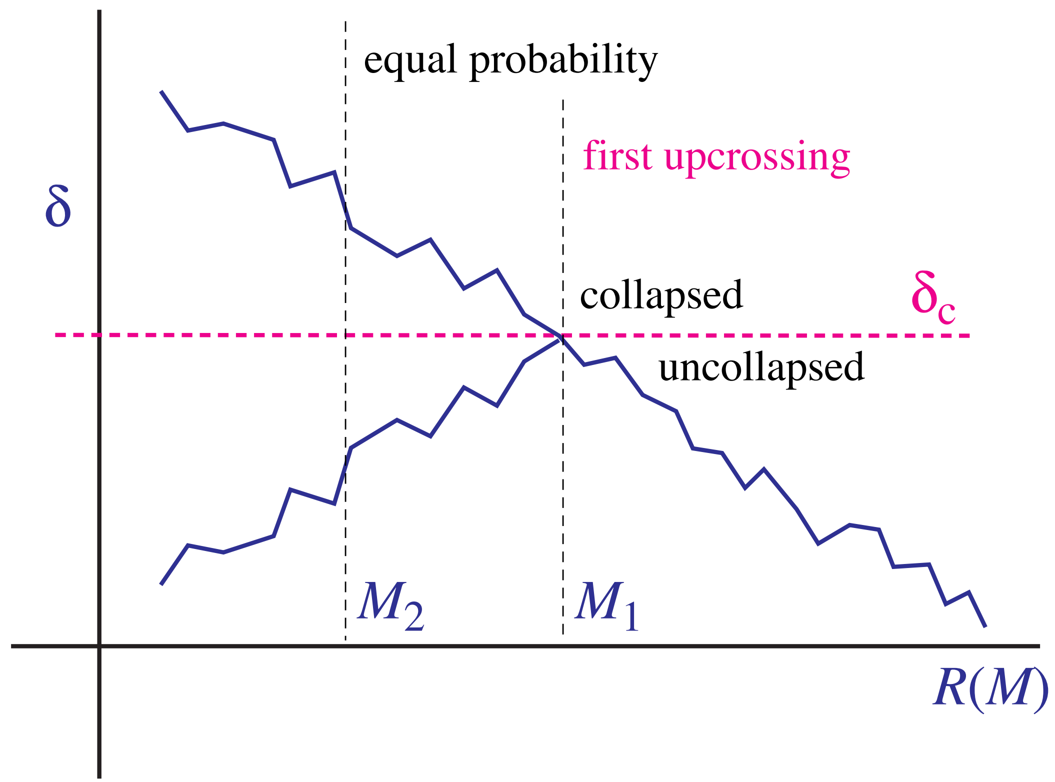
Extended Press-Schechter Formalism

- A region that is underdense when smoothed on the scale M_2 may be overdense on a scale of a larger $M > M_2$
- If smoothing is a tophat in k-space, independence of k-modes implies fluctuation executes a random walk



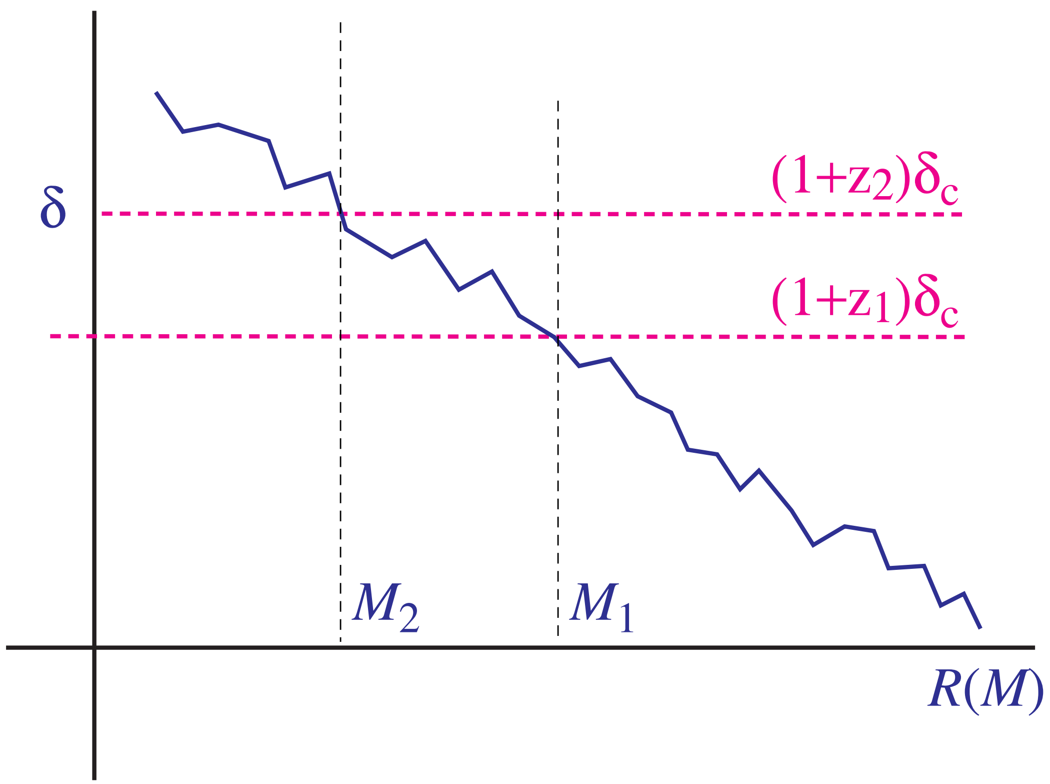
Extended Press-Schechter Formalism

- For each trajectory that lies above threshold at M_2, there is an equivalent trajectory that is its mirror image reflected around δ_c
- Press-Schechter ignored this branch. It supplies the missing factor of 2



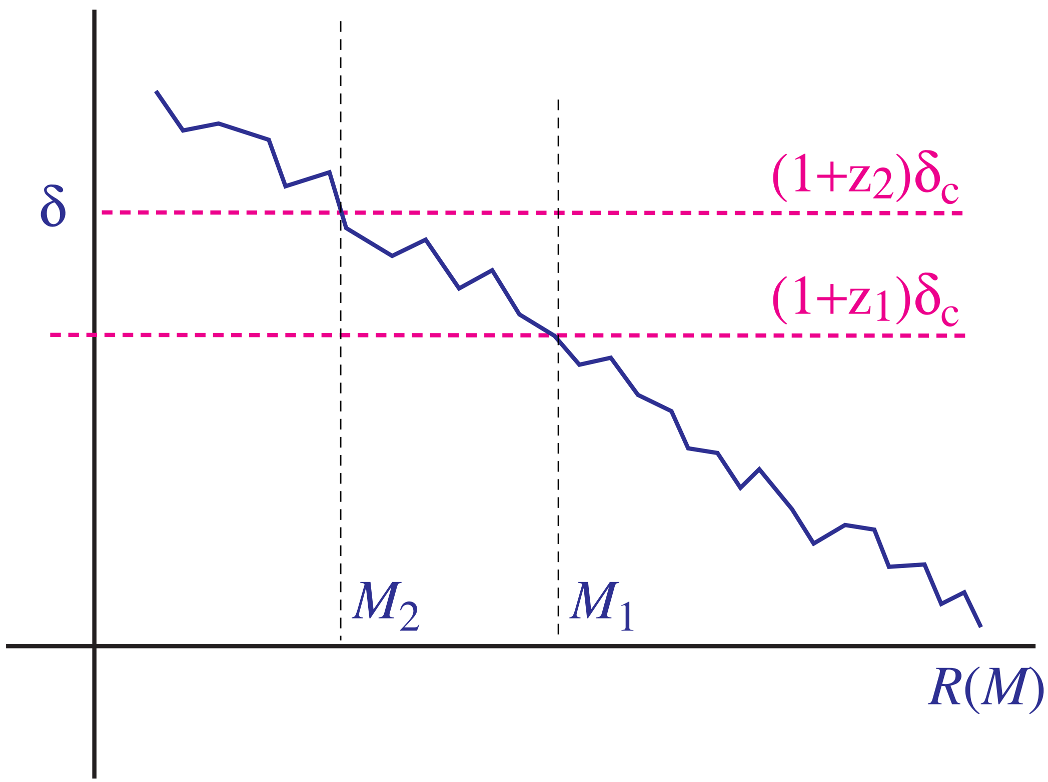
Conditional Mass Function

- Extended Press-Schechter also gives the conditional mass function, useful for merger histories.
- Given a halo of mass M_1 exists at z_1, what is the probability that it was part of a halo of mass M_2 at z_2



Conditional Mass Function

- Same as before but with the origin translated.
- Conditional mass function is mass function with δ_c and $\sigma^2(M)$ shifted



Magic "2" resolved?

- Spherical collapse is defined for a real-space not k-space smoothing. Random walk is only a qualitative explanation.
- Modern approach: think of spherical collapse as motivating a fitting form for the mass function

$$\nu \exp(-\nu^2/2) \rightarrow A[1 + (a\nu^2)^{-p}]\sqrt{a\nu^2} \exp(-a\nu^2/2)$$

Sheth-Torman 1999, $a = 0.75$, $p = 0.3$. or a completely empirical fitting

$$\frac{dn}{d\ln M} = 0.301 \frac{\rho_m}{M} \frac{d\ln \sigma^{-1}}{d\ln M} \exp[-|\ln \sigma^{-1} + 0.64|^{3.82}]$$

Jenkins et al 2001. Choice is tied up with the question: what is the mass of a halo?

Numerical Mass Function

- Example of difference in mass definition (from Hu & Kravstov 2002)