

The Boltzmann Equation

- Track the spatial and angular distribution of the photons under scattering
- Low order angular terms (monopole & dipole) return covariant energy momentum conservation
- Higher order angular terms simply encapsulate geometric effects from the free propagation of photons

Two Approaches

- Classical: start with differential cross section $\frac{d\sigma}{d\Omega}$
- Quantum: start with quantum mechanical matrix element (see Scott's Book)

• Liouville Equation (no scattering)

Phase space distribution of photons
conserved in the absence of scattering

$$f(\vec{x}, \eta; \vec{p}, p) = f(\vec{x}, \hat{n}, p; \eta)$$

↑ ↑ ↑
 space-time momentum energy (momentum)
 direction

$$\frac{d}{d\eta} f(\vec{x}, \hat{n}, p; \eta) = 0$$

$$\dot{f} + \frac{\partial \vec{x}}{\partial \eta} \cdot \frac{\partial}{\partial \vec{x}} f + \frac{\partial \hat{n}}{\partial \eta} \cdot \frac{\partial}{\partial \hat{n}} f + \frac{\partial p}{\partial \eta} \frac{\partial}{\partial p} f = 0$$

$$\dot{f} + \hat{n} \cdot \nabla f + \dot{P} \frac{\partial}{\partial p} f = 0$$

$$\dot{\frac{P}{P}} = -\frac{\dot{a}}{a} - \dot{\Phi} - \hat{n} \cdot \nabla \Psi \quad \leftarrow$$

↑ ↑
 curvature pert gravitational redshift
 cosmological redshift

scalar fluctuations
 (generalise to vector, tensor)

(these are the same terms that come into covariant energy-momentum conservation)

Integrate over energies

$$4\Theta = \frac{1}{\pi^2 \rho_8} \int P^3 dp f - 1$$

$$\dot{\Theta} + \hat{n} \cdot \nabla \Theta = -\dot{\Phi} - \hat{n} \cdot \nabla \Psi$$

"free streaming equation"

Decompose into normal modes

$$\Theta(\vec{x}, \hat{n}, \eta) = \int \frac{d^3 k}{(2\pi)^3} \sum_{lm} \Theta_e^{(m)} G_e^{(m)}(\vec{k}, \vec{x}, \hat{n})$$

$$(recall \quad G_e^{(m)} = (-i)^l \sqrt{\frac{4\pi}{2l+1}} Y_e^{(m)}(\hat{n}) \exp(i\vec{k} \cdot \vec{x}))$$

Power Spectrum

$$(2l+1)^2 C_e = \frac{2}{\pi} \int \frac{dk}{k} \underbrace{\sum_m k^3 \langle \Theta_e^{(m)*} \Theta_e^{(m)} \rangle}_{\text{defined as average over } m}$$

Expand free Streaming Equation

$$\hat{n} \cdot \nabla \exp(i\vec{k} \cdot \vec{x}) = i \hat{n} \cdot \vec{k} = i \sqrt{\frac{4\pi}{3}} k Y_1^0(\hat{n}) \cdot \exp(ik \cdot \vec{x})$$

↑
in the $\hat{k} \parallel \hat{e}_3$
frame

The Hierarchy Equation:

Addition of Angular Momentum (dipole \otimes plane wave)

$$\sqrt{\frac{4\pi}{3}} Y_l^0 Y_e^m = \frac{K_e^m}{(2l+1)(2e-1)} Y_{l-1}^m + \frac{K_{e+1}^m}{(2l+1)(2e+3)} Y_{e+1}^m$$

$$K_e^m = \sqrt{l^2 - m^2}$$

$l \rightarrow l \pm 1$ infinite hierarchy

$$\dot{\Theta}_e^{(m)} = K \left[\frac{K_e^m}{(2l+1)} \Theta_{e-1}^{(m)} - \frac{K_{e+1}^m}{(2l+3)} \Theta_{e+1}^{(m)} \right] + S_e^{(m)}$$

for scalar fluctuations in Newtonian Gauge

$$S_0^{(0)} = -\dot{\Phi} \quad S_i^{(0)} = K \nabla^i \Phi \quad \text{else zero}$$

generalize to $m=\pm 1$ vectors, $m=\pm 2$ tensors

The Integral Equation

Decompose the Source at a distance $\vec{x} = D\hat{n}$

$$G_e^m = (-i)^e \sqrt{\frac{4\pi}{2l+1}} Y_e^m(\hat{n}) \exp(i\vec{k} \cdot \vec{x})$$

$$\exp(i\vec{k} \cdot \vec{x}) = \sum_e (-i)^e \sqrt{\frac{4\pi(2e+1)}{(2e+1)!}} j_e(KD) Y_e^0$$

recouple $Y_{es}^{ms} Y_{e'}^0 \rightarrow Y_e^{ms}$

$$G_{es}^m = \sum_e (-i)^e \sqrt{\frac{4\pi(2e+1)}{(2e+1)!}} \alpha_{les}^{(m)}(KD) Y_e^m(\hat{n})$$

Sources :

$$l_s = 0 \quad m = 0 \quad (\text{scalar source})$$

$$\alpha_{0s}^0 = j_e$$

$$l_s = 1 \quad m = 0 \quad (\text{dipole source, potential})$$

$$\alpha_{1e}^0 = j'_e$$

$$l_s = 1 \quad m = \pm 1 \quad (\text{dipole source, vorticity})$$

derive in p.s #5

Projection of Source = Integral approach

$$\frac{\Theta_e^{(m)}(\eta_0, k)}{2l+1} = \int_0^{\eta_0} d\eta \sum_{l_s} S_{ls}^{(m)} \alpha_{lse}^{(m)} [k(\eta_0 - \eta)]$$

approximations: collisionless & flat

$$\frac{G_e^{(0)}(\eta_0, k)}{2l+1} = \int_0^{\eta_0} d\eta (-\dot{\Phi} j_e + k^2 \bar{\Psi} j'_e)$$

for scalar fluctuations

integrate by parts

$$\frac{G_e^{(m)}(\eta_0, k)}{2l+1} = \int_0^{\eta_0} d\eta (-\dot{\Phi} + \dot{\bar{\Psi}}) j_e [k(\eta_0 - \eta)]$$

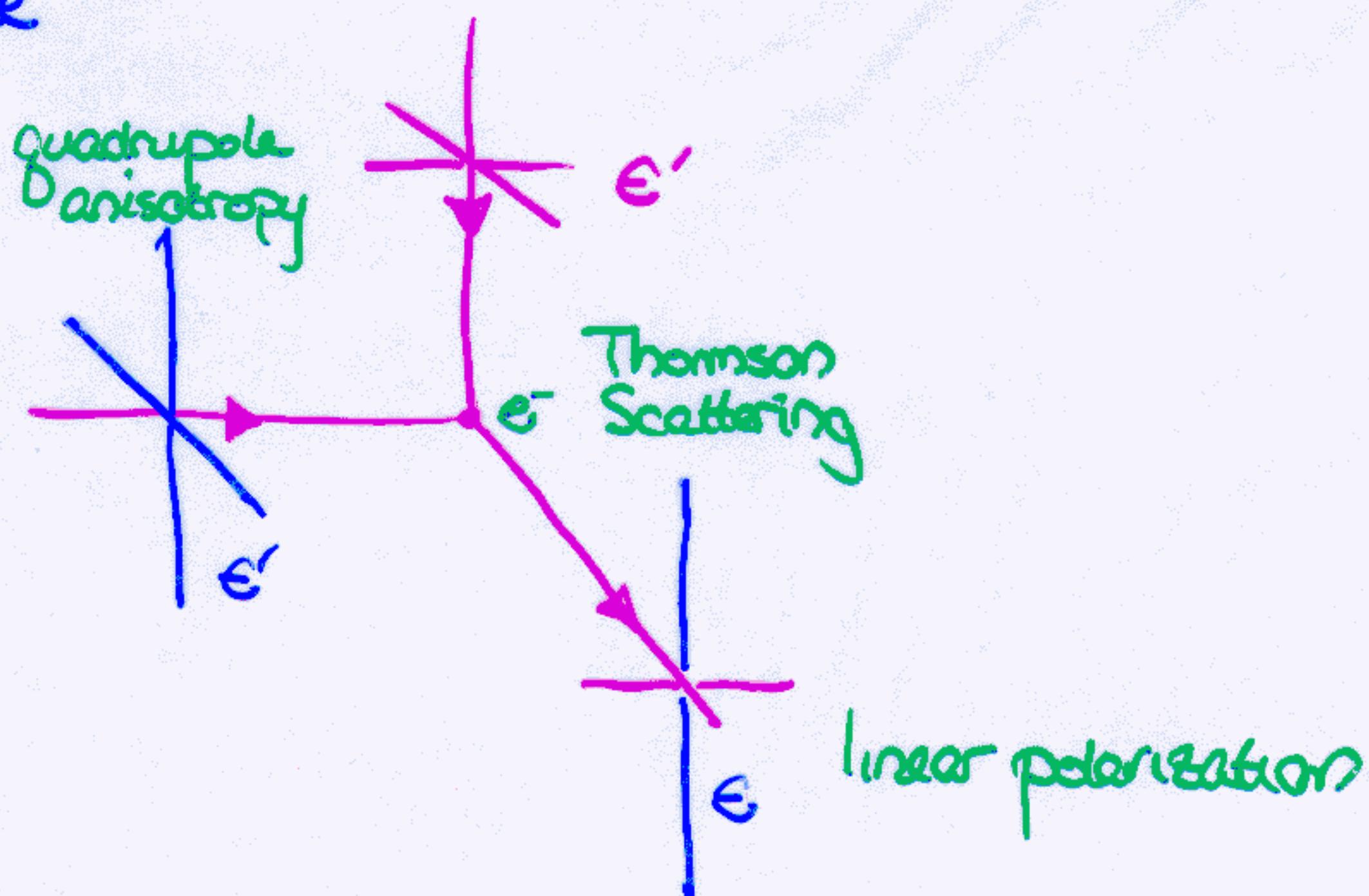
$\approx \partial \bar{\Psi}$

= integrated Sachs-Wolfe effect.

great conceptually but sources are implicit function of $G_e^{(m)}$ itself.

Thomson Collision Term

Thomson scattering depends on both the angular distribution & the polarization state



Quadrupole Anisotropy \Leftrightarrow Linear Polarization

Differential Cross Section

$$\frac{d\sigma}{d\Omega} \propto |\hat{e} \cdot \hat{E}'|^2$$

E = polarization vector

Polarization Description

$\Pi \propto \langle E_i^* E_j \rangle$ \vec{E} : electric field on 2D subspace

normalized & integrated over frequencies

$$\Pi = \Theta \mathbb{I} + Q \sigma_3 + U \sigma_1 + \sqrt{\Theta_2} \sigma_2$$

Circular

σ_i = Pauli Matrices

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Q = \text{Tr}(\Pi \sigma_3)/2 = \frac{T_{11} - T_{22}}{2}$$

$$\begin{array}{c} T_{11} \\ \times \\ T_{22} \end{array}$$

$$U = \text{Tr}(\Pi \sigma_1)/2 = \frac{T_{12} + T_{21}}{2} = T_{12} \quad \times$$

Q, U are spin 2 ($\frac{\pi}{2} \rightarrow$ sign flip, $\pi \rightarrow$ original)

$Q+iU$ transform into themselves under a rotation (phase factor $e^{i\pi/4}$)

$$\Pi = \Theta \mathbb{I} + (Q+iU) \underbrace{\left(\frac{\sigma_3 - i\sigma_1}{2} \right)}_{\text{rotate}}$$

$$+ (Q-iU) \underbrace{\left(\frac{\sigma_3 + i\sigma_1}{2} \right)}_{\text{sign flip}}$$

decompose as spin 2 object

decompose as spin 2 object

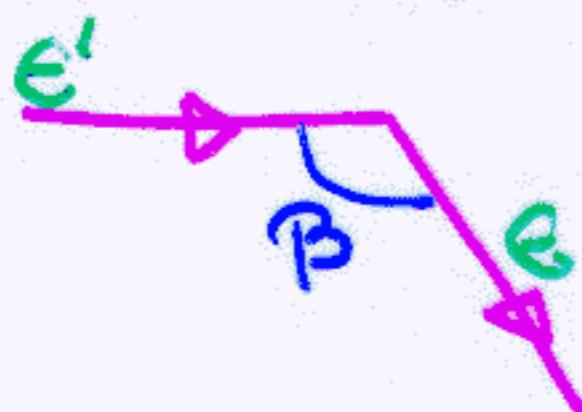
The Boltzmann Equation

$$\frac{d}{dm} f = C[f]$$

- = scattering into phase space
- scattering out of phase space

scattering into → angular & polarization dependence

1. Step 1: scattering in coordinates fixed by scattering plane & electron rest frame



incoming & outgoing directions define plane

$\Theta_{||}$: temperature fluctuation, in-plane polarization state

Θ_{\perp} : " , \perp -plane polarization

α : difference of $\pm 45^\circ$ from scattering plane

$$|\epsilon_{||} \cdot \epsilon'_{||}|^2 = \cos^2 \beta$$

$$\epsilon_1 = \frac{\epsilon_{||} + \epsilon_{\perp}}{\sqrt{2}} \quad \epsilon_2 = \frac{\epsilon_{||} - \epsilon_{\perp}}{\sqrt{2}}$$

$$|\epsilon_{\perp} \cdot \epsilon'_{\perp}|^2 = 1$$

$$\epsilon_1 \cdot \epsilon'_1 = \epsilon_2 \cdot \epsilon'_2 = \frac{\cos \beta + 1}{2}$$

$$\Theta_{||} \propto \cos^2 \beta \Theta'_{||}$$

$$\epsilon_1 \cdot \epsilon'_1 = \epsilon_2 \cdot \epsilon'_2 = \frac{\cos \beta - 1}{2}$$

$$\epsilon_{\perp} \propto \Theta'_{\perp}$$

$$U = (\theta_1 - \theta_2)/2$$

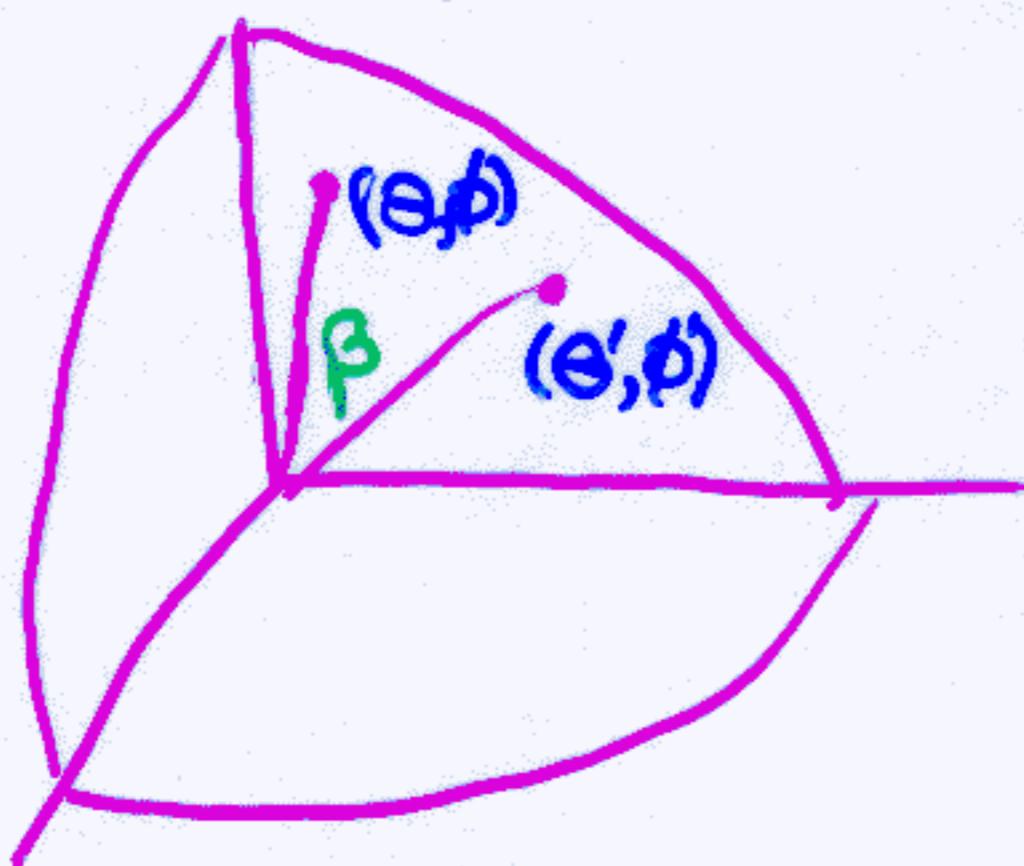
$$\Theta_1 \propto \frac{(\cos\beta + 1)^2}{4} \Theta'_1 + \frac{(\cos\beta - 1)^2}{4} \Theta'_2$$

$$\Theta_2 \propto \frac{(\cos\beta - 1)^2}{4} \Theta'_1 - \frac{(\cos\beta + 1)^2}{4} \Theta'_2$$

$$\frac{(\theta_1 - \theta_2)}{2} \propto \cos\beta \frac{(\Theta'_1 - \Theta'_2)}{2}$$

$$U \propto \cos\beta U'$$

2. Step 2: transform to $\Theta, Q+iU$ in fixed frame



and fix normalization to the scattering rate

$\dot{\tau}$

in terms of $\vec{T} = (\Theta, Q+iU, Q-iU)$

$$C[\vec{T}]_{\text{into}} = \frac{1}{10} \dot{\tau} \int d\Omega' \sum_{m=-2}^2 P^{(m)}(\hat{n}, \hat{n}') T(n')$$

all incoming directions
polarization matrix

$$+ \dot{\tau} \int \frac{d\Omega'}{4\pi} [\Theta(n'), Q, Q]$$

Isotropic scattering

$$P^{(m)} = \begin{pmatrix} Y_2^{m*}(n') Y_2^m(n) & -\sqrt{\frac{3}{2}} {}_2Y_2^{m*}(n') Y_2^m(n) & \sqrt{\frac{3}{2}} {}_{-2}Y_2^{m*}(n') Y_2^m(n) \\ -\sqrt{6} Y_2^{m*}(n') {}_2Y_2^m(n) & 3 {}_2Y_2^{m*}(n') {}_2Y_2^m(n) & 3 {}_{-2}Y_2^{m*}(n') {}_2Y_2^m(n) \\ -\sqrt{6} Y_2^{m*}(n') {}_{-2}Y_2^m(n) & 3 {}_2Y_2^{m*}(n') {}_{-2}Y_2^m(n) & 3 {}_{-2}Y_2^{m*}(n') {}_{-2}Y_2^m(n) \end{pmatrix}$$

$\pm 2 Y_2^m$ spin 2 quadrupole harmonics

$${}_2Y_2^{IR} = \frac{1}{8} \sqrt{\frac{5}{\pi}} (1 \mp \cos\theta)^2 e^{\pm 2i\phi}$$

$${}_2Y_2^{\pm 1} = \frac{1}{4} \sqrt{\frac{5}{\pi}} \sin\theta (1 \mp \cos\theta) e^{\pm i\phi} \quad \& \quad {}_2Y_2^{m*} = (-1)^{m+s} {}_{-2}Y_2^{-m}$$

$${}_2Y_2^0 = \frac{3}{4} \sqrt{\frac{5}{6\pi}} \sin^2\theta$$

3. Step 3: add in scattering out of mode
and transformation out of electron
rest frame.

$$C[\vec{T}] = -\dot{\tau} \vec{I}(\hat{n}) + \frac{1}{10} \dot{\tau} \int d\hat{n}' \sum_m P^{(m)}(\hat{n}, \hat{n}') T(n')$$

$$\vec{I}(\hat{n}) = T(\hat{n}) - \left(\int \frac{dn'}{4\pi} \Theta' + \hat{n} \cdot \vec{v}_B, 0, 0 \right)$$

describes isotropization in the rest frame

$\Rightarrow B_0$ not changed

Θ, \vec{v}_B not changed

$B_{e>2}$ exponentially destroyed

4 Full Boltzmann Equation

$$\dot{\hat{T}} + \hat{n} \cdot \nabla_i \hat{T} = C[\hat{T}] + \tilde{G}$$

$$\tilde{G} = (-\dot{\Phi} - \hat{n} \cdot \nabla_i \Psi, 0, 0) \quad \text{scalar fluctuations}$$

5. Normal Mode Decomposition

$$\Theta(\vec{x}, \hat{n}, \eta) = \int \frac{d^3 k}{(2\pi)^3} \sum_{lm} \Theta_e^{(m)} G_e^m$$

$$(Q \pm iU)(\vec{x}, \hat{n}, \eta) = \int \frac{d^3 k}{(2\pi)^3} \sum_{lm} (E_e^{(m)} \pm iB_e^{(m)}) \pm_2 G_e^m$$

$$\pm_2 G_e^m = (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} [\pm_2 Y_\ell(\hat{n})] \exp[i\vec{k} \cdot \vec{x}]$$

$$\dot{\Theta}_e^{(m)} = K \left[\frac{K_e^m}{(2\ell-1)} \Theta_{e-1}^{(m)} - \frac{K_{e+1}^m}{(2\ell+3)} \Theta_{e+1}^{(m)} \right] - \dot{\tau} B_e^{(m)} + S_e^{(m)}$$

$$S_0^{(0)} = \dot{\tau} B_0^{(0)} - \dot{\Phi} \quad S_1^{(0)} = \dot{\tau} V_0^{(0)} + K \bar{\Psi}$$

$$S_2^{(0)} = \frac{\dot{\tau}}{10} [B_2^{(0)} - \sqrt{6} E_2^{(0)}]$$

full hierarchy equations

$$\frac{\Theta_e^{(m)}(n_0, K)}{2\ell+1} = \int_0^{n_0} d\eta e^{-\tau} \sum_{ls} S_{les}^{(m)} \alpha_{les}^{(m)} [K(n_0 - \eta)]$$

full integral equations

Polarization Hierarchy

$$\dot{E}_e^{(m)} = K \left[\frac{a K_e^{(m)}}{(2\ell-1)} E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell+1)} B_\ell^{(m)} - \frac{a K_{\ell+1}^{(m)}}{(2\ell+3)} E_{\ell+1}^{(m)} \right] \\ - \dot{\tau} \left[E_\ell^{(m)} + \frac{\sqrt{6}}{10} (G_\ell^{(m)} - \sqrt{6} E_\ell^{(m)}) \delta_{0,2} \right]$$

$$\dot{B}_e^{(m)} = K \left[\frac{2 K_e^{(m)}}{(2\ell-1)} B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell+1)} E_\ell^{(m)} - \frac{2 K_{\ell+1}^{(m)}}{(2\ell+3)} B_{\ell+1}^{(m)} \right] \\ - \dot{\tau} B_\ell^{(m)}$$

Polarization Integral Solution

$$\frac{E_e^{(m)}(\eta_0, K)}{2\ell+1} = -\sqrt{6} \int_0^{\eta_0} d\eta \, \dot{\tau} e^{-\tau} \left[\frac{1}{10} (B_\ell^{(m)} - \sqrt{6} E_\ell^{(m)}) \right] E_\ell^{(m)} [K(\eta_0 - \eta)]$$

$$\frac{B_e^{(m)}(\eta_0, K)}{2\ell+1} = -\sqrt{6} \int_0^{\eta_0} d\eta \, \dot{\tau} e^{-\tau} \left[\frac{1}{10} (B_\ell^{(m)} - \sqrt{6} E_\ell^{(m)}) \right] B_\ell^{(m)} [K(\eta_0 - \eta)]$$

$$a K_e^{(m)} = \sqrt{(\ell^2 - m^2)(\ell^2 - 4)} \sqrt{\ell^2}$$

$$E_\ell^{(0)} = \sqrt{\frac{3(\ell+2)!}{8(\ell-2)!}} \frac{J_\ell(x)}{x^2}$$

$$B_e^{(0)} = 0$$