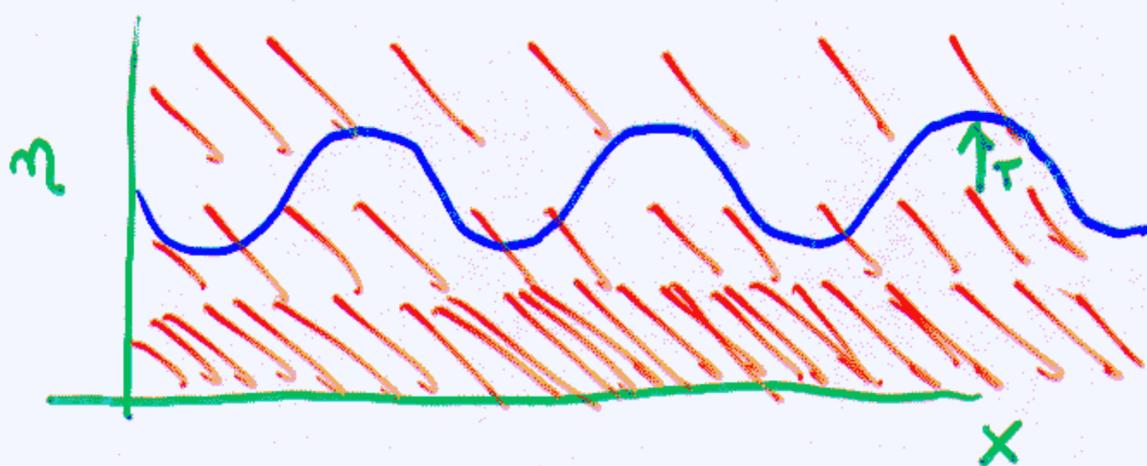


# Gauge

- Qualitative

- A redefinition of the temporal coordinate <sup>as a fn of position</sup>  $\Rightarrow$  redefinition of density perturbation due to expansion

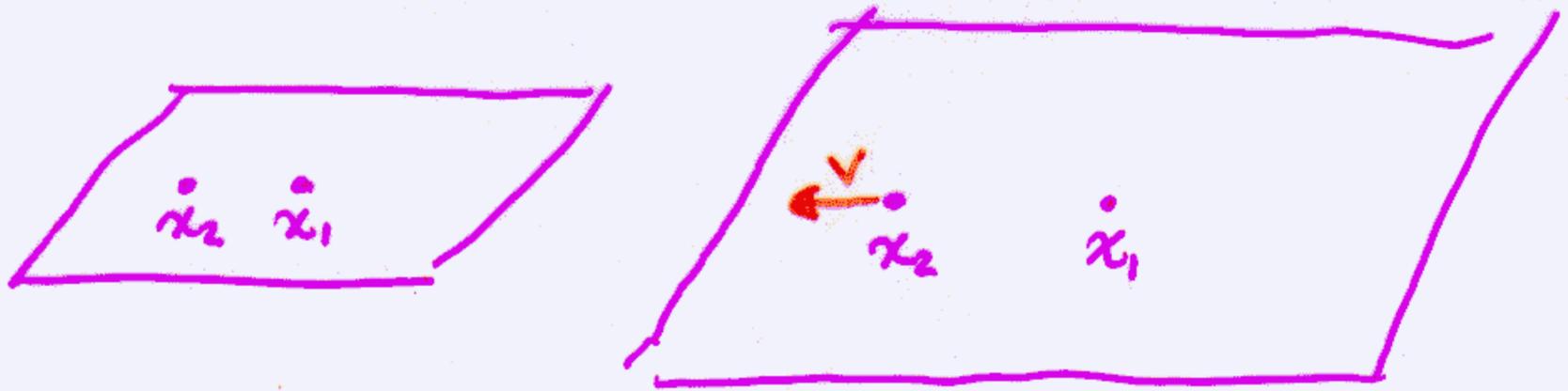


$$\tilde{\delta\rho} = \delta\rho - T\dot{\rho}$$

trivial example: CMB temperature is measured to be higher at higher redshift in molecular lines  
if we warp the time coordinate so that at the distance  $d_{\text{molecule}}$   $z_{\text{molecule}}$  is now

we assign a spatially varying temperature to CMB even though it is in fact homogeneous in the "right" coordinates

- A redefinition of the spatial coordinate as a function of time  
 $\Rightarrow$  redefinition of velocity field



trivial example: recession velocity

$$d = ax$$

$$v = \dot{d} = \dot{a}x = Hd$$

generally under a spatial coordinate shift  $\hookrightarrow$  peculiar velocities

$$\tilde{v} = v + \dot{L}$$

- Analogous changes in metric

eg.  $H_L$  is a spatially varying change to scale factor

$$\tilde{H}_L = H_L - \frac{\dot{a}}{a} T$$

under spatially varying time redefn.

## Gauge Freedom / coordinate transformation

Matter & Metric take on different values in different coordinate systems (no such thing as "gauge invariant" perturbation)

### General Coordinate Transformation

$$\begin{aligned}\tilde{\eta} &= \eta + T \\ \tilde{x}^i &= x^i + L^i\end{aligned}$$

Simple Case: scalar function

$$\begin{aligned}f(\tilde{\eta}, \tilde{x}^i) &= f(\eta, x^i) \\ &= f(\tilde{\eta}, \tilde{x}^i) - \frac{\partial f}{\partial \eta} T - \frac{\partial f}{\partial x^i} L^i\end{aligned}$$

↪ 2nd order

General Case: tensor function

$$\tilde{T}_{\mu\nu}(\tilde{\eta}, \tilde{x}^i) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} T_{\alpha\beta}(\eta, x^i)$$

### Fourier Modes

$$\begin{aligned}T(\eta, x) &= T(\eta, k) Q^{(0)} \\ L_i(\eta, x) &= \sum_{m=-1}^1 L_i^{(m)} Q_i^{(m)}\end{aligned}$$

- Metric

- Matter

Scalar

$$\begin{aligned}\tilde{A} &= A - \dot{T} - \frac{\dot{a}}{a} T \\ \tilde{B} &= B + \dot{L} + K T \\ \tilde{H}_L &= H_L - \frac{K}{3} L - \frac{\dot{a}}{a} T \\ \tilde{H}_T &= H_T + K L\end{aligned}$$

$$\tilde{\delta}_\rho = \delta_\rho - \dot{\rho} T$$

$$\tilde{\delta}_\phi = \delta_\phi - \dot{\phi} T$$

$$\tilde{v} = v + \dot{L}$$

(superscript 0 suppressed)

Vector

$$\begin{aligned}\tilde{B}^{(\pm 1)} &= B^{(\pm 1)} + \dot{L}^{(\pm 1)} \\ \tilde{H}_T^{(\pm 1)} &= H_T^{(\pm 1)} + K L^{(\pm 1)}\end{aligned}$$

$$\tilde{v}^{(\pm 1)} = v^{(\pm 1)} + \dot{L}^{(\pm 1)}$$

Gauge Choice fixes  $\underbrace{T, L}_{\text{Scalar}}, \underbrace{L^{(+1)}, L^{(-1)}}_{\text{Vector}}$  4 functions

Choose gauge so as to  
and/or

- 1) eliminate variables
- 2) simplify equations

go between gauges by defining  $T, L, L^{(+1)}, L^{(-1)}$   
( $\Rightarrow$  restoring general covariance = Bardeen's "gauge invariance")

- "Gauge Modes"

If  $T, L, L^{(+1)}, L^{(-1)}$  not completely specified  
gauge mode solutions (unphysical) appear

## • Newtonian Gauge

- Good: intuitive, Newtonian-like gravity  
matter & metric algebraically related
- Bad: numerically unstable
- Best: solve in another gauge  
transform to Newtonian to interpret

Defn:  $B = H_T = 0 \Rightarrow$  metric diagonal

$A \equiv \Psi \Rightarrow$  time-time pert; Newtonian Pot.

$H_L \equiv \Phi \Rightarrow$  space-space pert; Spatial Curvature

## Gauge Transformation into Newtonian

$$\dot{\tilde{H}}_T = 0 \Rightarrow \boxed{L = -H_T/k}$$

$$\dot{\tilde{B}} = 0 \Rightarrow kT = -B - \dot{L} \Rightarrow \boxed{T = -B/k + \dot{H}_T/k^2}$$

Gauge freedom entirely fixed  
(no gauge modes)

## EINSTEIN Eqs

concerns: unstable

$$(k^2 - 3K)\Phi = 4\pi G a^2 (\delta\rho + 3\frac{\dot{a}}{a}(\rho + p)v/k)$$

$$k^2(\Psi + \Phi) = -8\pi G a^2 p\pi \quad (\Rightarrow \Psi \approx -\Phi)$$

## Conservation

$$\left[\frac{d}{dt} + 3\frac{\dot{a}}{a}\right]\delta\rho + 3\frac{\dot{a}}{a}\delta p = -(\rho + p)(kv + 3\dot{\Phi})$$

$$\left[\frac{d}{dt} + 4\frac{\dot{a}}{a}\right][(\rho + p)v/k] = \delta p - \frac{2}{3}(1 - 3\frac{K}{k^2})p\pi + (\rho + p)\Psi$$

## Comoving Gauge

- Good: causality simple (Bardeen curvature conserved)
- Bad: time-space metric has no Newtonian analogue
- Best: use this gauge to prove general relations and then transform them to Newtonian

Defn:  $T^0_i = 0$  energy flux vanishes  $\Rightarrow B = v$

$H_T = 0$  no metric shear

$$A \equiv \xi$$

$$H_L \equiv \zeta \Rightarrow \text{"Bardeen Curvature"}$$

Gauge Transformation into Comoving

$$\tilde{v} - \tilde{B} = 0 \Rightarrow T = (v - B)/k$$

$$\tilde{H}_T = 0 \Rightarrow L = -H_T/k$$

Gauge freedom entirely fixed (no gauge modes)

Einstein Eqs:

$$\dot{\zeta} = \frac{\dot{a}}{a} \zeta - K v/k$$

$$(k^2 - 3K) \left[ \zeta + \frac{\dot{a}}{a} v/k \right]$$

$$= 4\pi G a^2 \delta\rho$$

Conservation Eqn

$$\left[ \frac{d}{dt} + 3\frac{\dot{a}}{a} \right] \delta\rho + 3\frac{\dot{a}}{a} \delta p = -(\rho + p)(k v + 3\dot{\zeta})$$

$$(\rho + p) \dot{\zeta} = -\delta p + \frac{2}{3} \frac{(1 - 3K)}{k^2} p \Pi$$

$$\dot{\zeta} = \frac{\dot{a}}{a} (\text{stress gradient}) = 0 \text{ as } k \rightarrow 0$$

## • Synchronous Gauge

- Good: stable, compatible with numerical codes
- Bad: off-diagonal metric - hard to interpret - no potential
- Best: numerically solve in this gauge  
interpret in Newtonian

Defn:  $A = B = 0 \Rightarrow$  metric pert one spatial

$$\begin{aligned}
 -H_L - \frac{1}{3} H_T &\equiv \eta \\
 H_T &\equiv h_T
 \end{aligned}
 \left\{ \begin{array}{l} h \equiv 6H_L \text{ also used} \\ \text{but less stable} \end{array} \right.$$

### Gauge Transformation into Synchronous

$$\tilde{A} = 0 \Rightarrow T = a^{-1} \int d\eta a A + c_1 a^{-1}$$

$$\tilde{B} = 0 \Rightarrow L = - \int d\eta (B + kT) + c_2$$

Gauge freedom remains in  $(c_1, c_2)$  fix through initial cond

### Einstein Eqns

$$-(k^2 - 3K) \left[ \eta + \frac{\dot{a}}{a} h_T / k^2 \right] = 4\pi G a^2 \left[ \delta\rho + 3 \frac{\dot{a}}{a} (\rho + p) v / k \right]$$

$$\dot{\eta} + \frac{K}{k^2} \dot{h}_T = 4\pi G a^2 (\rho + p) v / k$$

### Conservation Eqns

$$\left[ \frac{d}{d\eta} + 3 \frac{\dot{a}}{a} \right] \delta\rho + 3 \frac{\dot{a}}{a} \delta\rho = -(\rho + p) (k v - 3\dot{\eta} - \dot{h}_T)$$

$$\left[ \frac{d}{d\eta} + 4 \frac{\dot{a}}{a} \right] \left[ (\rho + p) v / k \right] = \delta\rho - \frac{2}{3} \left( 1 - 3 \frac{K}{k^2} \right) \rho \Pi$$

# Hybrid Formulation

- Gauge Transformation Properties
  - Defn of  $T, L$
- build fluctuation in one gauge from another

## Useful Examples

$$\begin{aligned} \delta\rho|_{\text{comoving}} &= \delta\rho|_{\text{arbit.}} - \dot{\rho} (v-B)|_{\text{arbit.}}/K \\ &= \delta\rho|_{\text{arbit.}} + \underbrace{3\frac{\dot{a}}{a}(\rho+p)(v-B)|_{\text{arbit.}}/K}_{\text{source of Poisson Eqn}} \end{aligned}$$

eg  $(K^2 - 3K)\Phi|_{\text{Newt}} = 4\pi G a^2 \delta\rho|_{\text{comoving}}$  hybrid

$$\mathcal{S}|_{\text{comov}} = \Phi|_{\text{Newt}} - \frac{\dot{\rho}}{a} v|_{\text{Newt}}/K$$

$$\mathcal{S}'|_{\text{comov}} = \Phi|_{\text{Newt}} + 2(\Psi|_{\text{Newt}} - \Phi'|_{\text{Newt}}) \frac{\rho + \rho_s}{\rho'}$$

$$\begin{aligned} ' &= \frac{d}{da} \\ \rho_s &= \frac{-3}{8\pi G a^2} K \end{aligned}$$

$$\mathcal{S}' - \mathcal{S}|_{\text{comov}} = (\Psi - \Phi')|_{\text{Newt}} \frac{\rho_s}{\rho'}$$

$\therefore$  Matter in Newtonian metric as labeled

and recall

$$\begin{aligned} \mathcal{S} &= -\text{stress fluctuation} \\ &= \frac{-\delta p}{\rho+p} + \frac{2}{3} \left(1 - 3\frac{K}{K^2}\right) \frac{P}{\rho+p} \Pi|_{\text{comov}} \end{aligned}$$