Astro 448: Problem Set 1

Due Jan. 22

1 Problem 1: Age, Conformal Time, and Distance

Consider a universe with no spatial curvature and a single matter species but with an arbitrary equation of state $w = p/\rho$.

- Write down the explicit expression for the Hubble parameter $H(a; \rho_0, w)$. Eliminate ρ_0 by requiring $H \to H_0$ as $a \to 1$.
- Write down the expression for $H_0 t_0$ as a function of w. What happens as $w \to -1$ a cosmological constant? Why is this not a problem in a universe with a matter and radiation as well as a cosmological constant? Convert your expression for the $w = 1/3$ case (radiation) into an explicit expression by back substituting $\rho_0 = aT^4$ in the expression for H_0. What is the age of the universe when $T = 10^9$K?
- Write down the expression for the conformal time $\eta_0 = \int_0^{t_0} dt/a(t)$ by changing variables from t to a using H. Evaluate $H_0\eta_0$ as a function of w. What happens as $w \to -1/3$?
- Write down the expression for the conformal time elapsed between an initial epoch a_i and a final epoch a_f and recall that it is also the comoving distance a particle going at the speed of light travels in this interval. For $w > -1/3$ what happens to this distance as $a_f \to \infty$? Discuss the implications for causal contact in such a universe. For $w < -1/3$ what happens? Again discuss the implications for causal contact. Note that $w < -1/3$ is an accelerating universe. Remember this for when we discuss inflation and dark energy. Recall also that the comoving distance is related to luminosity distance and physical angular diameter distance (in a spatially flat universe) by factors of a_i/a_f or redshift. Give these relations (you can find them in any cosmology book). Remember this when we discuss the location of the first peak in the CMB spectrum and the SNe results.

2 Problem 2: Compton-y distortions

Recall that the Kompaneets equation is given by

$$\frac{\partial f}{\partial t} = \frac{d\tau}{dt}\frac{k_B T_e}{m_e c^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial f}{\partial x} + f + f^2\right)\right], \tag{1}$$

where $x = h\nu/kT_e$.

Consider small deviations of the spectrum from the blackbody form

$$f = \frac{1}{e^{h\nu/k_B T} - 1}. \tag{2}$$

- Show
$$f + f^2 \approx -\frac{T}{T_e}\frac{\partial f}{\partial x}, \tag{3}$$
- Transform variables from time t to the Compton-y parameter
$$y = \int dt \frac{d\tau}{dt}\frac{k_B(T_e - T)}{m_e c^2}, \tag{4}$$
and write the Kompaneets equation in the form of a diffusion equation $\frac{\partial f}{\partial y} = \ldots$ The Kompaneets equation describes an upwards diffusion in energy of the photons via scattering off hotter electrons.
- Again assuming small deviations, insert the blackbody form eqn. (2) into the right hand side of the Kompaneets/diffusion equation. Trivially integrate the equation to show that the change in the distribution function is given by
$$\frac{\Delta f}{f} = -y x_\nu e^{x_\nu} f\left(4 - x_\nu \coth\frac{x_\nu}{2}\right), \tag{5}$$
where $x_\nu = h\nu/k_B T$.
- Define the effective thermodynamic temperature as the temperature of a blackbody that has the same f at a given frequency as the perturbed spectrum. Convert $\Delta f/f$ to $\Delta T/T$. What happens as $x_\nu \to 0$? What happens at $x_\nu \to \infty$. Argue that there must be a frequency (independent of y) at which $\Delta T/T = 0$. Numerically find this value of x_ν. Convert your answer to frequency (in GHz) and wavelength (cm) assuming $T = 2.726$K. This is known as the null in the thermal Sunyaev Zeldovich effect.