

Relativistic Perturbation Theory

Qualitative Discussion

- FRW lesson: detailed GR calculation has simple interpretations

Equations of motion = Einstein Eqs

+ (Covariant) Energy-Momentum Conservation

A prescription for the stresses $\omega = p/p$ (pressure) closes the system of equations

= redundancy since Bianchi identities automatically satisfied

- "Guess" the equations of motion in perturbation theory

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho \quad \text{Energy conservation for nr matter}$$

neglect gravitational perturbations temp

$$(\delta\rho)^* = -3\frac{\dot{a}}{a}\delta\rho - \nabla \cdot (\rho v) \quad \begin{matrix} \text{number} \\ \text{density} \end{matrix} \quad \begin{matrix} \text{divergence} \\ \text{dilution} \end{matrix} \quad \begin{matrix} \text{momentum} \\ \text{density} \end{matrix} \quad \text{energy conservation}$$

$$(\rho v)^* = -4\frac{\dot{a}}{a}(\rho v) + \nabla \delta p \quad \begin{matrix} \text{number} \\ \text{density} \end{matrix} \quad \begin{matrix} \text{pressure gradient} \\ \alpha \alpha' \end{matrix} \quad \begin{matrix} \text{momentum conservt.} \\ + \text{momentum decay} \end{matrix}$$

prescription for sound speed
 $\delta p/\delta \rho$ closed system

- Restore gravity in Newtonian limit

gravitational potential Ψ also

"Stretches" space by $\Phi \simeq -\Psi$

$$a \rightarrow a(1 + \Phi)$$

\therefore Continuity Equation

$$(\delta\rho)^* = -3\frac{\dot{a}}{a}\delta\rho - 3\dot{\Phi} - \nabla \cdot (\rho v)$$

perturbed
density dilution

\therefore Euler Equation

$$(\rho v)^* = -4\frac{\dot{a}}{a}(\rho v) + \nabla \delta p + \underline{\nabla \rho \Psi}$$

Potential
gradient

\therefore Poisson Equation

$$\frac{\nabla^2}{a^2} \Psi = 4\pi G \delta\rho$$

physical
derivative

Issues:

1. Coordinate ambiguity
e.g. time surface
2. Beyond the gravitational potential
e.g. Gravity Waves

Technical Details

- Covariant Perturbation Theory

Covariant = takes the same form in all coordinate systems

c.f. invariant = takes the same value in all coordinate systems

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\nabla_\mu T^{\mu\nu} = 0$$

covariant equations

preserve general covariance

- $g_{\mu\nu}$ is a 4×4 symmetric matrix

1	2	3	4
5	6	7	
8	9		
10			

= 10 degrees of freedom (functions of space & time)

- Bardeen (1980) decomposition

$$g^{00} = -\bar{a}^2 (1 - 2A) \quad \text{①}$$

$$g^{0i} = -\bar{a}^2 B^i \quad \text{③}$$

$$g^{ij} = \bar{a}^2 (\gamma^{ij} - 2H_L \gamma^{ij} - 2H_T^{ij})$$

↑ trace

1	2	3
4	5	
6		

↑ trace free

- Likewise, $T_{\mu\nu}$ is a 4×4 symmetric matrix

$$T^0_0 = -\bar{\rho} - \delta\rho \quad ①$$

$$T^0_i = (\rho + p)(v_i - B_i)$$

$$T^i_0 = -(ρ + p)v^i \quad ③$$

$$T^i_j = \underbrace{(\rho + \delta\rho)}_{\text{Isotropic Stress}} \delta^i_j + p \underbrace{\Pi^i_j}_{\text{Anisotropic Stress}}$$

- D.O.F. Counting:

20 variables (10 metric 10 matter)

- 10 Einstein Equations
 - 4 Conservation Equations
 - + 4 Bianchi Identities
-

10 D.O.F.

- 4 Gauge (coordinate choice: 1 time, 3 space)
-

6 D.O.F. \rightarrow 6 stress perturbations

$$(\delta p, \Pi^i_j)$$

completely general: any form of matter
e.g. matter, radiation, scalar fields, defects

• Scalar, Vector, Tensor

Fourier decomposition of a tensor
on a space of constant curvature K

a plane wave is an eigenfunction of
the Laplace operator $Q^{(0)} = e^{ikx}$ ($K=0$)

$$\nabla^2 Q^{(0)} = -K^2 Q^{(0)}$$

scalar density e^{ikx}

generalize

$$\nabla^2 Q_i^{(\pm 1)} = -K^2 Q_i^{(\pm 1)}$$

vector vorticity $\frac{-i}{\sqrt{2}} (\hat{e}_1 \pm i \hat{e}_2)_i e^{ikx}$

$$\nabla^2 Q_{ij}^{(\pm 2)} = -K^2 Q_{ij}^{(\pm 2)}$$

tensor gravity waves $-\sqrt{\frac{3}{8}} (\hat{e}_1 \pm i \hat{e}_2)_i (\hat{e}_1 \pm i \hat{e}_2)_j e^{ikx}$

Build Others

$$Q_i^{(0)} = -K^{-1} \nabla_i Q^{(0)}$$

(curl free vectors)

$$Q_{ij}^{(0)} = (K^2 \nabla_i \nabla_j - \frac{1}{3} \delta_{ij}) Q^{(0)}$$

$$Q_{ij}^{(\pm 1)} = \frac{-1}{2K} [\nabla_i Q_j^{(\pm 1)} + \nabla_j Q_i^{(\pm 1)}]$$

Fourier Amplitudes

$$A = \tilde{A} Q^{(0)}; H_L = \tilde{H}_L Q^{(0)}$$

$$\delta Q = \tilde{\delta} Q^{(0)}; \delta P = \tilde{\delta} P Q^{(0)}$$

$$H_{Tij} = \sum_{m=-2}^2 H_T^{(m)} Q_{ij}^{(m)}$$

$$B_i = \sum_{m \in I} \tilde{B}^{(m)} Q_i^{(m)}$$

$$V_i = \sum_{m \in I} \tilde{V}^{(m)} Q_i^{(m)}$$

$$\Pi_{ij} = \sum_{m=-2}^2 \tilde{\Pi}^{(m)} Q_{ij}^{(m)}$$

Covariant Scalar Equations

conservation

$$\left[\frac{d}{dt} + 3\frac{\dot{a}}{a} \right] \delta\rho + 3\frac{\dot{a}}{a} \delta p = -(ρ+P)(κv + 3\dot{H}_L) \quad \text{continuity}$$

$$\left[\frac{d}{dt} + 4\frac{\dot{a}}{a} \right] \left[(\rho+p) \frac{(v-B)}{\kappa} \right] = \delta p - \frac{2}{3} \left(1 - \frac{3K}{\kappa^2} \right) p\bar{\pi} \quad \text{Euler} \\ + (\rho+p) A$$

$$(κ^2 - 3K) \left[H_L + \frac{1}{3} H_T + \frac{\dot{a}}{a} \frac{1}{κ^2} (κB - \dot{H}_T) \right]$$

$$= 4\pi G a^2 \left[\delta\rho + 3\frac{\dot{a}}{a} (\rho+p)(v-B)/\kappa \right]$$

$$κ^2 (A + H_L + \frac{1}{3} H_T) + \left(\frac{d}{dt} + 2\frac{\dot{a}}{a} \right) (κB - \dot{H}_T) \\ = 8\pi G a^2 p\bar{\pi}$$

Poisson
(density egn)Shear
EgnMomentum
Egn

$$\frac{\dot{a}}{a} A - \dot{H}_L - \frac{1}{3} \dot{H}_T - \frac{K}{\kappa^2} (κB - \dot{H}_T)$$

$$= 4\pi G a^2 (\rho+p)(v-B)/\kappa$$

$$\left[2\ddot{\frac{a}{a}} - 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a} \frac{d}{dt} - \frac{K^2}{3} \right] A - \left[\frac{d}{dt} + \frac{\dot{a}}{a} \right] \left(\dot{H}_L + \frac{K}{6} B \right)$$

$$= 4\pi G a^2 \left(\delta p + \frac{1}{3} \delta\rho \right)$$

Pressure
Egn
(acceleration)

Metric: A, H_L time-time Gravitational
Space-space Potential

 B, H_T

Matter: $\delta\rho, \delta p$
 v, Π

$$6 \text{ egn} - 2 \text{ Bianchi Ids} = 4 \text{ egn}$$

$$\frac{8 \text{ var} - 2 \text{ gauge}}{2 \text{ d.o.f}} = 6 \text{ var}$$

$$\Rightarrow \delta\rho \Pi$$

Covariant Vector Equations

$$\left\{ \begin{array}{l} \text{conservation} \\ \left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] [(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k] \\ = -\frac{1}{2} (1 - \frac{2K}{\kappa^2}) p \bar{\Pi}^{(\pm 1)} \end{array} \right. \quad \text{Euler}$$

$$\left\{ \begin{array}{l} \text{momentum} \\ \left(1 - \frac{2K}{\kappa^2} \right) \kappa B^{(\pm 1)} - \dot{H}_T^{(\pm 1)} \\ = 16\pi G a^2 (\rho + p) (v^{(\pm 1)} - B^{(\pm 1)})/k \end{array} \right. \quad \text{Momentum Egn}$$

$$\left\{ \begin{array}{l} \text{shear Egn} \\ \left[\frac{d}{d\eta} + 2\frac{\dot{a}}{a} \right] (\kappa B^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) \\ = -8\pi G a^2 p \bar{\Pi}^{(\pm 1)} \end{array} \right.$$

Metric: $B^{(\pm 1)}, H_T^{(\pm 1)}$

6 egn - 2 Bianchi identities = 4 egn

Matter: $v^{(\pm 1)}, \bar{\Pi}^{(\pm 1)}$

8 var - 2 gauge = 6 var

2 d.o.f. $\Rightarrow \bar{\Pi}^{(\pm 1)}$

(Covariant) Tensor Equation

$$\left[\frac{d^2}{d\eta^2} + 2\frac{\dot{a}}{a} \frac{d}{d\eta} + (K^2 + 2K) \right] H_T^{(\pm 2)} = 8\pi G a^2 p \bar{\Pi}^{(\pm 2)} \quad \text{shear & g}$$

Metric: $H_T^{(\pm 2)}$

2 egn - 0 Bianchi ident = 2 egn

Matter: $\bar{\Pi}^{(\pm 2)}$

4 var - 0 gauge = 4 var
2 d.o.f. $\Rightarrow \bar{\Pi}^{(\pm 2)}$

\therefore Transverse traceless stresses \Rightarrow gravity waves

Multicomponent Generalization

Einstein Equations

$$\delta\rho = \sum_i \delta\rho_i$$

$$(\rho + p)v^{(m)} = \sum_i (\rho_i + p_i)v_i^{(m)}$$

$$\delta p = \sum_i \delta p_i$$

$$p\pi^{(m)} = \sum_i p_i \pi_i^{(m)}$$

Conservation Equations

continuity

+
Euler

for each component separately
conserved, i.e. interact only
through gravity

or

continuity

+
Euler

for each component

Supplemented by sources

- particle creation
- momentum exchange

Example Photon-Baryon

$$\left[\frac{d}{dt} + 4\frac{\dot{a}}{a} \right] \delta\rho_\gamma = -\frac{4}{3} \rho_\gamma (kv + 3\dot{H}_L)$$

$$\left[\frac{d}{dt} + 3\frac{\dot{a}}{a} \right] \delta\rho_b = -\rho_b (kv + 3\dot{H}_L)$$

$$\left[\frac{d}{dt} + 4\frac{\dot{a}}{a} \right] \frac{4}{3} \rho_\gamma v_\gamma = \frac{1}{3} k \delta\rho_\gamma - \frac{2}{9} k \left(1 - \frac{3K}{K^2}\right) \rho_\gamma \pi_\gamma - \frac{4}{3} \rho_\gamma S_\tau + \frac{4}{3} K \rho_\gamma A$$

$$\left[\frac{d}{dt} + 4\frac{\dot{a}}{a} \right] \rho_b v_b = k \rho_b A + \frac{4}{3} \rho_\gamma S_\tau$$

$$S_\tau \equiv \dot{\tau}(v_\gamma - v_b) \leftarrow \text{deriv: Doppler effect}$$