

# Tight Coupling Approximation

- Rapid Scattering limit

- No anisotropy of CMB in electron rest frame
- CMB dipole  $v_\delta = v_b$
- CMB quadrupole  $\Pi_\delta = 0$

- Equations of Motion in Newtonian Gauge

$$\left[ \frac{d}{dt} + 4\frac{\dot{a}}{a} \right] \delta \rho_\delta = -\frac{4}{3} \rho_\delta (\kappa v_\delta + 3\dot{\Psi})$$

$$\left[ \frac{d}{dt} + 3\frac{\dot{a}}{a} \right] \delta \rho_b = -\rho_b (\kappa v_b + 3\dot{\Psi})$$

$$\begin{aligned} \left[ \frac{d}{dt} + 4\frac{\dot{a}}{a} \right] \frac{1}{3} \rho_b v_\delta &= \frac{1}{3} \kappa \delta \rho_\delta - \frac{2}{3} \kappa \left( 1 - \frac{3K}{\kappa^2} \right) \rho_\delta \Pi_\delta \\ &\quad + \frac{4}{3} \kappa \rho_\delta \dot{\Psi} - \frac{4}{3} \dot{\tau} \rho_\delta (v_\delta - v_b) \end{aligned}$$

$$\left[ \frac{d}{dt} + 4\frac{\dot{a}}{a} \right] \rho_b v_b = \kappa \rho_b \dot{\Psi} + \frac{4}{3} \dot{\tau} \rho_\delta (v_\delta - v_b)$$

Simplify:  $\Pi_\delta = 0$   $v_b = v_\delta$  (formally lowest order in  $k/\dot{z} \ll 1$ , [optical depth through wavelength])

## • Lowest Order (perfect coupling)

- Add Euler Equations with  $v_b = v_\gamma$

$$\left[ \frac{d}{dm} + 4\frac{\dot{a}}{a} \right] \left( \frac{4}{3} \rho_\gamma v_\gamma + \rho_b v_\gamma \right) = \frac{\kappa}{3} \delta \rho_\gamma + k \left( \frac{4}{3} \rho_\gamma + \rho_b \right) \Psi$$

- Define Baryon-Photon momentum density ratio

$$R = \frac{3\rho_b}{4\rho_\gamma}$$

$$\left[ \frac{d}{dm} + 4\frac{\dot{a}}{a} \right] \left( \frac{4}{3} \rho_\gamma v_\gamma (1+R) \right) = \frac{\kappa}{3} \delta \rho_\gamma + \frac{4}{3} \rho_\gamma (1+R) \kappa \Psi$$

$$\frac{d}{dm} \rho_\gamma = -4\rho_\gamma \frac{\dot{a}}{a} \quad \Theta_0 = \frac{1}{4} \frac{\delta \rho_\gamma}{\rho_\gamma} = \frac{\Delta T}{T}$$

$$(1+R) \dot{v}_\gamma + R v_\gamma = \kappa \Theta_0 + (1+R) \Psi$$

$$\dot{v}_\gamma = -\frac{R}{1+R} v_\gamma + \frac{1}{1+R} \kappa \Theta_0 + \kappa \Psi$$

$\uparrow$                      $\uparrow$                      $\uparrow$   
 drag term    reduced pressure    gravity affects  
 behavior under expansion       $\gamma$  and  $b$  alike

$v_b \propto a^{-1}$ : particle momentum decay

$v_\gamma = \text{const}$  particle momentum

decay already in redshift of  $T$

dipole  $\frac{\Delta T}{T} \leftarrow$  both redshift

## • Continuity Equation / Adiabatic Evolution

$$\left[ \frac{d}{dt} + 4 \frac{\dot{a}}{a} \right] \delta \rho_\gamma = - \frac{4}{3} \rho_\gamma (\kappa v_\gamma + 3 \dot{\Phi})$$

$$\left[ \frac{d}{dt} + 3 \frac{\dot{a}}{a} \right] \delta \rho_b = - \rho_b (\kappa v_\gamma + 3 \dot{\Phi})$$

define fractional density fluctuation

$$\delta_b \equiv \frac{\delta \rho_b}{\rho_b} \quad \delta_\gamma \equiv \frac{\delta \rho_\gamma}{\rho_\gamma}$$

$$\dot{\Theta}_0 = - \frac{\kappa}{3} v_\gamma - \dot{\Phi}$$

$$\dot{\delta}_b = - \kappa v_\gamma - 3 \dot{\Phi}$$

adiabatic evolution

$$\dot{\Theta}_0 = \frac{1}{3} \dot{\delta}_b \quad \text{or} \quad \dot{\delta}_\gamma = \frac{4}{3} \dot{\delta}_b$$

local number density of baryons to photons  
conserved ("entropy" fluctuation)

$$\sigma \equiv \frac{\delta(n_b/n_\gamma)}{n_b/n_\gamma} = \delta_b - \frac{3}{4} \delta_\gamma$$

$\Rightarrow \dot{\sigma} = 0$  in tight coupling limit

## Oscillator Equations

$$\dot{\Theta}_0 = -\frac{\kappa}{3} v_\delta - \dot{\Phi}$$

$$\dot{v}_\delta = -\frac{R}{1+R} v_\delta + \frac{1}{1+R} \kappa \Theta_0 + \kappa \Psi$$

define effective mass of oscillator

$$m_{\text{eff}} \equiv 1 + R$$

$$(m_{\text{eff}} v_\delta)^\ddot{} = \kappa \Theta_0 + m_{\text{eff}} \kappa \Psi$$

$$m_{\text{eff}} \dot{\Theta}_0 = -\frac{\kappa}{3} m_{\text{eff}} v_\delta - m_{\text{eff}} \dot{\Phi}$$

$$(m_{\text{eff}} \dot{\Theta}_0)^\ddot{} = -\frac{\kappa}{3} (\kappa \Theta_0 + m_{\text{eff}} \kappa \Psi) - (m_{\text{eff}} \dot{\Phi})^\ddot{}$$

$$(m_{\text{eff}} \dot{\Theta}_0)^\ddot{} + \frac{\kappa^2}{3} \Theta_0 = -m_{\text{eff}} \frac{\kappa^2}{3} \Psi - (m_{\text{eff}} \dot{\Phi})^\ddot{}$$

Forced Oscillator Equation with  
Time Dependent Mass

## • Approximations

$$\frac{\dot{m}_{\text{eff}}}{m_{\text{eff}}} \approx 0$$

$\left\{ \begin{array}{l} R \ll 1 \\ \text{or} \\ \text{time scales short} \\ \text{compared with expansion} \end{array} \right.$

$$\ddot{\Theta}_0 + \frac{k^2}{3m_{\text{eff}}} \Theta_0 = -\frac{k^2}{3} \Psi - \ddot{\Phi}$$

adiabatic sound speed

$$\begin{aligned} c_s^2 &= \dot{P}/\dot{\rho} = \frac{\frac{1}{3} \dot{\rho}_s}{\dot{\rho}_s + \dot{\rho}_b} = \frac{1}{3} \frac{1}{1 + \dot{\rho}_b/\dot{\rho}_s} \\ &= \frac{1}{3} \frac{1}{1 + R} = \frac{1}{3m_{\text{eff}}} \end{aligned}$$

rewrite

$$\boxed{\ddot{\Theta}_0 + k^2 c_s^2 \Theta_0 = -\frac{k^2}{3} \Psi - \ddot{\Phi}}$$

oscillations propagate with speed of sound  
in medium

$$\text{Constant Curvature } \dot{\Phi} \approx 0 \quad \left\{ \begin{array}{l} km \ll 1 \text{ superhorizon} \\ \text{or} \\ a \gg a_{\text{eq}} \text{ matterdom. (stress free)} \end{array} \right.$$

$$\ddot{\Theta}_0 + k^2 c_s^2 \Theta_0 = -\frac{k^2}{3} \Psi$$

harmonic oscillator under constant grav. field

- Solution for ( $\dot{m}_{\text{eff}} \approx \dot{\Phi} \approx 0$ )

$$\Theta_0(\eta) = [\Theta_0(0) + (1+R)\Psi] \cos ks + \frac{1}{ks} \dot{\Theta}_0(0) \sin ks - (1+R)\Psi$$

define sound horizon

$$S \equiv \int_0^{\eta} d\eta' c_s(\eta')$$

- Inflationary Initial Conditions

$$\Theta_0 = -\frac{2}{3}\Psi \quad (\text{gauge transformation in matter dominated epoch})$$

or

$$\Theta_0 = -\frac{1}{2}\Psi \quad (\text{gauge transformation in radiation dominated epoch})$$

take initial conditions in matter dominated epoch for simplicity & consistency ( $\dot{\Phi} \neq 0$  for RD)

$$V_F = -\frac{3\dot{\Theta}_0}{k}; \dot{\Theta}_0(0) = 0$$

Inflectionary initial conditions select the Cosine phase of the oscillation

- Inflationary Mode (a.k.a. adiabatic mode)

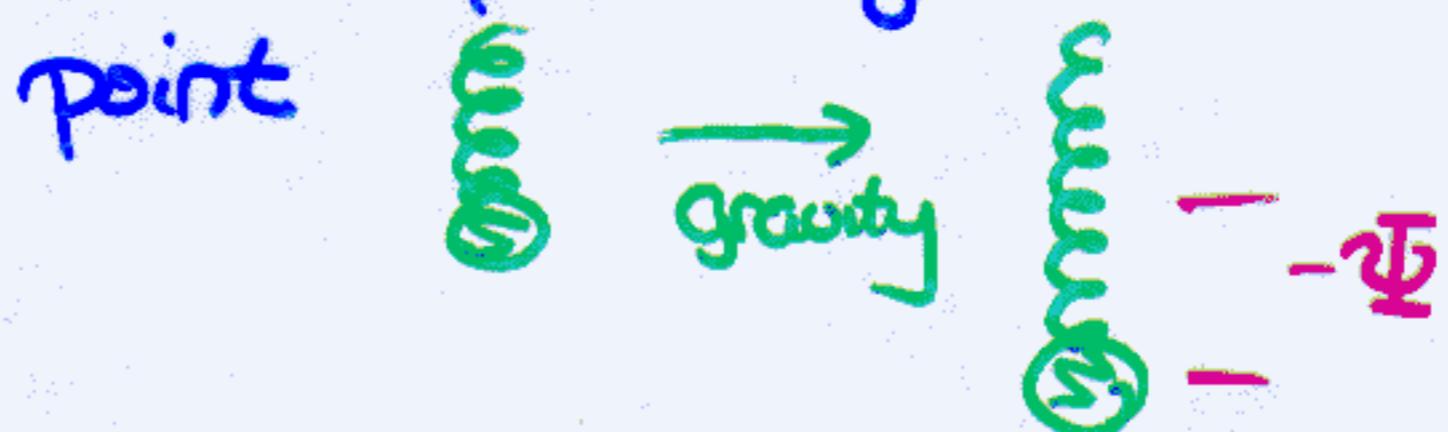
$$\Theta_0(\eta) = \left[ \frac{1}{3} + R \right] \bar{\Psi} \cos k\eta - (1+R) \bar{\Psi}$$

phenomenology:

- zero point of oscillation

$$\Theta_0 = -(1+R) \bar{\Psi}$$

- $-\bar{\Psi}$  piece: constant gravitational field displaced equilibrium point



- adding baryons increases shift by changing the effective mass

$$\Theta_0 = -m_{\text{eff}} \bar{\Psi}$$



- photons climbing out of  $\bar{\Psi}$  after recombination lose energy exactly cancelling the photon part of the zero point shift

$$\Theta_0 + \bar{\Psi} (= -R^2 \bar{\Psi} \text{ in equil}) \equiv \boxed{\text{effective temp}}$$

## Doppler Effect

- fluid in motion: doppler effect due to velocity along line of sight to observer

$$\hat{n} \cdot \vec{\nabla}_g = \frac{\Delta T}{T}$$

- rms amplitude:  $\frac{\Delta T}{T} = \frac{V_8}{\sqrt{3}}$

- solution:

$$\frac{V_8}{\sqrt{3}} = -\frac{\sqrt{3}}{K} \dot{\theta}_0$$

$$\frac{V_8}{\sqrt{3}} = -\frac{\sqrt{3}}{K} \left[ -\left(\frac{1}{3} + R\right) \Psi \sin k_s \text{ kGs} \right]$$

$$= \frac{1}{(1+R)^{1/2}} \left[ \left(\frac{1}{3} + R\right) \Psi \sin k_s \right]$$

- phenomenology

- doppler shift is  $\frac{1}{2}$  out of phase with temperature

- Amplitude is comparable to temperature oscillation (reduced by  $(1+R)^{1/2}$ )

- No zero point shift

## • Baryon/Photon time dependence

R d a

$m_{\text{eff}} = (1+R)$  grows with time

adiabatic approximation (WKB solution):  
mass changes slowly compared with oscillation

an oscillator with a time dependent mass  
has an adiabatic invariant

$$\frac{E}{\omega} = \text{const}$$

$$= \frac{1}{2} \frac{m_{\text{eff}} \omega^2 A^2}{\omega}$$

$$= \frac{1}{2} m_{\text{eff}} \omega A^2$$

$$A \propto \frac{1}{(m_{\text{eff}} \omega)^{1/2}} \propto \frac{1}{(1+R)^{1/4}}$$

1) Amplitude decays with increasing baryons

2) gravity-free solutions

$$\Theta_a = (1+R)^{-1/4} \cos(kS)$$

$$\Theta_b = (1+R)^{-1/4} \sin(kS)$$

[homogeneous]  
egn

these can be used to construct full solution  
if gravitational potentials known