A Parameterized Post-Friedmann Framework for Modified Gravity

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We develop a parameterized post-Friedmann (PPF) framework which describes three regimes of modified gravity models that accelerate the expansion without dark energy. On large scales, the evolution of scalar metric and density perturbations must be compatible with the expansion history defined by distance measures. On intermediate scales in the linear regime, they form a scalar-tensor theory with a modified Poisson equation. On small scales in dark matter halos such as our own galaxy, modifications must be suppressed in order to satisfy stringent local tests of general relativity. We describe these regimes with three free functions and two parameters: the relationship between the two metric fluctuations, the large and intermediate scale relationships to density fluctuations and the two scales of the transitions between the regimes. We also clarify the formal equivalence of modified gravity and generalized dark energy. The PPF description of linear fluctuation in $f(R)$ modified action and the Dvali-Gabadadze-Porrati braneworld models show excellent agreement with explicit calculations. Lacking cosmological simulations of these models, our non-linear halo-model description remains an ansatz but one that enables well-motivated consistency tests of general relativity. The required suppression of modifications within dark matter halos suggests that the linear and weakly non-linear regimes are better suited for making complementary test of general relativity than the deeply non-linear regime.

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I. INTRODUCTION

Theoretically compelling alternatives to a cosmological constant as the source of the observed cosmic acceleration are currently lacking. In the absence of such alternatives, it is useful to have a phenomenological parameterized approach for testing the predictions of a cosmological constant and phrasing constraints in a model-independent language. This approach parallels that of local tests of general relativity. The parameterized post-Newtonian description of gravity forms a complete description of leading order deviations from general relativity locally under a well-defined set of assumptions [1].

A parameterization of cosmic acceleration from the standpoint of dark energy is now well-established. The expansion history that controls distance observables is completely determined by the current dark energy density and its equation of state as a function of redshift. Structure formation tests involve additional parameters that control inhomogeneities in the dark energy. Covariant conservation of energy-momentum requires that the dark energy respond to metric or gravitational potential fluctuations at least on scales above the horizon. In a wide class of models where the dark energy remains smooth relative to the matter on small scales, the phenomenological parameter of interest is where this transition occurs [2, 3, 4].

A similar structure is imposed on modified gravity models that accelerate the expansion without dark energy. Requirements that gravity remain a metric theory where energy-momentum is covariantly conserved also place strong constraints their scalar degrees of freedom. On scales above the horizon, structure evolution must be compatible with the background expansion [5]. Intermediate scales are characterized by a scalar-tensor theory with a modified Poisson equation [6]. If these modifications are to pass stringent local tests of gravity then additional scalar degrees of freedom must be suppressed locally [7]. Two explicit models that exhibit all three regimes of modified gravity are the so-called $f(R)$ modified Einstein-Hilbert action models [8, 9, 10] and the Dvali-Gabadadze-Porrati (DGP) braneworld model [11].

Although several parameterized gravity approaches exist in the literature, none describe all three regimes of modified gravity (cf. [12, 13]) and most do not explicitly enforce a metric structure to gravity or energy momentum conservation (e.g. [14, 15, 16, 17, 18]).

In this paper, we develop a parameterized post-Friedmann (PPF) framework that describes all three regimes of modified gravity models that accelerate the expansion without dark energy. We begin in §II by describing the three regimes individually and the requirements they impose on the structure of such modifications. In §III, we describe a linear theory parameterization of the first two regimes and test it against explicit calculations of the $f(R)$ and DGP models. In §IV, we develop a non-linear ansatz for the third regime based on the halo model of non-linear clustering. In the Appendix, we clarify the formal relationship between modified gravity and dark energy beyond the smooth class of models.

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II. THREE REGIMES OF MODIFIED GRAVITY

In this section, we discuss the three regimes of modified gravity theories that accelerate the expansion without dark energy. We begin by reviewing the requirements on super-horizon metric perturbations imposed by compatibility with a given expansion history (§II A). Modifications that introduce extra scalar degrees of freedom then enter a quasi-static regime characterized by a modified Poisson equation (§II B). Finally stringent local tests of gravity require that modifications are suppressed in the non-linear regime within collapsed dark matter haloes (§II C).

A. Post-Friedmann Super-horizon Regime

Under the assumption that modified gravity remains a metric theory in a statistically homogeneous and isotropic cosmology where energy-momentum is covariantly conserved, a parameterization of the expansion history that is complete under general relativity is complete under modified gravity as well. A modified gravity model and a dark energy model with the same expansion rate $H = a^{-1}da/dt$ and spatial curvature predicts the same observables for any measure that is based on the distance-redshift relation (see e.g. [19]). Hence in terms of the background, modified gravity models can be parameterized in the same way as dark energy without loss of generality. Neglecting spatial curvature and radiation for simplicity here and throughout, we can assign an effective energy density

$$\rho_{\rm eff} = \frac{3}{8\pi G}(H^2 - H_m^2 a^{-3}) \, , \tag{1}$$

where

$$H_m^2 \equiv \frac{8\pi G}{3}\rho_m(\ln a = 0) \tag{2}$$

would be the contribution of matter to the expansion under the normal Friedmann equation. Alternately we can assign a current effective energy density in units of the critical density $\Omega_{\rm eff} = 8\pi G\rho_{\rm eff}/3H_0^2$ and an effective equation of state

$$1 + w_{\rm eff}(\ln a) \equiv -\frac{1}{3}\frac{\rho'_{\rm eff}}{\rho_{\rm eff}} = -\frac{1}{3}\frac{2HH' + 3H_m^2 a^{-3}}{H^2 - H_m^2 a^{-3}} \, . \tag{3}$$

Compatibility with this expansion combined with energy momentum conservation highly constrains the evolution of metric fluctuations above the horizon. Super-horizon metric fluctuations in a perturbed universe can be viewed as evolving as a separate universe under the same modified Friedmann equation but with different parameters.

Bertschinger [5] showed that consequently metric fluctuations in fact obey the same fundamental constraints as they do in general relativity. These constraints appear in different ways in different gauges as detailed in the Appendix. Under the assumptions of a metric theory and energy-momentum conservation, all of the usual gauge structure including the so-called "gauge invariant" approach used here apply to modified gravity as well.

In the comoving gauge of the matter, the constraint for adiabatic initial conditions looks particularly simple. The curvature or space-space piece of the metric fluctuation ζ remains constant to leading order ([20], see also (A14))

$$\zeta' = \mathcal{O}(k_H^2\zeta) \, , \tag{4}$$

where $' = d/d\ln a$ and $k_H = k/aH$ is the wavenumber in units of the Hubble parameter. In the more familiar Newtonian gauge where the curvature is denoted Φ and the time-time piece or gravitational potential Ψ, the gauge transformation equation (A19)

$$\zeta = \Phi - V_m/k_H \tag{5}$$

and the momentum conservation equation (A8)

$$V_m' + V_m = k_H\Psi \tag{6}$$

along with Eqn. (4) imply

$$\Phi'' - \Psi' - \frac{H''}{H'}\Phi' - \left(\frac{H'}{H} - \frac{H''}{H'}\right)\Psi = \mathcal{O}(k_H^2\zeta) \, . \tag{7}$$

Here V_m is the scalar velocity fluctuation of the matter in both the comoving and Newtonian gauges (see Eqn. (A5) and Eqn. (A10)). Eqn. (7) is also satisfied in general relativity [21].

These relations have been explicitly shown to hold for DGP braneworld gravity [22] and $f(R)$ modified action gravity [23]. What distinguishes a particular model of gravity or dark energy is the relationship between Φ and Ψ in the Newtonian gauge or equivalently ζ and V_m in the comoving gauge. We will parameterize this relation by

$$g \equiv \frac{\Phi + \Psi}{\Phi - \Psi} = \frac{k_H\zeta + V_m' + 2V_m}{k_H\zeta - V_m'} \, . \tag{8}$$

In terms of the post-Newtonian parameter $\gamma = -\Phi/\Psi$, $g = (\gamma - 1)/(\gamma + 1)$. Given that gravitational redshift and lensing effects involve the metric combination

$$\Phi_- = \frac{\Phi - \Psi}{2} \, , \tag{9}$$

we will typically state metric results in terms of Φ_-. It is useful to note that $\Phi = (g+1)\Phi_-$ and $\Psi = (g-1)\Phi_-$. Super-horizon scalar metric fluctuations for adiabatic perturbations are completely defined by the expansion history H and the metric ratio g.

In Fig. 2 we show the evolution of Φ_- given a metric ratio that evolves as $g = g_0 a$ for a ΛCDM expansion history defined by $w_{\rm eff} = -1$ and $1 - \Omega_{\rm eff} = \Omega_m = 0.24$. Given

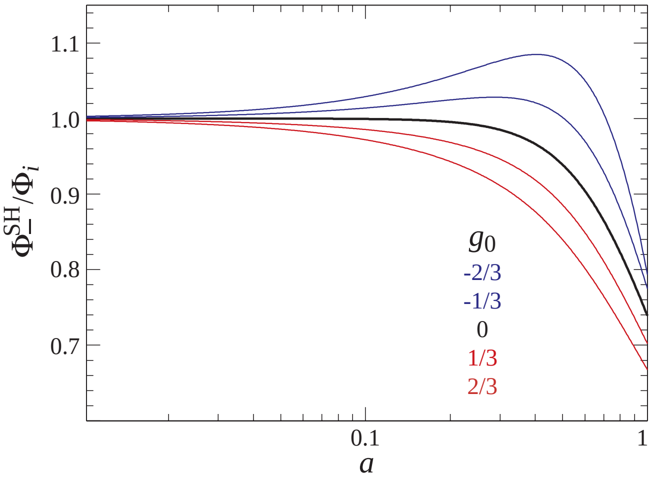

FIG. 1: Super-horizon (SH) metric evolution of $\Phi_- = (\Phi - \Psi)/2$ for various choices of $g = g_0 a$. For $g < 0$, Φ_- can actually grow near the onset of acceleration. The expansion history is fixed by $w_{\rm eff} = -1$ and $\Omega_m = 0.24$ in all cases.

that $g \to 0$ as $a \to 0$, we take matter dominated initial conditions of $\Phi_- = \Phi_i = 3\zeta_i/5$ and $\Phi_-' = 0$ as in general relativity. Note that if $g < 0$, Φ_- can actually grow during the acceleration epoch. Changing the evolution of gravitational potentials alters the low order multipoles of the CMB through the integrated Sachs Wolfe effect. For example it can be used to suppress the CMB quadrupole (see e.g. [23]).

Under general relativity, the metric ratio is determined by the ratio of anisotropic stress to energy density (see Eqn. (A21)). It supplements the information in the expansion history since in the background the anisotropic stress vanishes by assumption. In particular, with the matter carrying negligible anisotropic stress, this relationship is determined by the anisotropic stress of the dark energy. For models of dark energy based on scalar fields, this also vanishes in linear theory and hence $g = 0$. Nevertheless, a hypothetical dark energy component that produces the same g as a modified gravity model cannot be distinguished from super-horizon metric fluctuations. We further examine the relationship between modified gravity and dark energy in the Appendix.

B. Post-Newtonian Quasi-Static Regime

Well inside the horizon but still in the regime of linear fluctuations, theories that exhibit an extra scalar degree of freedom tend toward a post-Newtonian scalar-tensor description. As is the case in general relativity when time derivatives of the metric fluctuations can be ignored compared with spatial gradients, the modified field equations reduce to a modified Poisson equation

$$k^2 \Phi_- = \frac{4\pi G}{1+f_G} a^2 \rho_m \Delta_m \,, \tag{10}$$

where Δ_m is the fractional density perturbation (in the matter comoving gauge), f_G parameterizes a possibly time-dependent modification of the Newton constant, and the relationship between the two metric fluctuations is again parameterized by g as in Eqn. (8). We call this the quasi-static approximation. This quasi-static approximation has been explicitly shown to hold for both DGP braneworld gravity [22, 24] and $f(R)$ models [23, 25] once $k_H \gg 1$.

Density fluctuations on the other hand are determined by the Newtonian limit of the conservation equations

$$\begin{aligned} \Delta_m' &= -k_H V_m \,, \\ V_m' + V_m &= k_H \Psi = (g-1) k_H \Phi_- \,. \end{aligned} \tag{11}$$

Combining the modified Poisson equation (10) and the conservation equations (11) and taking f_G =const. for simplicity yields

$$\begin{aligned} &\Phi_-'' + \left(4 + \frac{H'}{H}\right)\Phi_-' \\ &\quad + \left[3 + \frac{H'}{H} + \frac{3}{2}\frac{H_m^2}{(1+f_G)H^2 a^3}(g-1)\right]\Phi_- = 0 \,. \end{aligned} \tag{12}$$

Note that this quasi-static equation is inequivalent to the super-horizon evolution (7) whenever $w_{\rm eff} \neq -1$, $g \neq 0$, or $f_G \neq 0$. The differences for $w_{\rm eff} \neq -1$ applies in general relativity as well and corresponds to the well-known fact that growth defined by a smooth dark energy component is inconsistent with conservation of energy momentum on super-horizon scales (e.g. [2, 3]). In Fig. 2, we compare the quasi-static and super-horizon evolution to the present. Note that in terms of the change in the gravitational potential from its initial value, important for gravitational redshift effects in the CMB, the differences are amplified. Compared with ΛCDM where $\Delta\Phi_- \approx -\Phi_i/4$ (see Fig. 1), these changes are enhanced by a factor of ~ 4 and cannot be neglected for large values of $|g|$.

C. General Relativistic Non-linear Regime

A successful modification of gravity must have a third regime where non-linearities in the modified field equations bring the dynamic back to general relativity. Scalar-tensor modifications of gravity that persist to small scales can be ruled out by stringent local tests of gravity. For example, the Cassini mission imposes the limit [1]

$$|g| < 1.2 \times 10^{-5} \tag{13}$$

for the metric ratio in the solar system.

In both the DGP braneworld and $f(R)$ modifications, the extra scalar degree of freedom obeys a non-linear equation that suppresses its effects within collapsed objects such as dark matter halos. In the DGP braneworld model, the scalar degree of freedom represents a brane

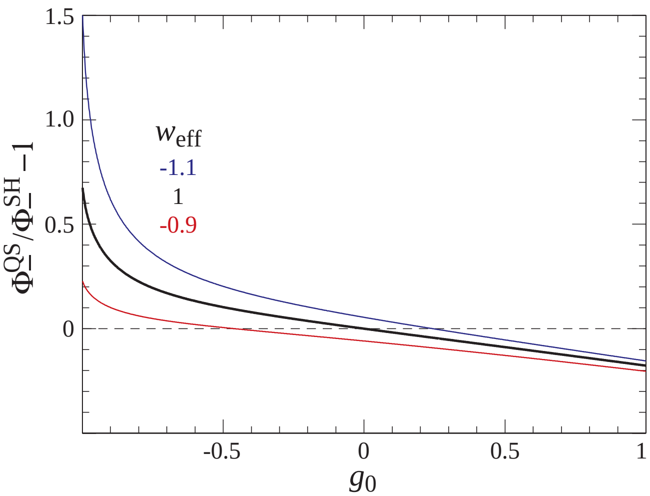

FIG. 2: Fractional difference between the quasi-static (QS) vs super-horizon (SH) metric at $a = 1$ for a metric ratio of $g = g_0 a$. The two evolutions differ unless $w_{\rm eff} = -1$ and $g_0 = 0$. $\Omega_m = 0.24$ for all cases.

bending mode. Non-linear interactions of this mode become important at the so-called Vainshtein radius $r_* \sim (2GMr_c^2)^{1/3}$ [7]. Here $r_c \sim H_0^{-1}$ is the crossover scale where gravity becomes fully 5D. This radius is comparable to the virial radius of dark matter halos and suggests that inside a halo gravitational interactions behave as in general relativity.

Similarly, for modified action $f(R)$ models the extra scalar degree of freedom corresponds to df/dR and has a mass that depends on the local curvature R. Deviations from general relativity can then be suppressed by the so-called chameleon mechanism [26, 27] so long as the gravitational potential is sufficiently deep [28, 29].

In the empirical PPF description that follows in §III and §IV we seek a description that joins these three regimes.

III. LINEAR THEORY PARAMETERIZATION

In this section, we construct a parameterized framework for linear perturbations in modified gravity models that joins the super-horizon and quasi-static regimes described in the previous section. We first describe the construction (§III A) and then test the parameterization against modified action $f(R)$ models (§III B) and the DGP braneworld model (§III C).

A. PPF Parameters

We have seen in §II that in both the super-horizon and quasi-static regimes, the evolution of metric fluctuations are primarily determined by the metric ratio g. However, the manner in which the metric ratio g determines metric evolution differs between the two regimes. The quasi-static regime also allows the freedom to change the effective Newton constant.

Let us introduce a parameterization that bridges the dynamics of the two regimes at a scale that is parameterized in units of the Hubble scale. The Newtonian-limit conservation equations (11) must first be corrected for metric evolution. The exact conservation equations imposed by $\nabla^\mu T_{\mu\nu} = 0$ and the metric is given by (see Eqn. (A15) and (A20))

$$\begin{aligned} \Delta_m' &= -k_H V_m - 3\zeta' \,, \\ V_m' + V_m &= (g-1)k_H\Phi_- \,, \end{aligned} \tag{14}$$

where the additional term involving evolution of the metric is given in the matter comoving gauge ζ' to match the definition of density perturbations Δ_m in this gauge. It is related to the evolution of the Newtonian metric by Eqn. (5)

$$\zeta' = (g+1)\Phi_-' + (1-g+g')\Phi_- - \frac{H'}{H}\frac{V_m}{k_H} \,. \tag{15}$$

In order to match the super-horizon scale behavior we introduce an additional term Γ to the modified Poisson equation (10)

$$k^2[\Phi_- + \Gamma] = 4\pi G a^2 \rho_m \Delta_m \,. \tag{16}$$

We now demand that as $k_H \to 0$ Γ enforces the metric evolution of Eqn. (7). In this limit, the derivative of Eqn. (16) gives an evolution equation for Γ given the conservation equations (14) and the required metric evolution. Aside from g, the only remaining freedom is determining the leading order behavior of ζ'. Without loss of generality, we can parameterize Eqn. (4) with a possibly time-dependent function f_ζ

$$\lim_{k_H\to 0} \zeta' = \frac{1}{3} f_\zeta k_H V_m \,. \tag{17}$$

Although the super-horizon metric is determined by H and g alone, its relationship to the comoving density perturbation is not. Since $k_H V_m = \mathcal{O}(k_H^2\zeta)$ and $\zeta' = \mathcal{O}(k_H^2\zeta)$, this degree of freedom enters into the conservation equation (14) at leading order. Combining these relations, we obtain the equation of motion for Γ

$$\Gamma' + \Gamma = S \,, \qquad (k_H \to 0) \,, \tag{18}$$

where the source is

$$\begin{aligned} S = &-\left[\frac{1}{g+1}\frac{H'}{H} + \frac{3}{2}\frac{H_m^2}{H^2 a^3}(1+f_\zeta)\right]\frac{V_m}{k_H} \\ &+ \left[\frac{g'-2g}{g+1}\right]\Phi_- \,. \end{aligned} \tag{19}$$

Here we have kept only the leading order term in k_H.

Note that the exact choice of f_ζ is rarely important for observable quantities. Any choice will produce the correct behavior of the metric evolution since that depends only on enforcing $\zeta' = \mathcal{O}(k_H^2\zeta)$. Hence observables associated with gravitational redshifts and lensing are not sensitive to this choice. Only observables that depend on the comoving density on large scales beyond the quasi-static regime are affected by this parameter. Furthermore the super-horizon density perturbation in Newtonian gauge or any gauge where the density fluctuation evolves as the metric fluctuation is also insensitive to f_ζ.

On small scales, recovery of the modified Poisson equation (10) from (16) implies

$$\Gamma = f_G\Phi_- \,, \qquad (k_H \to \infty)\,. \tag{20}$$

Finally to interpolate between these two limits we take the full equation of motion for Γ to be

$$(1 + c_\Gamma^2 k_H^2)\left[\Gamma' + \Gamma + c_\Gamma^2 k_H^2\left(\Gamma - f_G\Phi_-\right)\right] = S\,. \tag{21}$$

For models where $S \to 0$ as $a \to 0$ we take initial conditions of $\Gamma = \Gamma' = 0$ when the mode was above the horizon.

In summary, given an expansion history $H(a)$, our PPF parameterization is defined by 3 functions and 1 parameter: the metric ratio $g(\ln a, k_H)$, the super-horizon relationship between the metric and density $f_\zeta(\ln a)$, the quasi-static relationship or scaling of Newton constant $f_G(\ln a)$, and the relationship between the transition scale and the Hubble scale c_Γ. For models which modify gravity only well after matter radiation equality, these relations for the metric, density and velocity evolution combined with the usual transfer functions completely specify the linear observables of the model. In specific models, these functions can themselves be simply parameterized as we shall now show for the $f(R)$ and DGP models.

B. $f(R)$ Models

In $f(R)$ models, the Einstein-Hilbert action is supplemented by the addition of a free function of the Ricci scalar R. The critical property of these models is the existence of an extra scalar degree of freedom $f_R = df/dR$ and the inverse-mass or Compton scale associated with it. The square of this length in units of the Hubble length is proportional to

$$B \;=\; \frac{f_{RR}}{1+f_R}R'\frac{H}{H'}\,, \tag{22}$$

where $f_{RR} = d^2f/dR^2$. Below the Compton scale, the metric ratio $g \to -1/3$.

The evolution of B and the expansion history come from solving the modified Friedmann equation obtained by varying the action with respect to the metric. We follow the parameterized approach of [23] where a choice of the expansion history through $w_{\rm eff}$ and the Compton scale today $B_0 \equiv B(\ln a = 0)$ implicitly describes the $f(R)$ function and model. For illustrative purposes, we take $\Omega_m = 0.24$ and $w_{\rm eff} = -1$.

Given $H(\ln a)$ and $B(\ln a)$, the metric ratio at super-horizon scales comes from solving Eqn. (7)

$$\begin{aligned}\Phi'' &+ \left(1 - \frac{H''}{H'} + \frac{B'}{1-B} + B\frac{H'}{H}\right)\Phi' \\ &+ \left(\frac{H'}{H} - \frac{H''}{H'} + \frac{B'}{1-B}\right)\Phi = 0\,, \;\; (k_H \to 0)\,.\end{aligned} \tag{23}$$

We have used the $f(R)$ relation [23]

$$\Phi + \Psi = -B\frac{H'}{H}\frac{V_m}{k_H}\,, \tag{24}$$

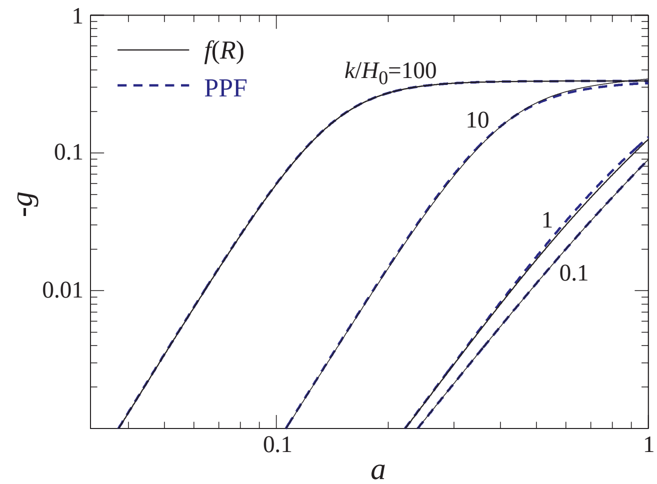

FIG. 3: Evolution and scale dependence of the metric ratio g in $f(R)$ models compared with the PPF fit. Here $B_0 = 0.4$, $w_{\rm eff} = -1$ and $\Omega_m = 0.24$.

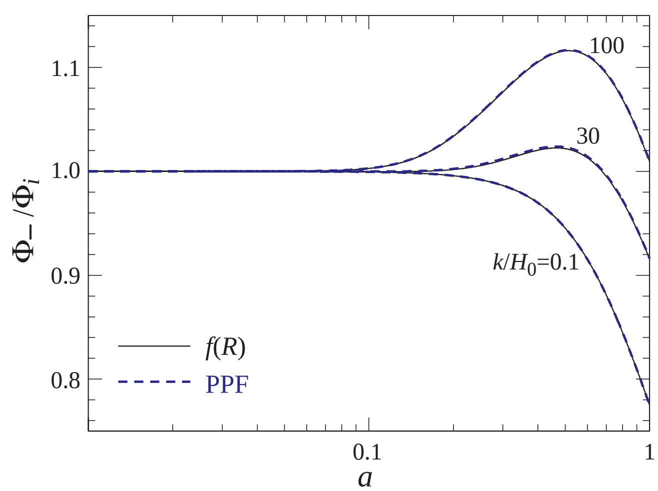

FIG. 4: Evolution and scale dependence of Φ_- in $f(R)$ models compared with the PPF fit. Here $B_0 = 0.4$, $w_{\rm eff} = -1$ and $\Omega_m = 0.24$.

which when combined with $\zeta' = 0$ and Eqn. (15) gives

$$\Psi = \frac{-\Phi - B\Phi'}{1-B}\,, \quad (k_H \to 0)\,. \tag{25}$$

The solution of Eqn. (24) together with (25) yields the metric ratio

$$g(\ln a, k_H = 0) = g_{\rm SH}(\ln a) = \frac{\Phi + \Psi}{\Phi - \Psi}\,. \tag{26}$$

The density evolution function f_ζ can be adequately described by noting that $\zeta' \propto k_H^2 B\zeta$ and that B also controls the behavior of g. We take

$$f_\zeta = c_\zeta g \tag{27}$$

with $c_\zeta \approx -1/3$.

For the transition to the quasi-static regime we take the interpolating function

$$g(\ln a, k) = \frac{g_{\rm SH} + g_{\rm QS}(c_g k_H)^{n_g}}{1 + (c_g k_H)^{n_g}}\,, \tag{28}$$

where $g_{\rm QS} = -1/3$. We find that the evolution is well described by $c_g = 0.71 B^{1/2}$ and $n_g = 2$. We show an example of this fit in Fig. 3.

Finally, the effective Newton constant is rescaled by f_R and the quasi-static transition takes place near the horizon scale

$$f_G = f_R\,, \qquad c_\Gamma = 1\,. \tag{29}$$

In Fig. 4, we show how well the PPF parameterization reproduces the full $f(R)$ metric evolution for scales that span the Compton wavelength transition in a $w_{\rm eff} = -1$ and $B_0 = 0.4$ model. We have checked that a wide range of $f(R)$ models including those of [29] produce comparable matches with these parameter choices.

C. DGP Model

In the DGP braneworld model, the transition between two different behaviors for g occurs at the horizon scale. Above the horizon, the propagation of perturbations into the bulk requires solving the full 5D perturbation equations [30]. Fortunately, above the horizon scale the evolution is scale free and can be solved using the iterative scaling method of [22]. Well below the horizon the evolution reaches the quasi-static limit where the equations can be effectively closed on the brane [6, 24]. We therefore take a similar approach to the $f(R)$ case of interpolating between these two well-defined regimes.

On super-horizon scales, the iterative scaling solution is well described by the fitting function

$$g_{\rm SH}(\ln a) = \frac{9}{8Hr_c - 1}\left(1 + \frac{0.51}{Hr_c - 1.08}\right)\,, \tag{30}$$

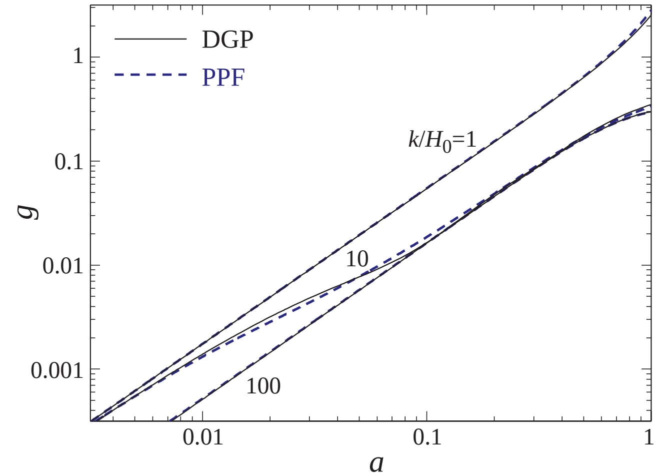

FIG. 5: Evolution and scale dependence of the metric ratio g in the DGP model compared with a PPF fit. Here $\Omega_m = 0.24$ and the PPF parameter $c_g = 0.4$.

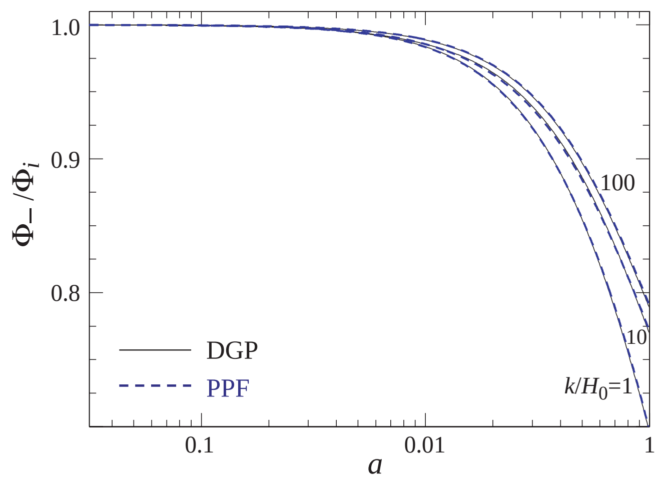

FIG. 6: Evolution and scale dependence of Φ_- in the DGP models compared a PPF fit. Here $\Omega_m = 0.24$ and the PPF parameter $c_g = 0.14$.

where recall that r_c is the crossover scale. In the DGP model ζ' is again related to g and so we take f_ζ to be defined by Eqn. (27) with $c_\zeta \approx 0.4$. The expansion history is given by

$$\frac{H}{H_0} = \sqrt{\Omega_{\rm rc}} + \sqrt{\Omega_{\rm rc} + \Omega_m a^{-3}}\,, \tag{31}$$

where

$$\Omega_{\rm rc} = \frac{1}{4r_c^2 H_0^2} = \frac{(1-\Omega_m)^2}{4}. \tag{32}$$

For illustrative purposes we take $\Omega_m = 0.24$.

In the quasi-static regime [24]

$$g_{\rm QS}(\ln a) = -\frac{1}{3}\left[1 - 2Hr_c\left(1 + \frac{1}{3}\frac{H'}{H}\right)\right]^{-1}. \qquad (33)$$

We employ the interpolation function (28) to join the two regimes. In Fig. 5, we show a fit to the results of [22] with $c_g = 0.4$ and $n_g = 3$ for several values of k_H that span the transition. The remaining parameters are

$$f_G = 0\,, \qquad c_\Gamma = 1\,. \qquad (34)$$

Around horizon crossing, the scaling assumption of [22] is briefly violated leading to possible numerical transients in the solution and an ambiguity in the exact value of c_g. The results for the metric evolution in [22] are best fit with $c_g = 0.14$ as shown in Fig. 6. The transition parameter c_g should therefore be taken as a free parameter in the range $c_g \sim 0.1 - 1$ until a more precise solution is obtained.

IV. NON-LINEAR PARAMETERIZATION

As discussed in §II C, we expect that a successful modification of gravity will have a non-linear mechanism that suppresses modifications within dark matter halos. In this section, we construct a non-linear PPF framework based on the halo model of non-linear clustering. Although a complete parameterized description of modified gravity in the non-linear regime is beyond the scope of this work, the halo model framework allows us to incorporate the main qualitative features expected in these models. Searching for these qualitative features can act as a first step for cosmological tests of gravity in the non-linear regime.

Under the halo model, the non-linear matter power spectrum is composed of two pieces (see [31] for details and a review). One piece involves the correlations between dark matter halos. As in general relativity, the interactions between halos should be well described by linear theory. The other piece involves the correlations within dark matter haloes. It is this term that we mainly seek to parameterize.

Specifically given a linear power spectrum of density fluctuations P_L, the halo model defines the non-linear spectrum as the sum of the one and two halo pieces

$$P(k) = I_1(k) + I_2^2(k)P_L(k)\,, \qquad (35)$$

with

$$I_1(k) = \int \frac{dM}{M}\left(\frac{M}{\rho_0}\right)^2\left[\frac{dn}{d\ln M}y^2(M,k)\right],$$
$$I_2(k) = \int \frac{dM}{M}\left(\frac{M}{\rho_0}\right)\frac{dn}{d\ln M}b(M)y(M,k)\,, \qquad (36)$$

where $\rho_0 = \rho_m(\ln a = 0)$. Here the integrals are over the mass M of dark matter halos and $dn/d\ln M$ is the mass function which describes the comoving number density of haloes. $y(M,k)$ is the Fourier transform of the halo density profile normalized to $y(M,0) = 1$ and $b(M)$ is the halo bias. Note that $I_2(k=0) = 1$ so that the linear power spectrum is recovered on scales that are larger than the extent of the halos.

A simple ansatz that restores general relativity in the non-linear regime is that the mass function and halo profiles remain unchanged from general relativity. Specifi-

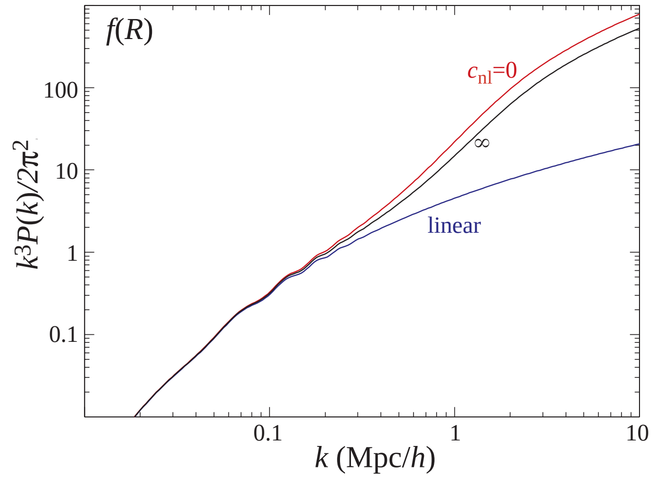

FIG. 7: PPF non-linear power spectrum ansatz for an $f(R)$ model. The non-linear power spectrum is constrained to lie between two extremes: defined by halo-model mass functions with the quasi-static growth rate [$c_{\rm nl} = 0$ or $P_0(k)$] and the smooth dark energy growth rate with the same expansion history [$c_{\rm nl} = \infty$ or $P_\infty(k)$]. Here $B_0 = 0.001$, $w_{\rm eff} = -1$ and $\Omega_m = 0.24$ with other parameters given in the text.

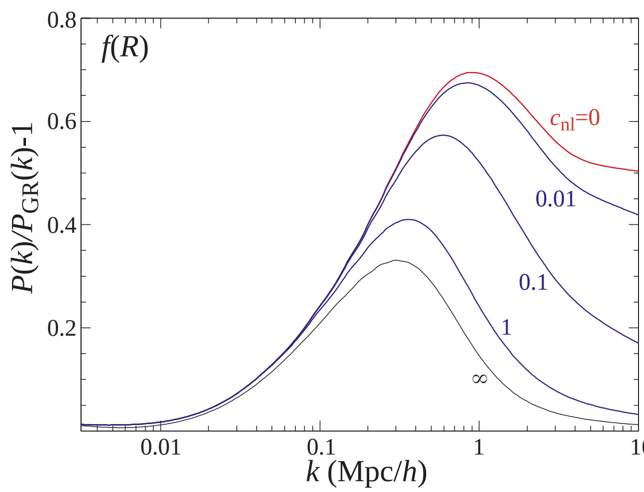

FIG. 8: Fractional difference in $P(k)$ of the PPF non-linear $f(R)$ ansatz from the smooth dark energy prediction with the same expansion history. As $c_{\rm nl} \to \infty$ deviations become confined to the weakly non-linear to linear regime. The model parameters are the same as in Fig. 7.

cally, whereas the mass function and halo profiles usually depends in a universal manner on $\sigma(M)$ the rms of the linear density field smoothed on a scale that encloses the mass M at the background density, we replace this with the rms of the linear density field of a smooth dark energy model with the same expansion history $\sigma_{\rm GR}(M)$. For definiteness, we adopt the Sheth-Torman mass function and bias [32]

$$\frac{dn}{d\ln M} = \frac{\rho_0}{M} f(\nu) \frac{d\nu}{d\ln M}\,,$$
$$b(M) = 1 + \frac{a\nu^2 - 1}{\delta_c} + \frac{2p}{\delta_c[1+(a\nu^2)^p]}\,, \tag{37}$$

where $\nu = \delta_c/\sigma(M)$ and

$$\nu f(\nu) = A\sqrt{\frac{2}{\pi} a\nu^2}[1+(a\nu^2)^{-p}]\exp[-a\nu^2/2]\,. \tag{38}$$

We choose $\delta_c = 1.68$, $a = 0.75$, $p = 0.3$, and A such that $\int d\nu f(\nu) = 1$. For the halo profiles we take the Navarro, Frenk and White (NFW) profile [33]

$$\rho \propto \frac{1}{cr/r_{\rm vir}(1+cr/r_{\rm vir})^2}\,, \tag{39}$$

where $r_{\rm vir}$ is the virial radius, [34]

$$c(M_v) = \frac{9}{1+z}\left(\frac{M}{M_*}\right)^{-0.13}\,, \tag{40}$$

and M_* is defined as $\sigma(M_*) = \delta_c$. We call the result of taking $\sigma(M) = \sigma_{\rm GR}(M)$ in the halo model equations (35) $P_\infty(k)$.

At the opposite extreme, we can make the ansatz that the usual mapping of the linear to nonlinear power found under general relativity remains unchanged in modified gravity. In this case, changes in the linear growth rate determine the non-linear power spectrum. This type of prescription of adopting the linear to nonlinear scaling of general relativity has been tested against cosmological simulations with various modified Poisson prescriptions [35]. It represents the case where gravity is modified down to the smallest cosmological scales. Specifically, in our halo model we take $\sigma(M) = \sigma_{\rm PPF}(M)$ as calculated from the linear power spectrum of the modified gravity model and employ the same Sheth-Torman and NFW prescriptions as before. Let us call the power spectrum in this limit $P_0(k)$.

We can parameterize an interpolation between these two extreme behaviors

$$P(k) = \frac{P_0(k) + c_{\rm nl}\Sigma^2(k)P_\infty(k)}{1+c_{\rm nl}\Sigma^2(k)}\,, \tag{41}$$

that is based on the degree of non-linearity defined by

$$\Sigma^2(k) \equiv \frac{k^3 P_L(k)}{2\pi^2}\,. \tag{42}$$

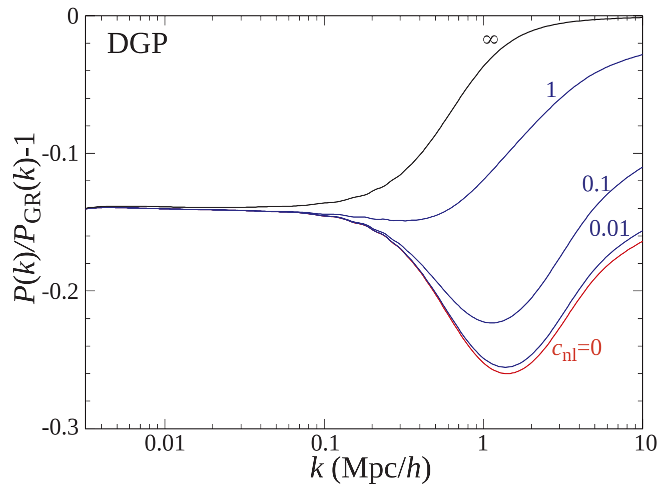

FIG. 9: Fractional difference in $P(k)$ of the PPF non-linear DGP ansatz from the smooth dark energy prediction. The DGP model has less power than the equivalent dark energy model in the linear and weakly non-linear regime. Here $\Omega_m = 0.24$ and other parameters are given in the text.

The analogous interpolation can also be used for the power spectrum of Φ_- that enters into gravitational lensing observables [36].

We show an example of this non-linear ansatz for an $f(R)$ model with $\Omega_m = 0.24$, $\Omega_m h^2 = 0.128$, $\Omega_b h^2 = 0.0223$, $n_s = 0.958$ and initial curvature fluctuation of $\delta_\zeta = A_S^{1/2} = 4.52 \times 10^{-5}$ at $k = 0.05$ Mpc^{-1}. We furthermore fix the expansion history with $w_{\rm eff=-1}$.

Unfortunately no cosmological simulations exist for $f(R)$ or DGP models against which to test the accuracy of this non-linear ansatz. Moreover Even under general relativity, the halo model model of Eqn. (35) does not exactly reproduce the non-linear spectra of cosmological simulations.

More robust in our parameterization is the relative change between a PPF power spectrum and the general relativistic prediction with smooth dark energy $P_{\rm GR}(k)$ and the same expansion history. This factor can then be applied to more exact results from cosmological simulations to search for deviations from general relativity. The difference between $P_{\rm GR}(k)$ and $P_\infty(k)$ is that the former uses the general relativistic linear power spectrum in Eqn. (35). This prescription can be further refined by calibrating $P_0(k)$ directly from simulations of the modified Poisson equation [35].

We show the fractional change between the PPF power spectra and the general relativistic power spectra for the $f(R)$ model in Fig. 8 and a DGP model with the same parameters but with the DGP expansion history of Eqn. (31) in Fig. 9. Note that as $c_{\rm nl} \to \infty$, deviations appear mainly in the linear to weakly non-linear regime. For the $f(R)$ model they appear as an enhancement of power and for the DGP model as a deficit of power reflecting the opposite sign of g in the linear regime of the two models.

V. DISCUSSION

We have introduced a parameterized framework for considering scalar modifications to gravity that accelerate the expansion without dark energy. This framework features compatibility in the evolution of structure with a background expansion history on large scales, a modification of the Poisson equation on intermediate scales, and a return to general relativity within collapsed dark matter halos. This return to general relativity is required of models to pass stringent local tests of gravity. We have also clarified the formal relationship between modified gravity and dark energy in the Appendix. A metric based modified gravity model can always be cast in terms of a dark energy component with a stress energy tensor defined to match its influence on the metric. However such a component would possess dynamics which are coupled to the matter.

Our parameterized post-Friedmann framework features several free functions even in the linear regime. The most important function is the relationship $g = (\Phi + \Psi)/(\Phi - \Psi)$ between the time-time and space-space pieces of the metric in Newtonian gauge. Supplementing these are two functions that link the metric to matter density perturbations: one on super-horizon scales and one on intermediate scales. Finally there is a parameter that controls the interpolation between these two regimes.

We have shown that with an appropriate choice of parameters this framework describes linear perturbations in the $f(R)$ modified action and DGP braneworld gravity models. It may be used in place of the more complicated 4th order and higher dimensional dynamics exhibited in these models respectively when studying phenomena such as the integrated Sachs-Wolfe effect in the CMB, large-scale gravitational lensing and galaxy clustering. We intend to explore these applications in a future work.

On non-linear scales our framework features an ansatz based on the requirement that scalar modifications should be suppressed locally in order to pass the stringent tests of general relativity in the solar system. Indeed the scalar degrees of freedom in both the $f(R)$ and the DGP models possess non-linearities that drive the dynamics back to general relativity in high curvature or high density regimes. Our ansatz is based on the halo model of non-linear clustering. It allows for a density dependent interpolation for the abundance and structure of dark matter halos between the expectations of general relativity and the modified Poisson equation on intermediate scales.

Due to the current lack of cosmological simulations in these modified gravity models, the accuracy of our simple ansatz remains untested. With cosmological simulations, our framework can be extended and refined by introducing more parameters that describe the potentially mass-dependent modification of dark matter haloes. In fact, our simple halo model parameterization is not even sufficient to accurately model non-linear effects in general relativity. Nonetheless phrased as a simple template form for relative deviations in the power spectrum between modified gravity and general relativity with smooth dark energy, our current ansatz can be used in conjunction with more accurate results from dark energy cosmological simulations. For example, it can be used to search for possible deviations of this type as a consistency check on dark energy inferences from expansion history tests with upcoming cosmic shear surveys.

While many such consistency tests have been proposed in the literature, it is important to incorporate a density dependence to the modifications as we have done here. The principle that non-linear scales should exhibit a return to general relativity itself suggests that mildly non-linear scales provide the most fruitful window for cosmological tests of gravity. Furthermore uncertainties in the baryonic influence on the internal structure of dark matter halos in the deeply non-linear regime even under general relativity (e.g. [37, 38]) make consistency tests in this regime potentially ambiguous. Our parameterized framework should enable studies of such issues in the future.

APPENDIX A: DARK ENERGY CORRESPONDENCE

Suppose we view the modifications to gravity in terms of an additional "dark energy" stress tensor. We are free to *define* the dark energy stress tensor to be

$$T_e^{\mu\nu} \equiv \frac{1}{8\pi G} G^{\mu\nu} - T_m^{\mu\nu} \,. \tag{A1}$$

Given this association, all of the familiar structure of cosmological perturbation theory in general relativity applies. In particular, covariant conservation of the matter stress energy tensor $T_m^{\mu\nu}$ and the Bianchi identities imply conservation of the effective dark energy

$$\nabla_\mu T_e^{\mu\nu} = 0 \,. \tag{A2}$$

The remaining degrees of freedom in the effective dark energy stress tensor can then be parameterized in the same manner as a general dark energy component [3]. Two models that imply the same $T_e^{\mu\nu}$ at all points in spacetime are formally indistinguishable gravitationally [21, 39].

Note however that this equivalence is only formal and two physically distinct models, e.g. $f(R)$ modified gravity and scalar field dark energy, will not in general imply the same effective stress energy tensor. The Einstein and conservation equations do not form a closed system and the distinction between modified gravity and dark energy lies in the closure relation. For dark energy that is not coupled to matter, the closure relationship takes the form of equations of state that define its internal dynamics. These micro-physical relations do not depend explicitly on the matter. For example for scalar field dark energy, the sound speed or the relationship between the pressure and energy density fluctuations is defined in the constant field gauge without reference to the matter [3], and is associated with the form of the kinetic term in the Lagrangian [40].

For modified gravity of the type described in this paper, we shall see that the closure relations must depend explicitly on the matter. The effective dark energy of a modified gravity model must be coupled to the matter. In other words, while the modification gravity can be modeled as fifth forces mediated by the effective dark energy, it cannot be viewed as a missing energy component that obeys separate equations of motion.

It is nonetheless useful to phrase the PPF parameterization in terms of an effective dark energy component. It enables the use of the extensive tools developed for cosmological perturbation theory and facilitates the development of PPF formalisms in different gauges.

1. Covariant Field and Conservation Equations

Following [20, 41], we parameterize linear scalar metric fluctuations of a comoving wavenumber k as

$$\begin{aligned} g^{00} &= -a^{-2}(1-2AY)\,, \\ g^{0i} &= -a^{-2}BY^i\,, \\ g^{ij} &= a^{-2}(\gamma^{ij} - 2H_L Y\gamma^{ij} - 2H_T Y^{ij})\,, \end{aligned} \tag{A3}$$

where the "0" component denotes conformal time $\eta = \int dt/a$ and γ_{ij} is the background spatial metric which we assume to be flat across scales comparable to the wavelength. Under this assumption, the spatial harmonics are simply plane waves

$$\begin{aligned} Y &= e^{i\mathbf{k}\cdot\mathbf{x}}\,, \\ Y_i &= (-k)\nabla_i Y\,, \\ Y_{ij} &= (k^{-2}\nabla_i\nabla_j + \gamma_{ij}/3)Y\,. \end{aligned} \tag{A4}$$

Likewise the components of the stress tensors can be parameterized as

$$\begin{aligned} T^0{}_0 &= -\rho - \delta\rho\,, \\ T_0{}^i &= -(\rho+p)vY^i\,, \\ T^i{}_j &= (p+\delta pY)\delta^i{}_j + p\Pi Y^i{}_j\,, \end{aligned} \tag{A5}$$

where we will use the subscripts m to denote the matter and e to denote the effective dark energy. When no subscript is specified we mean the components of the total or matter plus effective dark energy stress tensor. For simplicity we assume that the radiation is negligible during the epochs of interest.

By definition, Eqn. (A1) enforces the usual 4 Einstein field equations [21]

$$\begin{aligned} & H_L + \frac{1}{3}H_T + \frac{B}{k_H} - \frac{H_T'}{k_H^2} \\ &\quad = \frac{4\pi G}{H^2k_H^2}\left[\delta\rho + 3(\rho+p)\frac{v-B}{k_H}\right]\,, \\ & A + H_L + \frac{H_T}{3} + \frac{B'+2B}{k_H} - \left[\frac{H_T''}{k_H^2} + \left(3+\frac{H'}{H}\right)\frac{H_T'}{k_H^2}\right] \\ &\quad = -\frac{8\pi G}{H^2k_H^2}p\Pi\,, \\ & A - H_L' - \frac{H_T'}{3} = \frac{4\pi G}{H^2}(\rho+p)\frac{v-B}{k_H}\,, \\ & A' + \left(2 + 2\frac{H'}{H} - \frac{k_H^2}{3}\right)A - \frac{k_H}{3}(B'+B) \\ &\quad - H_L'' - \left(2+\frac{H'}{H}\right)H_L' = \frac{4\pi G}{H^2}(\delta p + \frac{1}{3}\delta\rho)\,, \end{aligned} \tag{A6}$$

where recall $' = d/d\ln a$ and $k_H = (k/aH)$. The conservation laws for the matter and effective dark energy become

$$\delta\rho' + 3(\delta\rho + \delta p) = -(\rho+p)(k_H v + 3H_L')\,, \tag{A7}$$
$$\frac{[a^4(\rho+p)(v-B)]'}{a^4k_H} = \delta p - \frac{2}{3}p\Pi + (\rho+p)A\,.$$

There are 4 metric variables and 4 matter variables per component that obey 4 Einstein equations and 2 conservation equations per component. However 2 out of 4 of the Einstein equations are redundant since the Bianchi identities are automatically satisfied given a metric. Furthermore, 2 degrees of freedom simply represent gauge or coordinate freedom. This leaves 2 degrees of freedom per component to be specified. Usually, this involves defining equations of state that specify the spatial stresses in terms of the energy density and velocities. As we shall see, it is this prescription that must be altered to describe modified gravity.

2. Gauge

The scalar gauge degrees of freedom are fixed by gauge conditions. Under a gauge transformation defined by the change in conformal time slicing T and spatial coordinates L

$$\begin{aligned} \eta &= \tilde\eta + T\,, \\ x^i &= \tilde x^i + LY^i\,, \end{aligned} \tag{A8}$$

the metric variables transform as

$$\begin{aligned} A &= \tilde A - aH(T'+T)\,, \\ B &= \tilde B + aH(L' + k_H T)\,, \\ H_L &= \tilde H_L - aH(T + \frac{1}{3}k_H)\,, \\ H_T &= \tilde H_T + aHk_H L\,, \end{aligned} \tag{A9}$$

and the matter variables transform as

$$\begin{aligned}
\delta\rho &= \widetilde{\delta\rho} - \rho' aHT\,,\\
\delta p &= \widetilde{\delta p} - p' aHT\,,\\
v &= \tilde{v} + aHL'\,,\\
\Pi &= \tilde{\Pi}\,.
\end{aligned} \tag{A10}$$

A gauge is fully specified if the functions T and L are uniquely defined.

In this paper we work in the matter comoving and Newtonian gauges. The matter comoving gauge is specified by the conditions

$$\begin{aligned}
B &= v_m\,,\\
H_T &= 0\,.
\end{aligned} \tag{A11}$$

They fully specify the gauge transformation from an alternate gauge choice

$$\begin{aligned}
T &= (\tilde{v}_m - B)/k\,,\\
L &= -\tilde{H}_T/k\,.
\end{aligned} \tag{A12}$$

To avoid confusion between fluctuations defined in different gauges, we will define

$$\begin{aligned}
\zeta &\equiv H_L\,,\\
\xi &\equiv A\,,\\
\rho\Delta &\equiv \delta\rho\,,\\
\Delta p &\equiv \delta p\,,\\
V &\equiv v\,.
\end{aligned} \tag{A13}$$

Δp should not be confused with $p\Delta = p(\delta\rho/\rho)$.

The appropriate Einstein and conservation equations for this gauge can be obtained by utilizing these definitions in Eqn. (A6) and (A8). For example, the third Einstein equation reads

$$\zeta' = -\frac{4\pi G}{H^2}(\rho_e + p_e)\frac{V_e - V_m}{k_H}\,, \tag{A14}$$

and the energy-momentum conservation equations for the matter become

$$\begin{aligned}
\Delta_m' &= -k_H V_m - 3\zeta'\,,\\
\xi &= 0\,.
\end{aligned} \tag{A15}$$

The dark energy momentum conservation equation

$$\frac{[a^4(\rho_e + p_e)(V_e - V_m)]'}{a^4 k_H} = \Delta p_e - \frac{2}{3}p_e\Pi_e\,, \tag{A16}$$

in conjunction with Eqn. (A14) and the first Einstein equation implies that unless Δp_e or $p_e\Pi_e > \mathcal{O}(\Delta\rho/k_H^2)$, $\zeta'/\zeta \to 0$ as $k_H \to 0$.

Similarly, the Newtonian gauge is defined by the condition $B = H_T = 0$ and the transformation

$$\begin{aligned}
T &= -\frac{\tilde{B}}{k} + \frac{\tilde{H}_T'}{k k_H}\,,\\
L &= -\frac{\tilde{H}_T}{k}\,.
\end{aligned} \tag{A17}$$

To avoid confusion we define

$$\begin{aligned}
\Phi &\equiv H_L\,,\\
\Psi &\equiv A\,.
\end{aligned} \tag{A18}$$

We refrain from utilizing matter variables in Newtonian gauge but note that velocities in the two gauges are the same. The relationship between the two metric fluctuations are

$$\begin{aligned}
\zeta &= \Phi - \frac{V_m}{k_H}\,,\\
\xi &= \Psi - \frac{V_m' + V_m}{k_H}\,.
\end{aligned} \tag{A19}$$

The matter momentum conservation law in Newtonian gauge becomes

$$V_m' + V_m = k_H\Psi\,. \tag{A20}$$

This equation can alternately be derived from the gauge transformation equation (A19) given that $\xi = 0$.

Finally the Einstein equations (A6) in Newtonian gauge imply

$$\frac{\Phi + \Psi}{2} = -\frac{4\pi G}{H^2 k_H^2} p\Pi = -\frac{4\pi G}{H^2 k_H^2} p_e\Pi_e\,, \tag{A21}$$

$$\begin{aligned}
\frac{\Phi - \Psi}{2} &= \frac{4\pi G}{H^2 k_H^2}\left(\rho\Delta + 3(\rho + p)\frac{V - V_m}{k_H} + p\Pi\right)\\
&= \frac{4\pi G}{H^2 k_H^2}\Big[\rho_m\Delta_m + \rho_e\Delta_e + 3(\rho_e + p_e)\frac{V_e - V_m}{k_H}\\
&\quad + p_e\Pi_e\Big]\,,
\end{aligned} \tag{A22}$$

where we have assumed that the anisotropic stress of the matter is negligible. A finite metric ratio parameter $g = (\Phi + \Psi)/(\Phi - \Psi)$ is thus associated with a non-vanishing effective anisotropic stress.

3. PPF correspondence

The system of equations defined by the field equations and the conservation equations are incomplete. To close the system of equations two more conditions must be required of the effective dark energy. It is this closure condition that the PPF parameterization must determine.

Given that the matter has no anisotropic stress, Eqn. (A21) defines the anisotropic stress of the effective dark energy in terms of the metric

$$p_e\Pi_e = -\frac{H^2 k_H^2}{4\pi G} g\Phi_-\,, \tag{A23}$$

where recall $\Phi_- = (\Phi - \Psi)/2$. This is the first of two closure relations.

The second closure relation comes from equating Eqn. (A22) and the modified Poisson equation (16)

$$\rho_e\Delta_e + 3(\rho_e + p_e)\frac{V_e - V_m}{k_H} + p_e\Pi_e = -\frac{k^2}{4\pi G a^2}\Gamma\,. \tag{A24}$$

The PPF equation of motion (21) for Γ is therefore the "equation of state" for the effective dark energy.

The conservation laws for the effective dark energy and/or remaining Einstein equations then define the other two components V_e and Δp_e. For example,

$$V_e = V_m - k_H \frac{H^2}{4\pi G a^2(\rho_e + p_e)}\zeta' \,, \tag{A25}$$
$$\Delta p_e = p_e \Delta_e - \frac{1}{3}\rho_e \Delta_e' - (\rho_e + p_e)(k_H V_e/3 + \zeta') \,.$$

The modification represented by this prescription obeys all 4 Einstein equations and both sets of conservation laws.

Unlike the case of a micro-physical candidate for dark energy such as a scalar field, the closure relations not only cannot be defined as direct relationships between the spatial stresses and the energy density and velocity, they here involve the matter and the metric fluctuations directly. Hence the effective dark energy is implicitly coupled to the matter and cannot be described as an independent entity.

On the other hand, the virtue of making this correspondence explicit is that with these relations all of the usual representations of perturbation theory can be reached by standard gauge transformations from our matter comoving and Newtonian representations.

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