Effects of Instrumental Noise on CMB Lensing Reconstruction

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Gravitational lensing of the cosmic microwave background results in the generation of higher-order correlations, which can be utilized to estimate the distribution of the intervening dark matter. Although techniques for constructing near-optimal estimators exist, an approach that takes issues related instrumental noise into account is necessary for a realistic assessment of the ability of future experiments to reconstruct gravitational lenses. We provide a quadratic estimator based on a likelihood approach that accounts for survey boundaries and inhomogeneous noise, and discuss the practical issues associated with implementing the estimator. We utilize the quadratic estimator to perform a preliminary study of the effects of survey boundaries and correlated noise on the reconstruction of lenses using the microwave background. We find that the presence of boundaries does not increase the bias in the reconstructed maps, and produce direction-dependent correlations between multipole moments separated by the fundamental mode of the region with missing data. We also examine a simple case of striping, and show that the method results in no additional bias. We conclude that the presence of non-ideal experimental noise does not significantly limit the ability to measure gravitational lensing in the cosmic microwave background.

I. INTRODUCTION

Weak gravitational lensing of the cosmic microwave background (CMB) anisotropies offers a unique window to studying the large scale distribution of dark matter at higher redshifts. By providing information about the gravitational potential at redshifts $z = 2 \sim 3$, a reconstruction of gravitational lenses using the CMB can complement cosmic shear surveys

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that probe lower redshifts. In addition, lensing reconstruction is essential in removing the contaminant to the gravitational wave component of CMB polarization, which is expected to provide information about the energy scale of inflation [1, 2].

Gravitational lensing not only induces modifications to the CMB temperature and polarization power spectra [3, 4], but also produces higher-order correlations between the CMB multipole moments, which can be used to reconstruct the distribution of the projected masses [5, 6]. The CMB observables are remapped according to the gradient of the projected gravitational potential, and act as a convolution in Fourier space. Using the fact that the correlations are proportional to the gravitational potential, estimators for the gravitational potential can be formed out of quadratic combinations of the CMB multipole moments. Refs. [7] and [8] have shown that estimators can be constructed out of gradients of temperature and polarization fields, albeit with high noise. Refs. [9, 10] constructed minimum variance combinations of the temperature and polarization fields that can map the gravitational potential to multipoles of $L \sim 1000$, and showed that the addition of polarization information greatly improves the reconstruction. An extension to a full sky formalism was considered in Ref. [11], and iterative improvements based on a likelihood function approach were presented in Refs. [12, 13].

As a step toward practical implementation, Ref. [14] investigated the effect of nonlinear lensing potentials, as well as the contamination due to Sunyaev-Zel’dovich (SZ) effects on reconstruction using the temperature field. However, experimental issues, such as the effect of survey boundaries and correlated noise, have not been explored systematically. Since the correlations that lensing reconstruction utilizes can also be generated from instrumental noise, they can lead to biases in the reconstruction. A treatment that properly accounts for these effects is thus essential for a realistic assessment of our ability to study lensing using the CMB. We provide such a method, based on a likelihood function approach, and present a preliminary study of the effects of instrumental noise on CMB lensing reconstruction.

We begin with a brief review of the effect of lensing on the CMB as well as previous works on lensing reconstruction in Sec. II. We construct an estimator incorporating survey boundaries and correlated noise in Sec. III. We consider practical issues in its implementation in Sec. IV, and assess the effects of boundaries and correlated noise on lensing reconstruction in Sec. V. We conclude in Sec. VI.
II. CMB LENSING

We review the effect of lensing on the microwave background temperature and polarization fields, as well as the construction of quadratic estimators for gravitational lenses. Details on the quadratic estimator approach can be found in Refs. [9–11], and iterative improvements in Refs. [12, 13].

The observed CMB temperature field is characterized by a scalar quantity \( \Theta(\hat{n}) \equiv \Delta T(\hat{n}) / T \), and the polarization anisotropy by a traceless, symmetric rank-2 tensor

\[
\mathcal{P}(\hat{n}) = \begin{pmatrix}
Q(\hat{n}) & U(\hat{n}) \\
U(\hat{n}) & -Q(\hat{n})
\end{pmatrix},
\]

in a fixed coordinate representation, where \( Q \) and \( U \) are Stokes parameters. The circular polarization \( V \) is assumed to be absent for the CMB. We will assume that the observed region is small enough that the sky can be considered flat. For convenience, we measure the Stokes parameters in a Cartesian coordinate system, where the basis vectors \( (\hat{e}_1, \hat{e}_2) \) are everywhere orthogonal and constant (i.e., the \( x \) and \( y \) coordinate directions).

In the flat-sky approximation, the temperature and polarization field have multipole expansions given by

\[
\Theta(\hat{n}) = \int \frac{d^2 l}{(2\pi)^2} \Theta(l)e^{il \cdot n},
\]

\[
(Q \pm iU)(\hat{n}) = \int \frac{d^2 l}{(2\pi)^2} (Q \pm iU)(l)e^{il \cdot n}.
\]

Although the values of Stokes parameters \( \{Q, U\} \) are coordinate dependent, a coordinate independent decomposition can be constructed by defining the “E modes” and “B modes” according to

\[
E(\hat{n}) = \int \frac{d^2 l}{(2\pi)^2} E(l)e^{il \cdot n},
\]

\[
B(\hat{n}) = \int \frac{d^2 l}{(2\pi)^2} B(l)e^{il \cdot n},
\]

\[
\begin{pmatrix}
E(l) \\
B(l)
\end{pmatrix} = \begin{pmatrix}
\cos 2\varphi_1 & \sin 2\varphi_1 \\
-\sin 2\varphi_1 & \cos 2\varphi_1
\end{pmatrix} \begin{pmatrix}
Q(l) \\
U(l)
\end{pmatrix},
\]

with \( \phi_l \) denoting the angle between \( l \) and \( \hat{x} \). The E modes correspond to the curl-free components of polarization, and the B modes are the pure curl components. For the rest of the work, we assume that the B mode contribution is negligible.
The effect of the gravitational potential \( \Phi(x, t) \) between the last scattering surface and the observer is encoded in the lensing potential \( \phi \), given in terms of the gravitational potential \( \Phi(x, t) \) by the line of sight integral

\[
\phi(\hat{n}) = -2 \int d\chi \frac{D_A(x_s - \chi)}{D_A(x)D_A(x_s)} \Psi(\chi \hat{n}, \chi),
\]

with \( D_A(\chi) \) denoting the angular diameter distance to coordinate distance \( \chi \), and \( x_s \) the coordinate distance to the last scattering surface. We assume a flat cosmology, so that \( D_A(\chi) = \chi \). The CMB photons are deflected according to the lensing equation

\[
n_o = n_s + \nabla \phi(\hat{n}_o),
\]

where \( n_o \) is the observed direction on the sky, \( n_s \) is the position on the source plane.

Gravitational lensing by the lensing potential \( \phi \) acts as a remapping of the CMB temperature and polarization fields according to \([7, 8]\)

\[
X(\hat{n}) = \tilde{X}(\hat{n} + \nabla \phi(\hat{n})),
\]

where \( X = \{\Theta, Q, U\} \), and tildes denote unlensed fields. If the deflection \( \nabla \phi \) due to lensing is small, we can expand Eq. (6) to first order in \( \phi \), so that

\[
X(\hat{n}) = \tilde{X}(\hat{n}) + \nabla \phi \cdot \nabla \tilde{X} + O(\phi^2).
\]

In multipole space,

\[
X(l) = \tilde{X}(l) + \int \frac{d^2\nu}{(2\pi)^2} \tilde{X}(\nu) W(\nu, l),
\]

\[
W(l, L) = -(1 \cdot L) \phi(L),
\]

where \( L = 1 + \nu \). In the \( \{E, B\} \) representation, the \( E \) and \( B \) modes receive additional lensing contributions according to

\[
\delta E(l) = \int \frac{d^2\nu}{(2\pi)^2} \left[ \tilde{E}(\nu) \cos 2\varphi_{\nu 1} - \tilde{B}(\nu) \sin 2\varphi_{\nu 1} \right] W(\nu, l),
\]

\[
\delta B(l) = \int \frac{d^2\nu}{(2\pi)^2} \left[ \tilde{E}(\nu) \sin 2\varphi_{\nu 1} + \tilde{B}(\nu) \cos 2\varphi_{\nu 1} \right] W(\nu, l),
\]

\[
\varphi_{\nu 1} \equiv \varphi_{\nu} - \varphi_1.
\]

Eq. (8) shows that lensing acts as a convolution in multipole space, with the convolution kernel \( W(l, L) \) derived from the lensing potential. Thus, in multipole space, the effect of a lens multipole moment \( \phi_L \) is to correlate CMB multipole moments separated by \( L \), so that

\[
\langle X(l) X'(l + L) \rangle|_{\text{lens}} \propto \phi(L),
\]

(10)
for CMB multipole moments $X$ and $X'$. In any given realization, the product $X(l)X'(l+L)$ is dominated by the random fluctuations of the CMB and instrumental noise. This noise can be overcome by creating weighted sums over all CMB multipole pairs separated by $L$.

Additionally, Eq. (9) shows that gravitational lensing tends to create B modes even if they are initially absent, and creates cross correlations between E modes and B modes. Since we assume that the amplitude of primordial B modes are low enough to be negligible, the presence of any correlations involving B modes in the observed map would be a signature of lensing, and leads to measurements of the lensing potential with high signal to noise ratios.

In practice, near-optimal estimators based on weighted quadratic pairs can be derived by constructing weightings that minimize the variance of the estimator. Minimum variance estimators were derived in Ref. [10], and are given by

$$\hat{\phi}^{(q)}_a(L) = A_a(L) \int \frac{d^2l_1}{(2\pi)^2} X(l_1)X'(l_2)F_a(l_1,l_2),$$

$$F_a(l_1,l_2) = \frac{C^{XX'}C^{XX}f_a(l_1,l_2) - C^{XX'}C^{XX}f_a(l_2,l_1)}{C^{XX'}C^{XX}C^{XX'}C^{XX} - (C^{XX'}C^{XX})^2},$$

$$A_a^{-1}(L) = \int \frac{d^2l_1}{(2\pi)^2} f_a(l_1,l_2)F_a(l_1,l_2),$$

(11)

where $l_2 = L - l_1$ and $a$ denotes a field pair $XX'$ (with $X,X' \in \{\Theta,E,B\}$). The lensing correlation factors $f_a$ are defined as

$$\langle X(l)X'(l') \rangle_\phi = f_a(l,l')\phi(L),$$

$$L = 1 + l',$$

(12)

and given in Table I for specific pairs.

The addition of polarization information aids in the reconstruction in several ways. As seen in Eq. (9), lensing creates B modes out of E modes (and vice versa). Since primordial B modes are expected to be small, B modes allow a cleaner reconstruction of lensing. As seen in Fig. 1, an estimator based on B mode information can improve the signal to noise by a factor of 3 relative to an estimator based on the temperature field alone.

### III. LEN S RECONSTRUCTION

Previous works [7–10, 12, 13] have presented several methods for the reconstruction of lensing using CMB observations under idealized conditions. Refs. [9, 10] developed estimators by on minimizing the expected variance in the estimator, whereas Refs. [12, 13] started
\begin{align*}
\alpha &\quad f_\alpha(l_1, l_2) \\
\Theta &\quad \tilde{C}^{\Theta\Theta}(L \cdot l_1) + \tilde{C}^{\Theta\Theta}(L \cdot l_2) \\
\Theta E &\quad \tilde{C}^{\Theta E} \cos \varphi_{l_1l_2}(L \cdot l_1) + \tilde{C}^{\Theta E}(L \cdot l_2) \\
\Theta B &\quad \tilde{C}^{\Theta E} \sin 2\varphi_{l_1l_2}(L \cdot l_1) \\
EE &\quad [\tilde{C}^{EE}(L \cdot l_1) + \tilde{C}^{EE}(L \cdot l_2)] \cos 2\varphi_{l_1l_2} \\
EB &\quad [\tilde{C}^{EE}(L \cdot l_1) - \tilde{C}^{BB}(L \cdot l_2)] \sin 2\varphi_{l_1l_2} \\
BB &\quad [\tilde{C}^{BB}(L \cdot l_1) + \tilde{C}^{BB}(L \cdot l_2)] \cos 2\varphi_{l_1l_2}
\end{align*}

TABLE I: $f_\alpha(l_1, l_2)$ for specific pairs. Here, $l_2 = L - l_1$.

FIG. 1: Noise in reconstructed mode for a full sky experiment with uniform pixel noise. Here, a noise of 1μK-arcmin and a Gaussian beam with full width half maximum of 4 arcmins is assumed. Use of the lensing-induced correlation between E modes and B modes significantly improves the reconstruction.

with a likelihood based approach, and provided reconstruction methods that doesn’t require
the small deflection approximation used in Eq. (7). In both cases, systematic treatments of
the presence of boundaries and correlated noise were absent. We now provide an likelihood-
based derivation for a quadratic estimator, which includes the effects of pixelization and
boundaries, as well as inhomogeneous and correlated noise in the observed CMB fields. The
derivation is similar to that presented in Ref. [12]; however, we explicitly keep terms that
are affected by the presence of boundaries in the observed map, and derive an estimator
that includes effects of non-diagonal pixel noise. We keep to a Taylor approximation to the
lensing equation, Eq. (7), since the quadratic estimator should be sufficient for most near
term experiments, as shown in Ref. [13].

A. Derivation

For the derivation, we will consider an experiment that observes an area $\Omega_{\text{obs}}$ with $N_{\text{pix}}$
pixels, with pixel size $A_{\text{pix}} = \Omega_{\text{obs}}/N_{\text{pix}}$. The sky is assumed to be pixelized so that $\Theta(n) \approx
\Theta_n$, and likewise for all other quantities.

The construction of the estimator is complicated by the fact that gravitational lensing
operates on the CMB through gradients. We choose to define the gradients using Fourier
transforms, and embed the observed sky in a larger periodic area $\Omega_{\text{tot}}$ with $N_{\text{tot}}$ pixels. The
necessary size of $\Omega_{\text{tot}}$ for a given map will be determined in a later section, by simulating
reconstructions of a large sky with various values of $\Omega_{\text{tot}}$.

The observed data vector consists of the signal part, which is the lensed sky convolved with
the instrumental beam $\mathcal{B}$, and pixel noise from the instrument, represented by the vector
$\hat{\nu}_n$. The noise properties are assumed to be known and characterized by its covariance matrix

$$N_{\hat{n}_n \hat{n}_b} = \langle \hat{\nu}_{\hat{n}_n} \hat{\nu}_{\hat{n}_b}^T \rangle$$

(13)

With a beam profile $\beta(n)$, the beam convolution is represented by a matrix

$$B_{\hat{n}_n \hat{n}_b} = \sum_i A_{\text{pix}} \beta_i e^{i \hat{n}_b \cdot (\hat{n}_n - \hat{n}_b)}$$

(14)

where $\beta_i$ is the discretized beam multipole moment. In our choice of discretization, discrete
moments are related to their continuous counterparts by

$$\beta_i = \frac{\beta(1)}{N_{\text{tot}} A_{\text{pix}}}.$$  

(15)

The data vector $m_{\hat{n}_n}$ can then be modeled as

$$m_{\hat{n}_n} = B_{\hat{n}_n \hat{n}_b} \Lambda_{\hat{n}_b \hat{n}_n} x_{\hat{n}_n} + \nu_{\hat{n}_n},$$  

(16)
where \( \Lambda_{n, n_0} \) represents the lensing of the signal vector \( \mathbf{x}_{n_0} \). For simplicity, we will neglect the presence of foregrounds in deriving the discretized estimator. We work in the \( \{ \Theta, Q, U \} \) representation. We also define the multipoles \( \mathbf{m}_n \) through Fourier transforms taken over \( \Omega_{\text{tot}} \)

\[
\mathbf{m}_n = \sum_l \mathbf{m}_l e^{il \cdot n}, \quad \mathbf{m}_n = \frac{1}{N_{\text{tot}}} \sum_l \mathbf{m}_l^* e^{-il \cdot n}.
\]

(17)

We define the signal covariance matrix as

\[
S_{n, n_0} \equiv \langle \mathbf{x}_{n_0} \mathbf{x}_{n_0}^T \rangle = \sum_l \frac{\tilde{C}_l}{N_{\text{tot}}^2} e^{il \cdot (n_0 - n_0)},
\]

(18)

with \( \tilde{C}_l \) denoting the matrix of unlensed CMB power spectra. The signal covariance matrix \( S_{n, n_0} \) is dependent on the vector difference \( \mathbf{n}_0 - \mathbf{n}_0 \), since the transformation of the correlation matrix from the \( \{ \Theta, E, B \} \) representation to the \( \{ \Theta, Q, U \} \) representation is direction-dependent. The covariance in the measured pixels is then given by

\[
C = B \Lambda S A^T B^T + N.
\]

(19)

Assuming that the pixel noise is Gaussian, the observed sky pixels are Gaussian under a fixed lens realization [12]. Therefore, for a fixed lens realization \( \phi \), the likelihood function can be written as

\[
\mathcal{L}(\phi) \equiv -\ln L(\phi) = \frac{1}{2} \ln \det C + \frac{1}{2} \mathbf{m}^T C^{-1} \mathbf{m}.
\]

(20)

The lens configuration is found by minimizing the gradient

\[
G_L(\phi) \equiv \frac{\partial \mathcal{L}}{\partial \phi_L^e} = -\frac{1}{2} \mathbf{m}^T C^{-1} P_L C^{-1} \mathbf{m} + \frac{1}{2} \text{Tr} \left[ C^{-1} P_L \right],
\]

(21)

where the projection operator \( P_L \) is the derivative of the covariance matrix:

\[
P_L = \frac{\partial C}{\partial \phi_L^e} = B \left[ \frac{\partial \Lambda}{\partial \phi_L^e} S A^T + \Lambda S \frac{\partial \Lambda^T}{\partial \phi_L^e} \right] B^T.
\]

(22)

The evaluation of the derivatives in Eq. (22) can be performed by discretizing the Taylor-approximated CMB lensing equation, Eq. (7). In this case, the lensing operator \( \Lambda \) and its derivative can be written as

\[
\Lambda_{n, n_0} = \delta_{n, n_0} + \frac{1}{N_{\text{tot}}} \sum_{L, l} (-L \cdot 1) \phi_Le^{i(L+1) \cdot n_0} e^{-l \cdot n_0},
\]

(23)

\[
\frac{\partial \Lambda_{n, n_0}}{\partial \phi_L^e} = \frac{1}{N_{\text{tot}}} \sum_{L, l} (L \cdot 1) e^{i(1-L) \cdot n_0} e^{-l \cdot n_0},
\]

(24)
where we used the fact that $\phi^*_L = \phi_{-L}$.

In general, quadratic estimators are iterative approaches, which approximate the local log likelihood as a quadratic function. Each $P$-th step of the iteration proceeds by expanding the gradient around $\phi = \phi^{(P)}$, so that the new approximation becomes

$$G_L(\phi) \approx G_L(\phi^{(P)}) + Q_{LL'}\phi_{L'},$$

where

$$Q_{LL'} = \frac{\partial^2 \mathcal{L}}{\partial \phi_L \partial \phi_{L'}}$$

is the curvature matrix evaluated at $\phi^{(P)}$. We further approximate the curvature matrix by its ensemble average $F_{LL'} \equiv \langle Q_{LL'} \rangle$, the Fisher information matrix. With these approximations, one step of the quadratic estimator for a discretized map is

$$\hat{\phi}_L^{(P+1)} = -(F^{-1})_{LL'} G_L,$$

where $G_L$ is the gradient (21), and $F_{LL'}$ is given by

$$F_{LL'} = \frac{1}{2} \text{Tr} \left[ C^{-1} P_L C^{-1} P_{-L} \right],$$

both evaluated at $\phi^{(P)}$. Iterations of Eq. (27) provides an estimate of $\phi$ that is accurate within the Taylor-approximated lensing equation.

For experiments with relatively large noise, however, a single step of the above iteration may be sufficient. In this case, we set $\phi^{(P)} = 0$ to be the starting point, and evaluate one step of the iteration (27). In addition to requiring only one iterative step, this has the advantage that the trace term in the gradient (21) and the Fisher matrix (28) are easier to compute, since the required inverse covariance matrix $C^{-1}$ does not include the lens operators $\Lambda$. In particular, the projection operator $P_L$ becomes

$$(P_L)_{\mathbf{n}_1, \mathbf{n}_2} = \sum_{\mathbf{l}_1 \mathbf{l}_2} e^{i\mathbf{l}_1 \cdot \mathbf{n}_1} e^{-i\mathbf{l}_2 \cdot \mathbf{n}_2} (N_{\text{tot}} A_{\text{pix}})^2 \delta_{\mathbf{l}_1, \mathbf{l}_2 - \mathbf{L}} \beta_{\mathbf{1}_1 \mathbf{1}_2}^* \left[ (\mathbf{L} \cdot \mathbf{l}_2) S_{\mathbf{l}_2} - (\mathbf{L} \cdot \mathbf{l}_1) S_{\mathbf{l}_1} \right],$$

with $S_{\mathbf{l}} \equiv \tilde{C}_{\mathbf{l}}/(N_{\text{tot}} A_{\text{pix}})$.

In most of the cases we consider below, the computed Fisher matrix shows degenerate directions, requiring some method for regularizing them. In these cases, one could consider a Gaussian prior for the lens,

$$\mathcal{P} = \frac{1}{2} \sum_{LL'} \left[ \ln S_{LL'}^\phi + \phi_L (S_{LL'}^\phi)^{-1} \phi_{L'}^* \right],$$
where
\[ S_{\mathbf{L}'}^\phi = \frac{C_{\phi}^L}{N_{\text{tot}} A_{\text{pix}}} \delta_{\mathbf{L}'}, \]  
(31)
and \( C_{\phi}^L \) is the assumed power spectrum for the lensing potential. Minimizing the (negative log) posterior probability \( \mathcal{L} + \mathcal{P} \), the quadratic estimator Eq. (27) is modified to
\[ \phi_{\mathbf{L}}^{(w)(P+1)} = - \left[ (S_{\phi})^{-1} + F \right]^{-1}_{\mathbf{L}' \mathbf{L}} G_{\mathbf{L}'} \].  
(32)

Eq. (32) can be recognized as a Wiener filtered quadratic estimator, and therefore does not result in a loss of information\[15]. We will utilize this Wiener filtered estimator for the rest of the work.

In both cases, we estimate the reconstruction error using the inverse of the Fisher matrix. It is easy to see that the covariance matrix
\[ C_{\phi} = \left( S^{-1} + F \right)^{-1} F \left( S^{-1} + F \right)^{-1} \]  
(33)
for the Wiener-filtered estimator Eq. (32), and reduces to \( C_{\phi} = F^{-1} \) for estimator Eq. (27).

Before discussing the practical issues in implementing the above, we note one caveat regarding the use of the Taylor approximated lensing equation Eq. (7) in deriving the estimator. Because the Taylor approximation fails for sufficiently small angles (or high multipole moments) \[14\], the reconstruction becomes biased for data with high angular resolution and low noise \[12\]. This bias can in principle be removed by using the full lensing operator derived in Ref. \[13\] and using it to iterate Eq. (27). However, as we will see in Sec. IV, the computational cost of a single estimate is already somewhat prohibitive without an implementation optimized for the specific experiment and observation geometry. We therefore do not perform a full iteration, as it is not a practical option for this work. We will explore the limitations imposed by this choice in Sec. IV B.

**B. Connection to earlier work**

The estimator Eq. (27) and its Wiener-filtered generalization Eq. (32) are generalized estimators that account for boundaries and correlations in pixel noise. We now show that the flat-sky estimators in Ref. \[10\] can be derived as a special case of estimator Eq. (27) applied to a periodic sky with uniform noise.
Let us suppose that the observation is made over a square region that can be regarded as periodic, and that the pixel noise is uniform over this region. In this case, \( N_{\text{tot}} = N_{\text{pix}} \), and the pixel noise is given by

\[
N_{\delta_{\text{tot}}} = \sigma_{\text{pix}}^{2} \delta_{\text{tot}}^{2},
\]

where \( \sigma_{\text{pix}}^{2} \) is the noise variance in each pixel. We also utilize a parametrization of the noise variance that is independent of the pixel size, defined as \( \Delta^{2} = \sigma_{\text{pix}}^{2} A_{\text{pix}}^{2} \).

Assuming that the observation is made with a Gaussian beam

\[
\beta(I) = \exp \left[ \frac{-l^{2} \theta_{I}^{2}}{16 \ln 2} \right],
\]

with the beam width characterized by full width half maximum (FWHM) \( \theta_{I} \), the covariance matrix at \( \phi = 0 \) and its inverse are given by

\[
C_{\mathbf{n}_{1} \mathbf{n}_{2}} = \sum_{1} \frac{\left| \beta(I) \right|^{2}}{N_{\text{tot}} A_{\text{pix}}} \hat{C}_{1} e^{i \mathbf{\hat{n}}_{1} \cdot \mathbf{\hat{n}}_{2}},
\]

\[
C_{\mathbf{n}_{1} \mathbf{n}_{2}}^{-1} = \sum_{1} \frac{A_{\text{pix}}}{N_{\text{tot}} |\beta(I)|^{2}} \hat{C}_{1}^{-1} e^{i \mathbf{\hat{n}}_{1} \cdot \mathbf{\hat{n}}_{2}},
\]

where

\[
\hat{C}_{1} = \hat{C}_{1} + |\beta(I)|^{-2} \Delta^{2}.
\]

Direct substitution of this into Eqs. (21) and (28) yield

\[
G_{\mathbf{L}} = -\frac{N_{\text{tot}} A_{\text{pix}}}{2} \sum_{1} \mathbf{x}_{1}^{T} \hat{C}_{1}^{-1} \left[ (\mathbf{L} \cdot \mathbf{l}_{1}) \hat{C}_{1} + (\mathbf{L} \cdot \mathbf{l}_{2}) \hat{C}_{1} \right] \hat{C}_{1}^{-1} \mathbf{x}_{1},
\]

\[
F_{\mathbf{L}_{1} \mathbf{L}_{2}} = \frac{\delta_{\mathbf{L}_{1} \mathbf{L}_{2}}}{2} \sum_{1} \text{Tr} \left\{ \hat{C}_{1}^{-1} \left[ (\mathbf{L} \cdot \mathbf{l}_{1}) \hat{C}_{1} + (\mathbf{L} \cdot \mathbf{l}_{2}) \hat{C}_{1} \right] \hat{C}_{1}^{-1} \left[ (\mathbf{L} \cdot \mathbf{l}_{1}) \hat{C}_{1} + (\mathbf{L} \cdot \mathbf{l}_{2}) \hat{C}_{1} \right] \right\},
\]

where we have defined \( \mathbf{x}_{1} \equiv \mathbf{m}_{1}/\beta(I) \), the beam-deconvolved data vector, and \( \mathbf{l}_{2} = \mathbf{L} - \mathbf{l}_{1} \). Transforming the sums over multipole moments back into integrals with the use of relation Eq. (15), we find that the estimator for this case is

\[
\hat{\phi}(\mathbf{L}) = A(\mathbf{L}) \int \frac{d^{2} \mathbf{l}_{1}}{(2\pi)^{2}} \frac{1}{2} e^{\mathbf{x}^{T}(\mathbf{l}_{1})} \hat{C}_{1}^{-1} \left[ (\mathbf{L} \cdot \mathbf{l}_{1}) \hat{C}_{1} + (\mathbf{L} \cdot \mathbf{l}_{2}) \hat{C}_{1} \right] \hat{C}_{1}^{-1} \mathbf{x}(\mathbf{l}_{2}),
\]

\[
A^{-1}(\mathbf{L}) = \int \frac{d^{2} \mathbf{l}_{1}}{(2\pi)^{2}} \frac{1}{2} \text{Tr} \left\{ \hat{C}_{1}^{-1} \left[ (\mathbf{L} \cdot \mathbf{l}_{1}) \hat{C}_{1} + (\mathbf{L} \cdot \mathbf{l}_{2}) \hat{C}_{1} \right] \hat{C}_{1}^{-1} \left[ (\mathbf{L} \cdot \mathbf{l}_{1}) \hat{C}_{1} + (\mathbf{L} \cdot \mathbf{l}_{2}) \hat{C}_{1} \right] \right\}.
\]

This estimator is similar to the minimum variance estimator Eq. (11) presented in Ref. [10], except for the use of the unlensed CMB power spectra in the inverse of the covariance matrix, as opposed to the lensed power spectra. As we will show in a later section, the errors derived from the Fisher matrix for Eq. (40) are much smaller than that for Eq. (11), since Eq. (40) does not include the variance in the lensed CMB in the estimator.
IV. IMPLEMENTATION

The estimator derived above is similar in construction to the quadratic estimators for power spectrum estimation [16–18]. However, the number of quantities being estimated for lensing reconstruction is much greater than that in power spectrum estimation. Furthermore, the projection operator Eq. (29) for lensing reconstruction involves filtering with direction-dependent quantities, and require different computational techniques to be developed.

Of particular concern is the computational time required for the estimation process. Current CMB experiments result in observed skies with close to a million pixels, and require computational methods that scale well with the number of pixels. We provide an estimate for the scaling properties of this estimator, provide some identities that aid in the computation, and identify the most time-consuming steps in the analysis.

The use of the Taylor approximated lensing equation Eq. (7) and the single iteration approximation in evaluating Eq. (27) can lead to bias in the estimated lens. We quantify the bias in the estimated lens by considering the average reconstructed lens for the simplest case of an observation on the full periodic sky. Additionally, we compute the actual covariance matrix in the reconstructed lens from the same sample, and compare against the Fisher matrix errors in Eq. (33).

A. Computational simplifications and scaling

The evaluation of the estimator Eq. (27) requires the calculation of a weighted quadratic quantity $m^T C^{-1} P_L C^{-1} m$, as well as the calculation of two traces. Although the form of the estimator is similar to quadratic estimators for the power spectrum [17], the projection operator $P_L$ carries nontrivial geometric dependencies. We show, however, that the computational cost of this estimator is roughly the same as that in power spectrum estimation.

The first step in the estimation process is the computation of the product $x^T P_L x$, where the vector $x = C^{-1} m$ is a filtered data vector that lives on $\Omega_{\alpha_0}$. Leaving aside the computation of $C^{-1}$ for the moment, we simplify the product by expanding $x$ in multipole moments and using expression Eq. (29), and obtain

$$x^T P_L x = 2 N_{\text{tot}} \sum_{i_1,i_2}^{\delta_{\text{i1},\text{i2}}} \left[ N_{\text{tot}} A_{\text{pix}/i_1,i_2}^x \right]^T (L \cdot L_2) S_{\text{i2}} \left[ N_{\text{tot}} A_{\text{pix}/i_2}^x \right].$$

(41)
Defining
\[ \tilde{x}_n \equiv \sum_l N_{\text{tot}} A_{\text{pix}/l} \beta_1 x_l e^{i \frac{e_n}{n}}, \]
and noting that multiplication by \( \mathbf{1} \) translates to a gradient operation in real space, the above expression can be re-expressed as
\[ -\frac{1}{2} x^T P_L x = N_{\text{tot}} \sum_n \left[ \nabla \cdot \left( \tilde{x}^T \nabla \tilde{x} \right) \right]_n e^{i \hat{\mathbf{L}} n}, \]
which equivalent to that presented in Ref. [11] for the continuous, full-sky case. This allows the calculation of these products in \( \mathcal{O}(N_{\text{tot}} \log N_{\text{tot}}) \) operations by performing the gradients of \( \mathbf{x} \) using fast Fourier transforms on the region \( \Omega_{\text{tot}} \).

The computational time for the evaluation of \( \mathbf{x}^T P_L \mathbf{x} \) is dominated by the evaluation of \( \mathbf{x} = \mathbf{C}^{-1} \mathbf{m} \), the other step being performed via Fourier transforms as described above. Following Ref. [17], we compute the inverse using the conjugate gradient method. The conjugate gradient method solves \( M \mathbf{x} = \mathbf{y} \) for \( \mathbf{x} \) iteratively, given a symmetric matrix \( M \), and typically converges in less time than a direct inversion method. In our case, we solve \( C \mathbf{x} = \mathbf{m} \), where \( \mathbf{m} \) is the data vector.

The speed of the conjugate gradient method depends on both the computational cost of each iteration and the rate of convergence. Calculation of a single step requires multiplication of a vector by the matrix \((S + N)\), where the signal matrix is diagonal in Fourier space. For most situations, noise is diagonal in the time-ordered data, or nearly diagonal in pixel space. As a result, \((S + N)\mathbf{x}\) can be computed in \( \mathcal{O}(N_{\text{tot}} \log N_{\text{tot}}) \) using FFT, if the noise is diagonal in pixel space, or \( \mathcal{O}(N_{\text{TOV}} \log N_{\text{TOV}}) \) for multiplication using time-ordered data.

The rate of convergence of the conjugate gradient method scales as the condition number (the ratio of the largest and smallest eigenvalues) of the matrix \( M \). For \( M = S + N \), the eigenvalues span many orders of magnitude; to compress the range of the eigenvalues, we rewrite \( C \mathbf{x} = \mathbf{m} \) in the form
\[ (1 + N^{-1/2} SN^{-1/2}) N^{1/2} \mathbf{x} = N^{-1/2} \mathbf{m}, \]
and apply the conjugate gradient method with \( M = 1 + N^{-1/2} SN^{-1/2} \) and solve for \( \mathbf{z} = N^{1/2} \mathbf{x} \). Note that our decomposition of \((S + N)\) is different from that presented in Ref. [17] and other standard choices, which require computation of \( S^{-1/2} \). Because our analysis includes the polarization modes, the signal covariance may have zero eigenvalues due to the
absence of power in the $B$ modes. This prevents us from utilizing the previously developed decompositions that rely on the invertibility of the signal covariance.

The convergence properties can be improved by choosing a preconditioner matrix $\tilde{M}$, such that $MM^{-1} \approx 1$, and solving $(MM^{-1})x = y$. We choose a simple preconditioner with

$$\tilde{M}_{n_1 n_2} = \sum_i \left( S_i + \bar{N}_i \right) e^{i \left( n_1 - n_2 \right)},$$

where $\bar{N}_i = \sigma^2 = \text{const.}$, and $\sigma^2$ is the largest diagonal value of the pixel noise matrix.

With the above steps, we find that our implementation of the conjugate gradient method allows the computation of $C^{-1}m$ in roughly $O(N_{\text{tot}} \log N_{\text{tot}})$ for a case with uncorrelated noise in a bounded region, albeit with a large prefactor (about a 1000 for complicated boundaries). It should be possible to improve on this by using a preconditioner that is tailored for the specific experiment, as is the case in Ref. [17], or by using alternate methods for computing the inverse (see Ref. [18], for example).

The computation of the traces $\text{Tr} \left[ C^{-1}P_L \right]$ and $F = \text{Tr} \left[ C^{-1}P_{L_1}C^{-1}P_{-L_2} \right] / 2$ are performed by noting that

$$\frac{1}{2} \text{Tr} \left[ C^{-1}P_L \right] = \langle d_L \rangle,$$

$$F = \langle d_{L_1} d_{-L_2} \rangle - \langle d_{L_1} \rangle \langle d_{-L_2} \rangle,$$

with $d_L \equiv -\frac{i}{2} m^T C^{-1}P_L C^{-1}m$, and averages taken over an ensemble of undensed maps $m$ with the same pixel noise properties as the observed map. This suggests that we can compute both traces stochastically by generating $N_{\text{MC}}$ samples of the simulated observation. The overall computation time therefore scales as $N_{\text{MC}}$ multiplied by the time required for one computation of $d_L$, so the time required for the estimation of the lensing potential scales as $O(N_{\text{MC}}N_{\text{tot}} \log N_{\text{tot}})$, possibly with a large prefactor coming from the computation of $C^{-1}$.

What criteria can be used to choose an appropriate value of $N_{\text{MC}}$? First, we choose $N_{\text{MC}}$ to be large enough so that the error in trace evaluation is small. This can be quantified by noting that the covariance matrix in $\phi$ when using the trace average Eq. (46) is related to the full covariance matrix by [17]

$$\hat{C}_{LL'}^{\phi} = \left( 1 + \frac{1}{N_{\text{MC}}} \right) C_{LL'},$$

where $C_{LL'}^{\phi}$ denotes the true covariance in $\phi$. 
We estimate the error in $\hat{\phi}$ arising from the stochastic evaluation of the Fisher matrix by numerically evaluating the error in $F$ due to the use of Eq. 47 with a finite number of samples, then propagating this error into $C^\phi$. Given an error $\delta F$ in the computation of $F$, $\hat{\phi}^{(MC)}$ becomes

$$\hat{\phi}^{(MC)} \approx -\left[ I + (S^{-1} + F)^{-1} \delta F \right] \hat{\phi},$$

where we have assumed that $(S^{-1} + F)^{-1} \delta F \ll I$, and $\hat{\phi}$ is the estimator assuming perfect knowledge of the trace and Fisher matrix. The fractional error due to the stochastic Fisher matrix evaluation is therefore given by $(S^{-1} + F)^{-1} \delta F$.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig2a.png}
\includegraphics[width=0.4\textwidth]{fig2b.png}
\caption{Fractional error ($\sigma_C$) in the diagonal element ($L = 200$ chosen) (left), and mean correlation between multipoles. Both cases are calculated using polarization data for uniform noise $\sigma_P = \sqrt{2} \mu K$ and a 4 arcminute beam.}
\end{figure}

For Fisher matrices which are dominated by the diagonal elements, we can directly compute the error in $F^{-1}$, by calculating the error in diagonal elements

$$\sigma_C^2 = \frac{\delta F_{\ell\ell}^{-1}}{F_{\ell\ell}^{-1}},$$

at a typical value of $L$, and $\delta F^{-1}$ evaluated using varying numbers of samples. To diagnose convergence of the off-diagonals, we choose to compute the mean correlation between
multipoles at a fixed separation,

$$ R_L \equiv \frac{1}{N} \sum_{i} \frac{|C_{ii}^\phi|^2}{|C_{ii}^\phi| |C_{yy}^\phi|}, $$

$$ Y' = 1 + L \quad (51) $$

at a nonzero separation $L$, again for several values of $N_{MC}$. As seen in Fig. 2, the errors fall off fairly rapidly, so that $N_{MC} = 5000$ is sufficient.

### B. Bias and errors from quadratic reconstruction

The estimator derived in Sec. IIIA is optimal under the assumption of small deflections, and if we are able to iterate Eq. (27) to convergence. However, Eq. (8) suggests that for high multipole moments, the assumption of small deflection fails. This, in addition to the use of a single step estimator, implies that Eq. (27) is expected to be biased. We quantify this bias by considering the ensemble averaged quadratic estimator.

The estimator contains errors both from the Gaussian nature of the CMB and instrumental noise. Taking an average over both, but keeping the lens fixed, we expect that

$$ \langle \hat{\phi} \rangle_{\text{ens}} = \phi, \quad (52) $$

if the estimator is unbiased. To test whether our estimator Eq. (27) is biased, we generate 10,000 samples of the CMB sky, lensed using the same lens $\phi_0$, with uniform noise added. We reconstruct the lens for each sample, and compute the above average, as well as the measured covariance

$$ C_{\text{ens}}^\phi = \langle \hat{\phi} \hat{\phi}^\dagger \rangle_{\text{ens}} - \langle \hat{\phi} \rangle_{\text{ens}} \langle \hat{\phi}^\dagger \rangle_{\text{ens}}, \quad (53) $$

which is a measure of the true covariance in our estimator. We compute the correlation between our average Eq. (52) and the true lens $\phi_0$ to quantify the bias, and compare the diagonal elements of $C_{\text{ens}}^\phi$ to the Fisher matrix errors given by Eq. (33).

Fig. 3 shows the cross correlation between the estimator average Eq. (52) and the true lens, as well as its autocorrelation, normalized to the true lens power spectrum. As pointed out in Ref. [14], the autocorrelation shows stronger biasing, primarily due to the presence of higher-order contributions to the power spectrum, and can be accounted for by subtracting the nonlinear noise contribution using the form derived in Ref. [19]. In addition, both the autocorrelation and the cross correlation show a lack starting at around $L \sim 800$. For these
FIG. 3: Cross-correlation between the averaged estimator and the true lens (solid line) and the autocorrelation in the averaged estimator (dashed line), normalized to the true lens power spectrum. 

high multipole moments, the use of the Taylor approximated lensing equation Eq. (7) fails, leading to inaccurate estimates.

Additionally, we compare the Fisher covariance Eq. (33) to the measured covariance Eq. (53) in Fig. 4. It is evident that the Fisher estimate of the error significantly underestimates the true error; the result is that the true error (in the sense of 1-σ error bars) on each multipole is roughly twice that predicted by the Fisher matrix. This should not pose a problem for reconstruction, however, as a high signal to noise ratio is maintained out to $l \sim 500$. As is also evident from the bias plot (Fig. 3), the Fisher matrix still serves as the correct normalization for the lens estimate and does not result in additional bias.

In each of the remaining sections, we compare maps produced from the same underlying CMB realization, but with different noise realizations and geometries. The differences between maps therefore do not arise from differences in the CMB realization, but purely from the noise and geometric properties we are attempting to investigate.
FIG. 4: Noise variance estimate, using the true ensemble average over the lensed CMB realizations (dotted line), and the Fisher estimate (dashed line).

V. APPLICATIONS

Studies of the expected noise properties from CMB lensing reconstruction surveys suggest that such measurements can be made with high signal to noise ratios in the near future [10]. However, the systematic effects from the presence of survey boundaries and correlated noise in the measurement may make the measurements difficult, or result in biased reconstructions.

We first investigate the effect of boundaries on lensing reconstruction. In particular, we consider the form of the bias and correlations induced by the presence of boundaries, and determine the optimal size of the embedding region used for analysis of a bounded region.

We also consider the effect of correlated noise on the measurement of lensing by examining simple cases of striped data, and discuss the extent to which correlated noise complicates the interpretation of the reconstruction.

Unless otherwise stated, we assume a reference experiment with white noise amplitudes $(\Delta_T, \Delta_P) = (1, \sqrt{2}) \, \mu\text{K}-\text{arcmin}$, and a Gaussian beam with $\theta_f = 4$ arcminutes.
FIG. 5: Reconstructed lenses with a cut in one direction. The upper left panel displays the original lens. The dashed lines in the other panels indicate the location of the boundary, with 20% (upper right), 30% (lower left), and 40% (lower right) cuts shown.

A. Boundary-induced bias and correlations

The first issue that arises in applying estimator Eq. (27) to realistic experiments is the treatment of cuts in the data. Even for all sky experiments, the presence of galactic foregrounds necessitates the removal of a significant fraction of the sky; for ground-based experiments, the area outside the survey boundary needs to be treated appropriately.

We first assess the impact of simple boundaries by considering an idealized sky cut that
preserves translational symmetry in the direction orthogonal to the cut. We generate a periodic sky $N_x = N_y = 512$ pixels, each pixel having an area of $(1 \text{ arcmin})^2$. We remove a $f = (20\%, 30\%, 40\%)$ of the sky with the boundary parallel to the vertical axis of the map, and reconstruct the lens using the Wiener filtered estimator Eq. (32).

Fig. 5 shows the reconstructed lenses, along with the original lens. Although the reconstructed lenses are visually similar to the input lens, the effect of Wiener filtering is evident in the smoothing of the region to the right of the boundary.

We determine whether the presence of a boundary introduces an additional bias in the reconstruction by computing the cross correlation between the reconstructed lenses and the true lens. As shown earlier in Fig. 3, the estimator applied to the full sky produces lens estimates that are biased by approx. 5\%, and we expect the cut sky estimator to be similarly biased. As seen in Fig. 6, there is minimal bias in the low multipole regime, and the bias increases toward $l \sim 800$, where we expect to see higher bias due to the failure of the gradient approximation.

![Cross correlation between lens reconstruction on data with sky cuts and the original lens, for cut fractions $f \in \{0\%, 20\%, 30\%, 40\%\}$, indicating the absence of substantial bias in the reconstruction due to pixel removal.](image)

FIG. 6: Cross correlation between lens reconstruction on data with sky cuts and the original lens, for cut fractions $f \in \{0\%, 20\%, 30\%, 40\%\}$, indicating the absence of substantial bias in the reconstruction due to pixel removal.

Although the reconstruction is free of additional bias, the resulting maps will be correlated in the direction perpendicular to the boundary. We quantify this by using the mean
correlation between pixels Eq. (51) defined earlier,

\[ R_L \equiv \sum_l \frac{|C_{ll}^\phi|^2}{|C_{ll}^\phi||C_{yy}^\phi|}, \]

\[ Y' = 1 + L. \]  

(54)

This measures the mean squared correlation between multipole moments separated by \( L \); if the covariance matrix \( C^\phi \) is diagonal, \( R_L \) is zero, except at \( L = 0 \). We quantify the RMS correlation between pixels separated by \( L \) using \( \chi(L) \equiv \sqrt{R_L/R_0} \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig7.png}
\caption{Correlation as a function of multipole separation in the \( x \) direction, along \( L_y = 0 \) and \( L_y = 253 \). The latter is a typical sample of the correlation function along \( L_y \neq 0 \), and shows that the correlation is purely between multipoles separated in the \( \hat{x} \) direction.}
\end{figure}

Fig. 7 shows \( \chi(L) \) for the three maps, evaluated along the \( L_y = 0 \) line, as well as along a randomly chosen \( L_y \neq 0 \) line for comparison. Because the data cuts preserve statistical translational symmetry in the \( \hat{y} \) direction, correlations are only evident between multipoles separated in the \( \hat{x} \) direction. The width of the correlation function \( \chi(L) \) is given by approximately \( 2\pi/\theta_c \), where \( \theta_c \) is the length of the cut in the \( \hat{x} \) direction. Stated differently, a data cut strongly correlates multipole moments separated by less than the fundamental mode of the cut.
FIG. 8: Reconstruction of a lens on a $6^\circ \times 6^\circ$ region (top left), using embedding regions with sizes $12^\circ \times 12^\circ$ (top right), $15^\circ \times 15^\circ$ (bottom left), and $18^\circ \times 18^\circ$ (bottom right). The maps have been filtered to remove modes with $l < 60$.

B. Observing non-periodic skies

In the above example, we have treated the $(512$ arcmin$)^2$ sky as being periodic, so that the data cut was analogous to a galaxy cut in a full sky CMB experiment. However, if the experiment does not cover most of the sky, it becomes impractical to apply the reconstruction method over the full sky. In this case, as mentioned in the derivation of Eq. (27), we assume that the observed sky could be embedded in a larger region that is considered periodic (but
much smaller than the full sky). For power spectrum estimation, such an assumption is valid provided that each side of the larger region is taken to be at least twice the longest distance in the observed area. Although the CMB lensing estimator Eq. (27) follows from the same analysis steps as the power spectrum estimators usually considered, the lensing estimator is more sensitive to the survey geometry, due to the filtering present in the projection operator Eq. (29). We test the validity of the embedding method by reconstructing a lens using different sizes of the embedding region, and estimate the required size of the region.

We create a signal sky composed of $4096 \times 4096$ pixels with $A_{\text{pix}} = 1$ arcmin$^2$, and create a data vector consisting of $360 \times 360$ pixels. We then reconstruct the lens, assuming embedding sizes $\Omega_{\text{tot}} = (12^\circ \times 12^\circ, 15^\circ \times 15^\circ, \text{ and } 18^\circ \times 18^\circ)$, and compare the reconstructed lens within the data region.

Two issues arise in quantifying the accuracy of the reconstruction. First, the reconstructions using different embedding regions sample the multipole space with differing spacings, since the multipole moments are spaced apart by $2\pi/\theta$, where $\theta$ is the angular extent of the embedding region on which Fourier transforms are taken. To avoid comparing multipole moments with different spacings, we only consider the reconstruction of the lens within the observed $6^\circ \times 6^\circ$ region.

Second, the reconstructed lenses contain poorly constructed modes that are on larger scales than the fundamental mode of the observed region, which appear as large scale variations in the lens amplitude. Since results from Sec. V A indicates that only the modes within the survey boundary are reconstructed well, we remove modes that are larger than the fundamental mode with $l \approx 60$ by applying a high pass filter to both the original and reconstructed lenses. As before, we compute the cross correlation between reconstructed maps and the input map, as well as the RMS deviations, to assess the accuracy of the reconstruction.

<table>
<thead>
<tr>
<th>Map</th>
<th>Deviation ($\gamma$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$12^\circ \times 12^\circ$</td>
<td>1216.8</td>
</tr>
<tr>
<td>$15^\circ \times 15^\circ$</td>
<td>692.3</td>
</tr>
<tr>
<td>$18^\circ \times 18^\circ$</td>
<td>488.3</td>
</tr>
</tbody>
</table>

TABLE II: RMS difference between reconstruction and original lens, computed using the filtered maps.
FIG. 9: Cross correlation between the reconstructed lenses and the original lens. Although use of the $12^\circ \times 12^\circ$ embedding results in a bias across all multipole moments, the map amplitudes for reconstructions using $15^\circ \times 15^\circ$ and $18^\circ \times 18^\circ$ are unbiased.

The filtered maps are shown in Fig. 8, and the map power spectra in Fig. 9. Although the map made from the $12^\circ \times 12^\circ$ reconstructs the general features of the original lens, significant bias is evident. On the other hand, the bias in the $15^\circ \times 15^\circ$ reconstruction, as measured in the power spectrum, is equal to that in the $18^\circ \times 18^\circ$ reconstruction, and are consistent with the fundamental bias as discussed in Sec. IV B. We also compute the RMS deviations according to

\[ \gamma^2 = \sum_n \left( \frac{\phi_n^{(\text{recon})} - \phi_n^{(\text{orig})}}{\phi_n^{(\text{orig})}} \right)^2 \]  

(55)

The deviations, shown in Table II, indicate that the $18^\circ \times 18^\circ$ reconstruction is a more accurate estimate of the original lens, and a marked improvement over the $12^\circ \times 12^\circ$ map. These results suggest that the small scale modes are reconstructed well with an analysis region with sides that are 2.5 times the extent of the observed region. However, contrary to expectations, the embedding region needs to have sides that are at least 3 times that of the observed region to provide a reasonable reconstruction.
C. Effects of correlated noise

CMB lensing reconstruction takes advantage of the fact that gravitational lensing produces correlations between CMB multipole moments that are initially absent, and therefore are sensitive to contamination from other sources of such correlations. One such source of correlations is the noise itself; if the noise is correlated in the pixel domain, it typically manifests itself as a cross correlation between multipole moments. We investigate the effects of correlated noise on lensing reconstruction by considering the simplest instance of striped noise.

We consider an experiment with both a white and $1/f$ noise component in the time-ordered domain (TOD), characterized by a TOD noise matrix

$$N(t - t') = \int df \sigma_f^2 \left[ 1 + \frac{\knee f}{f} \right] e^{2 \pi i f(t-t')},$$

where $f_{\knee}$ is the frequency where the $1/f$ component is equal to the white noise component.

We deliberately induce striping in the data by observing with a simple vertical scan strategy, where lines of pixels in the $y$ direction are scanned successively, such that the time for a single scan line is close to $1/f_{\knee}$. We consider a (360 arcmin)$^2$ map, with white noise amplitude $\sigma_r = 11.3\mu K\sqrt{s}$ for the polarization channels, each pixel being observed for 64 seconds. We choose $f_{\knee} = 3.84 \times 10^{-4}$ Hz or roughly 10 times the scan frequency.
these parameters, the total noise variance in each pixel is $\sigma_P = 2.24\mu K$, with the white noise component having $\sigma_P^{(\text{white})} = 1.414\mu K$.

![Graph of cross correlation between the original lens and the reconstruction from striped data (solid line) and data with homogeneous noise (dashed line).](image)

FIG. 11: Cross correlation between the original lens and the reconstruction from striped data (solid line) and data with homogeneous noise (dashed line).

We apply the estimator Eq. 27 to the resulting striped polarization maps; the resulting lensing potential map, Fig. 10, shows that the reconstruction from the striped data is visually comparable to that from a “sample” experiment with homogeneous noise with $\sigma_P = 2.24\mu K$. Turning to the cross correlation between the original lens and the reconstruction to estimate the bias in the measurement (see Fig. 11), we find that the striped data does not result in additional bias relative to the reconstruction on data with homogeneous noise. However, the noise variance in the reconstruction is much higher than that for the “sample” experiment, as seen in Fig. 12.

Although striping of the noise results in larger noise variance for the reconstructed multipoles compared to the case with homogeneous noise, it does not seem to result in correlations between the reconstructed multipole moments. With our particular form of the noise correlation Eq. (56) and scan strategy, the noise contribution from a particular TOD frequency $f$ maps on to a constant correlation between pixels separated by a fixed separation $\delta \hat n$. In this case, the pixel noise matrix becomes characterized by

$$
N_{\hat n_1, \hat n_2} = N_{\hat n_1, \hat n_2}^{(\text{white})} + N^{(1/f)}(\hat n_1 - \hat n_2),
$$

(57)
FIG. 12: Fisher matrix errors for the “sample” experiment and the striped experiment, showing that the striped experiment results in much higher reconstruction noise than the “sample” experiment with equivalent pixel noise variance.

where \( N^{(\text{white})}_{\mathbf{n}_1 \mathbf{n}_2} \equiv \sigma^2 \delta_{\mathbf{n}_1 \mathbf{n}_2} \) is the contribution from the white noise piece, and \( N^{(1/f)}(\mathbf{\hat{n}}_1 - \mathbf{\hat{n}}_2) \) denotes the 1/f noise, which is purely a function of the pixel separation. In this case, the noise matrix in multipole space becomes

\[
N_{l_1 l_2} = \left[ \frac{\sigma^2}{\bar{N}_{\text{tot}}} + N_{1}^{(1/f)} \right] \delta_{l_1 l_2}, \tag{58}
\]

with

\[
N^{(1/f)}(\mathbf{\hat{n}}) \equiv \sum_{\ell} N_{\ell}^{(1/f)} e^{i \ell \mathbf{n}}, \tag{59}
\]

so that the 1/f piece only adds to the diagonal part of the noise. With a pixel covariance matrix that is diagonal in multipole space, it can easily be shown that the Fisher matrix for the reconstructed lens Eq. (28) becomes diagonal. As a result, in this simplest case, striped noise acts essentially to increase the noise in the reconstruction, but does not add off-diagonal elements to the reconstruction noise covariance matrix. It should be noted that this conclusion applies to cases where the pixel noise covariance matrix \( N_{\mathbf{n}_a \mathbf{n}_b} \) only depends on the vector separation \( \mathbf{n}_a - \mathbf{n}_b \). In such a case, the pixel noise covariance matrix in multipole space takes the form in Eq. (58), and the noise matrix for the reconstructed lens multipole moments is diagonal.
VI. DISCUSSION

Gravitational lensing of the CMB offers the possibility of measuring the distribution of matter projected from high redshifts to today. Several methods to estimate the lensing potential exist [9, 10, 12, 13], and can measure the projected potential to arcminute scales in principle. In practice, however, instrumental effects as well as foreground contaminants and systematic effects in the matter distribution can potentially hinder efforts to measure the lensing effect. Of these, Ref. [14] has taken a step toward understanding the latter two, namely contamination from Sunyaev-Zel’dovich effects and non-Gaussian lensing potential. However, methods to understand the effect of experimental artifacts such as survey boundaries and correlated noise on CMB lensing reconstruction have not been available.

We have developed a framework for CMB lensing reconstruction that explicitly incorporates correlated noises and survey boundaries, and outlined a possible implementation scheme, with the intent of highlighting some of the computational difficulties. Due to its similarity to likelihood approaches used in power spectrum estimation [17, 18], our method shares many of the computational difficulties of the former, and techniques developed for power spectrum estimations are applicable to the quadratic estimators for CMB lensing.

With the computational tool in hand, we have undertaken a preliminary study of the effects of survey boundaries and striping on CMB lensing reconstruction using polarization data. As seen in Sec. V A, the primary effect of boundaries is to produce direction-dependent correlations between the reconstructed multipoles, which can be as large as several tens of percents. The characteristic scale of the data cut provides a correlation length in multipole space, where modes separated by less than the characteristic scale $2\pi /\theta_c$ are more strongly correlated.

Similarly, correlated noise in the pixel domain, in the form of striping induced by $1/f$ noise does not result in an overall bias. In the case of a simple unidirectional scan that was considered, we found that the effect of striping was to increase the noise variance in the estimator, without any additional structure in the correlation.

Lastly, we investigated the reconstruction of lenses using data on regions much smaller than the entire sky. By embedding the observed region on a larger, zero-padded sky, it is possible to recover information about the lens on scales smaller than $l \sim 2\pi /\theta_c$, where $\theta_c$ is the longest extent of the observed region. However, the reconstruction requires an
embedding region with sides at least three times $\theta_c$ in order to be unbiased relative to the usual quadratic estimator.

This work serves as a preliminary study to understand the general properties of noise contamination in CMB lensing. Although detailed investigations using realistic noise properties would be required, the results of this work suggests that the use of CMB polarization data for lensing reconstruction is promising.

Acknowledgments

TO thanks W. Hu for invaluable advice and support; and D. Nagai, E. A. Lim, and J. Chen for useful comments. This work was supported under NASA NAG 5-10840, the DOE OJI program, and partially by the Kavli Institute for Cosmological Physics.
