

Stability of cosmological solutions in $f(R)$ models of gravity

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We reconcile seemingly conflicting statements in the literature about the behavior of cosmological solutions in modified theories of gravity where the Einstein-Hilbert Lagrangian for gravity is modified by the addition of a function of the Ricci scalar, $f(R)$. Using the example of $f(R) = \pm \mu^4/R$ we show that only such choices of $f(R)$ where $d^2f/dR^2 > 0$ have stable high-curvature limits and well-behaved cosmological solutions with a proper era of matter domination. The remaining models enter a phase dominated by both matter and scalar kinetic energy where the scalar curvature remains low.

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The addition of a new term $f(R)$ to the Einstein-Hilbert action provides a possible explanation for the late-time acceleration of the universe [1–3]. However there is some confusion in the literature as to what kind of functions $f(R)$ result in solutions that are stable and cosmologically viable. In [4], the authors claim that a cosmology which behaves like Λ CDM at early times and is stable to perturbations requires $f_{RR} \equiv d^2f/dR^2 \geq 0$. On the other hand, Bean *et al.* [5] have demonstrated that linear perturbations are stable in the CDTT model [1] for which $f_{RR} < 0$. In addition, Amendola *et al.* [6,7] have claimed that $f(R)$ models do not have a normal matter-dominated epoch.

The aim of this brief report is to clarify the situation. We show that the sign of f_{RR} *does* determine whether the theory approaches the general relativity (GR) limit at high curvatures. For $f_{RR} > 0$, the theory at high curvatures behaves very close to GR and is stable. For $f_{RR} < 0$, the GR limit is unstable. In an $f(R)$ model, high density does not necessarily correspond to high curvature. We will show that the Ricci scalar rapidly evolves to a low-curvature solution at high redshift in this case. This is the instability discovered in [4]. However, the low-curvature solution itself is stable, and hence it has well-behaved linear perturbations [5]. The background cosmology in this low-curvature phase does not have the usual period of matter domination with $H \propto a^{-3/2}$. Instead it enters a ϕ MDE phase [6,7] where $H \propto a^{-2}$, and is not observationally viable.

Preliminaries.—We consider modifications to the Einstein-Hilbert action of the form [8]

$$S = \int d^4x \sqrt{-g^J} \left[\frac{R_J + f(R_J)}{2\kappa^2} + \mathcal{L}_m \right], \quad (1)$$

where R_J is the Ricci scalar, $\kappa^2 = 8\pi G$, and \mathcal{L}_m is the matter Lagrangian. We define $f_R \equiv df/dR_J$ and $f_{RR} \equiv d^2f/dR_J^2$. “J” denotes the Jordan frame, to be distin-

guished from the Einstein frame below. We consider only the metric-formalism version of the theory, in which the Ricci scalar is a fully dynamical degree of freedom.

In order for an $f(R_J)$ model to be consistent with cosmological observations, the evolution of the scale factor a_J must remain close to that of the concordance Λ CDM model: i.e. a period of radiation domination, followed by high-curvature matter domination, ending in an era of acceleration. In [4], the authors showed that there exists an $f(R_J)$ that reproduces any desired expansion history. The function is parametrized by

$$B(a_J) \equiv \frac{f_{RR}}{1 + f_R} \frac{dR_J}{d \ln a_J} \left(\frac{d \ln H_J}{d \ln a_J} \right)^{-1}, \quad (2)$$

where H_J is the Hubble parameter. However, models with $B < 0$ were found to be unstable to linear perturbations at high curvature and hence to not reproduce the desired phenomenology.

We compare this finding with that in the Einstein frame in which stable cosmological solutions for $B < 0$ models have been found [5]. As emphasized by [9,10], in the Einstein frame, gravity is unmodified but there is an additional minimally coupled scalar which evolves in a potential. Provided that $f_{RR} \neq 0$ and $1 + f_R > 0$, we can use the conformal transformation to the Einstein metric, $g_{\mu\nu}^E = \exp(\sigma\psi)g_{\mu\nu}^J$ with the new scalar field defined as

$$\psi \equiv \sigma \ln(1 + f_R), \quad (3)$$

where $\sigma = \text{sgn}(f_R)$ ensures that $\psi \geq 0$. The Jordan-frame curvature $R_J(\psi)$ can then be found by inverting the relation (3). The resulting Einstein-frame action is one for a scalar-tensor theory of gravity [11],

$$S = \int d^4x \sqrt{-g^E} \left\{ \frac{1}{2\kappa^2} \left[R_E - \frac{3}{2} (\nabla_E \psi)^2 - V(\psi) \right] + \exp(-2\sigma\psi) \mathcal{L}_m[g_{\mu\nu}^E, \exp(-\sigma\psi)] \right\}, \quad (4)$$

where

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$$V(\psi(R_J)) = \frac{R_J f_R - f}{(1 + f_R)^2}. \quad (5)$$

“E” denotes variables in the Einstein frame. Here the matter is not minimally coupled, which results in the need to transform back to the Jordan frame for the interpretation of physical results. Henceforth all derivatives, with the exception of f_R and f_{RR} , will be taken with respect to Einstein-frame variables.

High-curvature stability.—For expository purposes we use a generalization of the CDTT model [1]:

$$f(R_J) = -\sigma \frac{\mu^4}{R_J}, \quad (6)$$

where $\sigma = \pm 1$ and $\mu \sim H_0$. The $\sigma = +1$ model is the original CDTT choice with $B < 0$ and has a period of late-time acceleration with an effective $w = -2/3$. As in all inverse curvature models, the behavior in the low-curvature acceleration phase where $|f_R| \gg 1$ is independent of σ [12]. However, the acceleration phase for $\sigma = -1$ (mCDTT) eventually causes a sign change in $1 + f_R$ evolving *past* a coordinate singularity in the Einstein frame. Moreover, a sign change in $1 + f_R$ and hence B at low curvature would lead to Jordan-frame instabilities in a perturbed universe. Nevertheless, the mCDTT model is illustrative and its pathologies are avoidable. Any function $f(R)$ such that $\lim_{R_J \rightarrow \infty} f_R = 0$ could have been chosen for this analysis. For example, the model $f(R) = \mu^2(R/\mu^2)^m$ with $0 < m < 1$ has $B > 0$ and exhibits stable and observationally viable acceleration [13].

We analyze the model in the Einstein frame, with the scalar defined as in (3). At high curvature, ψ is inversely related to the Jordan-frame Ricci scalar R_J ,

$$\lim_{R_J \rightarrow \infty} \psi \sim \frac{\mu^4}{(R_J)^2}. \quad (7)$$

Small ψ implies a high curvature. If ψ evolves away from zero, the expansion enters a low-curvature phase. The potential for the scalar, as defined in (5), is

$$V(\psi) = 2\sigma\mu^2 e^{-2\sigma\psi} \sqrt{\sigma(e^{\sigma\psi} - 1)}. \quad (8)$$

We assume that the matter is a sum of a pressureless component, $p_m = 0$, with density ρ_m and an ultrarelativistic component, $p_r = 1/3$, with density ρ_r . Nonminimal coupling to the trace of the energy-momentum tensor requires that the evolution of ψ is driven by an effective potential

$$\square_E \psi = \frac{1}{3} \frac{\partial V}{\partial \psi} - \frac{\sigma}{3} \kappa^2 \rho_m^J e^{-2\sigma\psi} \equiv \frac{\partial V_{\text{eff}}}{\partial \psi}, \quad (9)$$

where we have used $\rho_m^J = \rho_m^E e^{2\sigma\psi}$ to bring out the dependence on ψ . Neither of the two quantities is conserved in the Robertson-Walker background of the Einstein frame. For example, ρ_m^E evolves according to

$$\rho_m^{E'} + 3\rho_m^E = -\frac{\sigma}{2} \psi' \rho_m^E. \quad (10)$$

Assuming that $\psi \ll 1$, i.e. that the curvature is high— $R_J \gg \mu^2$ —we can expand the effective potential around $\psi = 0$:

$$V_{\text{eff}} \approx \frac{\kappa^2 \rho_m^J}{6} + \frac{2}{3} \sigma \mu^2 \sqrt{\psi} - \frac{1}{3} \sigma \kappa^2 \rho_m^J \psi + \mathcal{O}(\psi^{3/2}). \quad (11)$$

This potential has an extremum at $\psi_{\text{GR}} \approx \mu^4 / (\kappa^2 \rho_m^J)^2$, i.e. at the GR limit of $R_J \approx \kappa^2 \rho_m^J$. For the CDTT model this is a maximum, whereas for the mCDTT model this is a minimum. The potentials for both models are presented in Fig. 1.

We can easily demonstrate the instability of the high-curvature solution for the CDTT model in the Einstein frame. For $\psi \ll 1$, the equation of motion of the scalar field becomes (9)

$$\psi'' + \left(3 + \frac{H'_E}{H_E}\right) \psi' + \frac{\sigma}{3H_E^2} \left(\frac{\mu^2}{\sqrt{\psi}} - \kappa^2 \rho_m^J\right) \approx 0, \quad (12)$$

where $' \equiv d/d \ln a_E$, and H_E obeys the Friedmann equation

$$H_E^2 = H_E^2 \frac{\psi^2}{4} + \frac{1}{6} V(\psi) + \frac{\kappa^2 \rho_m^E}{3} + \frac{\kappa^2 \rho_r^E}{3}. \quad (13)$$

We now assume that there exists a well-behaved high-curvature solution to (12), say ψ_0 . We then perturb around this solution by defining $\psi = \psi_0 + \delta\psi$ which results in the linearized equation of motion for the perturbation,

$$\delta\psi'' + \left(3 + \frac{H'_E}{H_E}\right) \delta\psi' - \frac{\sigma\mu^2}{6H_E^2 \psi_0^{3/2}} \delta\psi = 0. \quad (14)$$

For $\sigma = +1$, $\delta\psi$ is unstable on very short time scales since $\psi_0 \ll 1$. However, for the mCDTT model the high-curvature solution is stable, provided it exists. This analy-

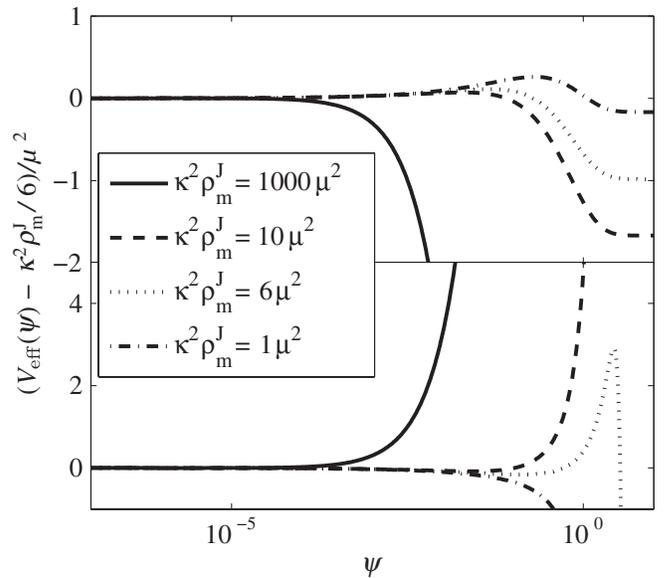


FIG. 1. Exact effective potentials with constant term subtracted, for a range of matter densities for scalar field, in CDTT (top panel) and mCDTT (bottom panel) models. Both models exhibit extrema at $R_J = \kappa^2 \rho_m^J$: a maximum for CDTT and a minimum for mCDTT.

sis is related to [10] where the mass of the scalar around $\psi = 0$,

$$m_\psi^2 = V_{,\psi\psi}|_{\psi=0} = f_{RR}^{-1} - 3R - 4f, \quad (15)$$

gave a large tachyonic value $m_\psi^2 < 0$ for the CDTT model at high curvature. The field rapidly rolls away from this maximum to high values, causing the Jordan-frame curvature to be significantly below that of GR. We illustrate this with a numerical solution in Fig. 2. The calculation is initialized at the GR value of the Ricci scalar, which quickly drops to a small and slowly varying value. Correspondingly, $H_J \propto a_J^{-2}$ as in the radiation-dominated epoch where $R \ll \kappa^2 \rho$.

We now need to establish the existence of the high-curvature solution in the case of mCDTT, i.e. that corrections to GR remain small throughout the matter-dominated phase. We continue to assume that $\psi, \psi' \ll 1$ (implying that Jordan- and Einstein-frame quantities are approximately equal) and we ignore the contribution of radiation. We take $\psi_{GR} = \mu^4 / (\kappa^2 \rho_m^J)^2$ as the GR solution for the Jordan-frame Ricci scalar. Defining the perturbation away from this solution as $\delta\psi \equiv \psi - \psi_{GR}$, we obtain the linearized equation for these perturbations from (12) during matter domination,

$$\delta\psi'' + \frac{3}{2}\delta\psi' + \frac{9}{2\alpha^4 a_E^6} \delta\psi = -45\psi_{GR} \approx -\frac{45\alpha^4 a_E^6}{9}, \quad (16)$$

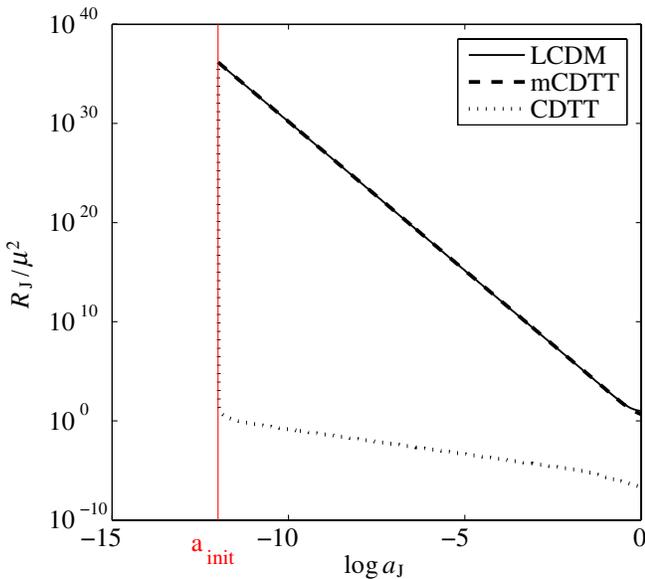


FIG. 2 (color online). Evolution of the Jordan-frame Ricci scalar for cold dark matter with cosmological constant (LCDM, solid line; $\mu = H_0$), mCDTT (dashed line; $\alpha = 2$), and CDTT (dotted line; $\alpha = 2$). All models start with the same GR initial conditions. This solution is unstable in the CDTT model and the curvature rapidly relaxes to the ϕ MDE phase where it is much lower than for the corresponding GR solution. This phase replaces the standard matter domination for $f(R)$ models with $B < 0$. mCDTT follows standard LCDM evolution until late times, where it asymptotes to a milder acceleration phase.

where $\alpha^2 \equiv 3\mu^2 / \kappa^2 \rho_m^J (a_J = 1)$. This is a damped harmonic oscillator with an extremely high frequency, $\omega \sim 1/\alpha^2 a_E^3$. The amplitude is driven by the inhomogeneous term, giving

$$\frac{\delta\psi}{\psi_{GR}} = -\frac{10\alpha^4}{a_E^6}. \quad (17)$$

This remains extremely small throughout matter domination, confirming that for the mCDTT model the cosmology remains extremely close to that for GR until the acceleration phase.

Low-curvature ϕ MDE phase.—We have demonstrated above that in the $B > 0$ mCDTT model the high-curvature solution is stable, whereas it is not in the $B < 0$ CDTT model. The evolution of $f(R)$ theories generically exhibits a fixed point in which the Einstein-frame scalar field is kinetic-energy dominated and remains a constant fraction of the energy density, together with the matter [6,7]. This phase replaces the ordinary matter-dominated expansion for $B < 0$ and is called the ϕ MDE solution. The analysis was extended in [14] to show that $B > 0$ models do have a usual matter-dominated era for at least some initial conditions.

In this section we will demonstrate—using a slight modification of the method originally introduced in [15] and then used in [5]—that indeed the ϕ MDE phase is a saddle point in the evolution for both CDTT and mCDTT, but it can only be achieved for CDTT.

We introduce the dimensionless variables

$$x = \frac{\psi'}{2}, \quad y = \frac{V}{6H_E^2}. \quad (18)$$

Assuming that radiation is negligible, we can rewrite the Friedmann equation (13) as an equation for the matter energy density and its derivative,

$$\frac{\kappa^2 \rho_m^E}{3H_E^2} = 1 - x^2 - y, \quad \frac{H_E'}{H_E} = -\frac{3}{2}(1 + x^2 - y). \quad (19)$$

We assume that the potential for the scalar field is of exponential form, $V(\psi) = A \exp(\gamma\psi)$, which is valid when $\psi \gg 1$. From (8), $\gamma = -3/2$ for CDTT, while $\gamma = 2$ for mCDTT. This allows us to recast (9) and (10) as evolution equations for x and y :

$$x' = \frac{\sigma}{2} - \frac{3}{2}x - \frac{\sigma}{2}x^2 + \frac{3}{2}x^3 - y\left(\gamma + \frac{\sigma}{2} + \frac{3}{2}x\right), \quad (20)$$

$$y' = y(2\gamma x + 3(1 + x^2 - y)). \quad (21)$$

The fixed points of the evolution occur wherever $x' = y' = 0$. In particular, the point corresponding to ϕ MDE lies at $(x, y) = (\sigma/3, 0)$. At this point, the scalar field energy density is completely dominated by the kinetic term and the fraction of the total energy in matter is $\Omega_m = 8/9$. From Eq. (19), the effective equation of state is $w = 1/9$, resulting in the Einstein-frame Hubble parameter evolving as $H_E \propto a_E^{-5/3}$. The Jordan- and Einstein-frame Hubble

parameters are related by

$$H_J = H_E e^{\sigma\psi/2} \left(1 - \frac{\sigma\psi'}{2}\right). \quad (22)$$

Since $\psi \sim 2 \ln a_E/3$ at this fixed point, $H_J \propto a_J^{-2}$, just as during radiation domination.

Using (12) and (13), we can express the Jordan-frame Ricci scalar as

$$R_J = e^{\sigma\psi} \left(4V + 2\sigma \frac{\partial V}{\partial \psi}\right), \quad (23)$$

and hence both the models under consideration have $R_J \ll H^2$ at the ϕ MDE fixed point.

The eigenvalues at this point are $-4/3$ and $2/3(5 + \alpha\sigma)$. For both models ϕ MDE is a saddle point. However, only for the CDTT model is this stage of evolution possible to achieve. Here, the field has a large value and is running off to positive infinity, exponentially suppressing the contribution of the potential to the Hubble parameter. In the mCDTT model, $V \propto e^{2\psi}$: the large-field limit is also the large-potential limit. The existence of the ϕ MDE phase therefore requires that $\psi \ll 0$, which is not a valid limit of the Jordan-frame theory and is a result of the use of the exponential approximation to the potential.

Discussion.—We have shown that, in the case of the generalized CDTT model, the sign of B determines the behavior of the model and leads to a qualitative change in the Einstein-frame potential for the scalar field. This scalar field has a large negative mass squared in the high-curvature limit for $B < 0$ models, resulting in unstable solutions in the GR limit. The field runs off to a large value, where the potential is effectively exponential. The cosmology enters a phase of simultaneous scalar field and matter domination with the physical Hubble parameter H_J

proportional to a_J^{-2} and a very small value of the curvature scalar in the Jordan frame.

The situation is radically different in the case of mCDTT, where $B > 0$ at high curvature. The GR-like solution is stable; the cosmology goes through a phase of standard matter domination where corrections to GR remain small. The acceleration behavior is asymptotically the same as CDTT but this specific model will develop instabilities deep in the acceleration regime.

Another important difference between the two model classes parametrized by B is their behavior around mass sources, i.e. their predictions for solar-system tests. Indeed, in $B < 0$ models the Ricci scalar is perturbed only slightly from its low background value by sources of matter density, $R_J \ll \kappa^2 \rho_m$. Parametrized post-Newtonian parameters can then be calculated using the weak-field approximations [16,17] which results in a highly excluded value of $\gamma = 1/2$ [10]. However, this is not necessarily true for $B > 0$ models. For these models, the value of the Ricci scalar can be high, reflecting the local matter density. It is therefore inappropriate to linearize the perturbations to R_J . As shown in [13,18], the chameleon mechanism resulting from the nonminimal coupling of the theory to matter and the existence of a minimum in the scalar potential around the GR value of the Ricci scalar allows for the theory to evade solar-system tests, at least for certain functions $f(R)$ and certain ranges of parameters. We will elaborate in detail on the constraints put on $f(R)$ theories by solar-system tests of gravity in a future work [19].

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