Near-horizon solution for Dvali-Gabadadze-Porrati perturbations

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We develop a scaling ansatz for the master equation in Dvali, Gabadadze, Porrati cosmologies, which allows us to solve the equations of motion for perturbations off the brane during periods when the on-brane evolution is scale free. This allows us to understand the behavior of the gravitational potentials outside the horizon at high redshifts and close to the horizon today. We confirm that the results of Koyama and Maartens are valid at scales relevant for observations such as galaxy-ISW correlations. At larger scales, there is an additional suppression of the potential which reduces the growth rate even further and would strengthen the integrated Sachs-Wolf effect.

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I. INTRODUCTION

That cosmic acceleration is a fact appears indubitable. Instead of an exotic new form of dark energy driving the acceleration, it may be caused by a modification of gravity. Precise measurements for gravity are only available in the range of scales from a millimeter to that of the solar system—we do not have any direct probe of Einstein gravity beyond these boundaries. Cosmic acceleration may originate in a breakdown of Einstein gravity at distances beyond the range above.

Dvali, Gabadadze, and Porrati (DGP) [1] have proposed a braneworld theory in which our universe is a $(3 + 1)$-dimensional brane embedded in an infinite Minkowski bulk. Gravity propagates everywhere, but, on the brane, an additional $(3 + 1)$-dimensional gravitational interaction is induced. This allows for gravitational potentials on the brane of a $(3 + 1)$-dimensional form at small distances to evolve into $(4 + 1)$-dimensional form beyond a cross-over scale determined by the unknown energy scale for the bulk gravity. The cosmological solution of this theory was shown to exhibit accelerated cosmic expansion without the aid of an exotic energy component like dark energy [2,3].

It has been shown already that the linearized field theory as defined by the DGP model contains ghost degrees of freedom [4–8], or even may violate causality in certain limits [9]. It is known, for instance, that the de Sitter background is unstable to classical linear perturbations; however, it is claimed in [10] that strong coupling effects at small radii around matter sources ensure that the theory remains stable.

The point of view of our work is to assume that linear perturbation theory remains valid on the largest scales. This is motivated by the fact that the late universe is dominated by the gravitational interaction of dark-matter halos. The internal structure of the halos is controlled by the strongly coupled nonlinear theory. On the other hand, the radius below which strong coupling is important for halos is approximately equivalent to their size and therefore their interactions should be driven by the linear theory analyzed in this work.

We do find that deep into the accelerated era the spacetime becomes unstable on the timescale of the expansion. However, this is an effect that only becomes important far into the future and is negligible as far as the observational impact today is concerned. We therefore assume that during the early universe, when the theory does not exhibit instabilities, the analysis for DGP proceeds in exactly the same way as that for general relativity. Then, deep during the acceleration era, instabilities develop and the theory may or may not be saved by nonlinear effects—an issue on which we remain agnostic. This evolutionary history appears to be the only one which is capable of reproducing the Universe as we see it. If strong coupling effects are important straight away and at all scales, the approximation of a homogeneous background cosmology is completely inapplicable and the DGP model would not be able to reproduce observations such as supernova luminosities. We therefore effectively assume a best-case scenario for DGP: should this analysis fail to predict the observations, the model is excluded. If it passes the observational tests, a more careful study of the effects of the strong coupling regime during the acceleration era would be required to understand fully the future evolution of the Universe in the DGP model.

Under the above assumptions, the equations of motion for the theory of gravity on the brane, pertinent to the study of cosmology, do not close owing to the interaction with the bulk at first order in perturbation theory. Koyama and Maartens [11] have used a quasistatic approximation, valid well within the horizon, to investigate structure formation at smaller scales. This solution shows the essential role that the bulk plays in correcting the gravitational potentials,

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reducing the growth rate. Also, Lue *et al.* [12] have reached a similar conclusion using a different approach, including nonlinearities in their calculations.

In the following, we present a new scaling ansatz for the master equation, allowing us to solve the equation and calculate the resulting cosmological evolution at all scales for high redshifts, and close to the horizon today. In Sec. II, we review the linearized equations of motion for DGP. We present our scaling solution in Sec. III and discuss its cosmological implications in Sec. IV. We study the robustness of the scaling solution in the appendix and discuss these results in Sec. V.

II. DGP EQUATIONS OF MOTION

A. Background

In the DGP model, gravity alone propagates in the bulk, and the 5D gravitational theory is complemented by an induced 4D Ricci scalar restricted to the brane. We assume that both the bulk and the brane have zero tensions, i.e. the cosmological constants are zero. We thus start off with the basic DGP action:

$$\mathcal{S} = \int \mathrm{d}^5x\sqrt{-g}\left[\frac{^{(5)}R}{2\kappa^2} + \delta(\chi)\left(\frac{^{(4)}R}{2\mu^2} + \mathcal{L}_{\mathrm{SM}}\right)\right], \tag{1}$$

a 3-brane embedded in an empty bulk, with all the standard model fields localized on the brane at $\chi = 0$. The constants μ^2 and κ^2 define the energy scales of the theories of gravity: one is Newton's constant, $\mu^2 = 8\pi G$, the other represents the energy scale of the bulk gravity.

As shown by Dvali *et al.* [1], the ratio of the two scales defines a cross-over radius beyond which the four-dimensional gravitational theory transitions into a five-dimensional regime

$$r_{\mathrm{c}} = \frac{\kappa^2}{2\mu^2}. \tag{2}$$

This scale is chosen to be of the order of the current Hubble length so that the acceleration of the expansion today results from the 4D to 5D transition. For illustrative purposes, we will take in our calculations $r_{\mathrm{c}}H_0 = 1.32$, in a matter and radiation universe with $\Omega_{\mathrm{m}} = 0.24$ and $h = 0.66$. When comparing with ΛCDM, the concordance set of parameters will be used: $\Omega_{\mathrm{m}} = 0.25$, $h = 0.72$. The two sets of parameters represent cosmologies which are best fits to supernova luminosity data (SNLS [13]) and the distance to the last-scattering surface (WMAP [14]), but assuming that the Universe is flat. We discuss the fits to these data in a companion paper, Song *et al.* [15].

For the purpose of cosmological calculations, we would like to reduce the five-dimensional braneworld to an effective theory on the brane, which could be studied using the usual range of four-dimensional tools. Using a 4 + 1 decomposition of the theory [16–18], we can derive the effective on-brane equations of motion—a set of modified Einstein equations

$$G_{\mu\nu} = 4r_{\mathrm{c}}^2 f_{\mu\nu} - E_{\mu\nu}, \tag{3}$$

where the Greek indices range across all the dimensions, $\mu, \nu = 0, 1, \ldots, 4$. $f_{\mu\nu}$ is a tensor quadratic in the four-dimensional Einstein and energy-momentum tensors,

$$f_{\mu\nu} \equiv \frac{1}{12}AA_{\mu\nu} - \frac{1}{4}A_{\mu}{}^{\alpha}A_{\nu\alpha} + \frac{1}{8}g_{\mu\nu}\left(A_{\alpha\beta}A^{\alpha\beta} - \frac{A^2}{3}\right), \tag{4}$$

$$A_{\mu\nu} \equiv G_{\mu\nu} - \mu^2 T_{\mu\nu}, \tag{5}$$

while $E_{\mu\nu}$ is the bulk Weyl tensor projected onto the brane using the vector normal to the brane, n^μ,

$$E_{\mu\nu} \equiv C_{\alpha\mu\beta\nu}n^\alpha n^\beta. \tag{6}$$

It can be shown that for branes with maximally symmetric spatial hypersurfaces, the projected Weyl tensor must take the form Ca^{-4}, with C a constant (see [19]: it is effectively a constant of integration for the background). Since the Weyl tensor is traceless, it will dilute as rapidly as radiation and will become irrelevant at late times, if it does not completely dominate the dynamics initially. We will therefore set $C = 0$, allowing us to find the modified Friedman equation for the background evolution of the cosmology on the brane. In the case of the flat brane, which is the only one to be considered here,

$$H^2 \mp \frac{H}{r_{\mathrm{c}}} = \frac{\mu^2\rho}{3}, \tag{7}$$

leading to the result that, for the upper-sign selection, the cosmology tends to a de Sitter phase as the matter density gets diluted away, potentially providing a model for the observed acceleration [20]. This is the choice we will make henceforth.

In addition to r_{c}, there is another scale present in the theory, which thus far we have ignored: the strong coupling scale given by

$$r_* = (r_{\mathrm{c}}^2 r_{\mathrm{g}})^{1/3}, \tag{8}$$

where r_{g} is the Schwarzschild radius of the point source under consideration. Beneath this radius, the linear approximation developed above is not valid and the theory returns to 4D general relativity with small corrections. It is not yet understood how two sources superimpose when the linear regime is not valid. However, the linear density fields are actually constructed out of a spatial average of dark-matter halos, each of which has r_* comparable to its radius. Thus we should be able to use the linear theory to discuss the gravitational dynamics of the spatial averages of the halos, but most likely not their internal structure.

B. Linear perturbations

An observer restricted to the brane will perceive the universe as purely four dimensional. Therefore, the most general linear scalar perturbations to the induced flat four-dimensional metric can be parameterized by

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(1 + 2\Phi)d\boldsymbol{x}^2, \tag{9}$$

while the linearized energy-momentum tensor can be written down as

$$T^0{}_0 = -\rho(1 + \delta), \tag{10}$$

$$T^0{}_i = (1 + w)\rho\partial_i q, \tag{11}$$

$$T^i{}_j = \rho(w + c_s^2\delta)\delta^i{}_j + w\rho\left(\partial^i\partial_j - \frac{1}{3}\delta^i{}_j\right)\pi. \tag{12}$$

Here ρ is the density of the cosmological background, $w \equiv p/\rho$ is the background equation of state parameter, and $c_s^2 = \delta p/\delta\rho$ is the sound speed for the pressure perturbations.

The full 5D perturbations of the bulk are richer: Deffayet in [21] has shown that their effect on the brane can be reduced to the presence of perturbations to the Weyl tensor as an additional source of stress energy. The precise relationship between the various components of the Weyl tensor is determined by their relationship to a master variable (see Sec. II C), which in turn encodes all the gauge-invariant perturbations to the full 5D metric throughout the bulk. We define the linear scalar perturbations to the Weyl tensor by

$$E^0{}_0 = -\mu^2\rho\delta_E, \tag{13}$$

$$E^0{}_i = \mu^2\rho\partial_i q_E, \tag{14}$$

$$E^i{}_j = \mu^2 w_E\rho\left[\delta_E\delta^i{}_j + \left(\partial^i\partial_j - \frac{1}{3}\delta^i{}_j\partial^2\right)\pi_E\right]. \tag{15}$$

The Weyl tensor is traceless, and hence the pressure perturbation will behave like radiation, i.e. $w_E = 1/3$. The linearized Poisson and anisotropy equations then are

$$\frac{k^2}{a^2}\Phi = \frac{\mu^2\rho}{2}\frac{2Hr_c}{2Hr_c - 1}\Delta + \frac{\mu^2\rho}{2}\frac{1}{2Hr_c - 1}\Delta_E, \tag{16}$$

$$\Phi + \Psi = -\left[1 + \frac{1}{2r_cH(1 + \frac{\dot{H}}{2H^2}) - 1}\right]\mu^2\rho a^2 w\pi - \frac{1}{2r_cH(1 + \frac{\dot{H}}{2H^2}) - 1}\frac{\mu^2\rho}{3}a^2\pi_E, \tag{17}$$

with Δ the comoving density contrast and the equivalent definition for Δ_E:

$$\Delta = \delta - 3H(1 + w)q, \qquad \Delta_E = \delta_E - 3Hq_E. \tag{18}$$

The equations of motion for the energy-momentum tensor are given by the conservation law $\nabla^\mu T_{\mu\nu} = 0$ supplemented by equations of state that define the stress fluctuations

$$\frac{d}{dt}\left[\frac{\delta}{1 + w}\right] - \frac{k^2}{a^2}q = -3\dot{\Phi}, \tag{19}$$

$$\dot{q} - 3c_s^2 qH + c_s^2\frac{\delta}{1 + w} - \frac{2}{3}\frac{w}{1 + w}k^2\pi = -\Psi. \tag{20}$$

We have here assumed adiabatic pressure fluctuations in the multicomponent matter system $c_s^2 = \dot{p}/\dot{\rho}$.

On the other hand, the Weyl tensor is not separately conserved and its equations of motion come from the Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$, as applied to Eq. (3),

$$\nabla^\mu E_{\mu\nu} = 4r_c^2\nabla^\mu f_{\mu\nu}. \tag{21}$$

We can rewrite this as

$$\dot{\delta}_E + (1 - 3w)H\delta_E - \frac{k^2}{a^2}q_E = 0, \tag{22}$$

$$\dot{q}_E - 3wHq_E + \tfrac{1}{3}\delta_E - \tfrac{2}{9}k^2\pi_E = S, \tag{23}$$

where the source term

$$S \equiv \frac{2r_c\dot{H}}{3H}\left[\frac{\Delta + \Delta_E}{1 - 2Hr_c} + \frac{k^2(w\pi + \pi_E/3)}{1 - 2Hr_c(1 + \frac{\dot{H}}{2H^2})}\right]. \tag{24}$$

Thus a nonzero Weyl tensor is unavoidably generated by matter perturbations in linear theory [11].

In order to close the above equations, we need the analogue of an equation of state to relate the Weyl anisotropic stress π_E to the other components of the Weyl tensor δ_E and q_E. Unlike the relationship between the Weyl pressure and energy density, this relation requires a consideration of perturbations in the bulk.

C. Master equation

In [22], Mukohyama showed that for maximally symmetric five-dimensional space-times, the full five-dimensional linear scalar perturbations in the bulk can be described using a master variable Ω. Since the bulk being considered is just Minkowski, and the brane is assumed flat, we can parameterize the unperturbed background 5D metric by [23]

$$ds^2 = -n(y, t)^2dt^2 + b(y, t)^2d\boldsymbol{x}^2 + dy^2, \tag{25}$$

where the brane sits at $y = 0$ and

$$b = a(1 + H|y|), \qquad n = 1 + \left(\frac{\dot{H}}{H} + H\right)|y|. \tag{26}$$

This parameterization is also valid for branes with nonzero curvature provided that $|\Omega_k| \ll 1$. The master variable then obeys a hyperbolic equation of motion:

$$-\frac{\partial}{\partial t}\left(\frac{\dot{\Omega}}{nb^3}\right)+\frac{\partial}{\partial y}\left(\frac{n}{b^3}\frac{\partial\Omega}{\partial y}\right)-\frac{nk^2}{b^5}\Omega=0. \tag{27}$$

We can then express all the gauge-invariant perturbations to the 5D bulk as functions of derivatives of the master variable. In particular, Deffayet [21] has shown that the components of the Weyl tensor evaluated on the brane can be expressed as

$$\mu^2\rho\delta_E=-\frac{k^4\Omega}{3a^5}\bigg|_{y=0}, \tag{28}$$

$$\mu^2\rho q_E=-\frac{k^2}{3a^3}(\dot{\Omega}-H\Omega)\bigg|_{y=0}, \tag{29}$$

$$\mu^2\rho\pi_E=-\frac{1}{2a^3}\left(3\ddot{\Omega}-3H\dot{\Omega}+\frac{k^2}{a^2}\Omega-\frac{3\dot{H}}{H}\frac{\partial\Omega}{\partial y}\right)\bigg|_{y=0}. \tag{30}$$

We will hereafter implicitly assume evaluation at $y=0$ for the master variable in the on-brane equations where no confusion might arise. We can now rewrite the Bianchi identity in terms of the master variable, obtaining, after assuming that the cosmological fluid has no anisotropy stress,

$$\ddot{\Omega}-3HF(H)\dot{\Omega}+\left(F(H)\frac{k^2}{a^2}+\frac{H}{K(H)r_c}+\frac{2Hr_c-1}{r_c}RH\right)\Omega=\frac{2a^3}{k^2K(H)}\mu^2\rho\Delta, \tag{31}$$

where R expresses the derivative across the brane

$$R\equiv\frac{1}{H\Omega}\frac{\partial\Omega}{\partial y}\bigg|_{y=0}. \tag{32}$$

We have defined two new functions of the Hubble parameter

$$F(H)\equiv\frac{2Hr_c(1+\frac{\dot{H}}{3H^2})-1}{2Hr_c-1}, \tag{33}$$

$$K(H)\equiv\frac{2Hr_c-1}{2Hr_c(1+\frac{\dot{H}}{2H^2})-1}. \tag{34}$$

Were it not for the $\partial\Omega/\partial y$ derivative across the brane in R, this equation would be a simple dynamic equation for Ω, which, given an evolution equation for the source Δ, could be solved as a coupled equation. The role of the master equation is to define R, the relationship between $\partial\Omega/\partial y$ and Ω.

Koyama and Maartens [11] adopted a quasistatic approach to solve these equations. The master equation then implies that the gradient

$$R=-\frac{k}{aH}. \tag{35}$$

In the Bianchi identity the time-derivative and brane-derivative terms are neglected compared to those of order $(k/aH)^2\Omega$. This quasistatic approach leads to their solution to which we will refer henceforth as "QS"

$$\Omega_{QS}=\frac{2a^5}{k^4F(H)K(H)}\mu^2\rho\Delta. \tag{36}$$

This solution is equivalent to a closure relationship for π_E in terms of δ_E through Eqs. (28) and (30). Using this, we can define the QS limit of DGP gravity where the Poisson and anisotropy equations become

$$\frac{k^2}{a^2}(\Phi-\Psi)=\frac{\mu^2\rho}{2}\Delta \tag{37}$$

$$\frac{k^2}{a^2}(\Phi+\Psi)=-\frac{\mu^2\rho}{6\beta}\Delta, \tag{38}$$

with

$$\beta\equiv1-2r_cH\left(1+\frac{\dot{H}}{3H^2}\right). \tag{39}$$

These equations are equivalent to the linear limit of the results obtained by Lue *et al.* [12].

In the next two sections, we will show how the quasistatic solution is dynamically achieved for the perturbations shortly after horizon crossing and discuss large-scale deviations from this solution.

III. SCALING SOLUTION TO MASTER EQUATION

A. Causal horizon

The master equation is a wave equation sourced by the comoving density perturbations on the brane through the Bianchi identity (31).

Since, in appropriate coordinates, the bulk is just Minkowski, the evolution of Ω in the bulk can be seen as a normal propagating wave given a boundary condition from the behavior on the brane. Beyond the causal horizon, the bulk should remain unperturbed. This causal horizon must be invariant in all coordinizations of the bulk; therefore, we can locate it by finding the null geodesic of Eq. (25), giving us the y position of the horizon as

$$\xi=y_{\rm hor}H=aH^2\int_0^a\frac{d\tilde{a}}{\tilde{a}^2H(\tilde{a})^2}. \tag{40}$$

Before the acceleration epoch, this reduces to $\xi=1/(2+3w)$ for a cosmology with a constant equation of state parameter, i.e. $\xi=1/3$ for a radiation-only cosmology and $\xi=1/2$ for a matter-only cosmology. We are making the assumption that the universe has not gone through a period of inflation, which would have moved the horizon much further out. If inflation did take place, it is not unreasonable to expect that any perturbations that existed prior to the inflationary phase will have been pushed far away and the bulk will start in an unperturbed state in the vicinity of the brane at the beginning of radiation domination, resulting in a causal horizon equivalent to that of the cosmology with no inflation.

The constancy of the horizon during the domination of a particular fluid suggests that we can define a new variable, in which the horizon will remain fixed at all times, lying at $x = 1$

$$x \equiv \frac{yH}{\xi}. \tag{41}$$

We can then recast the master equation in x and solve it as a boundary-value problem with the value of Ω at the horizon, $x = 1$, set to zero.

B. Scaling ansatz

The second boundary condition needed to solve the master equation is the behavior of the master variable on the brane. During epochs when the source remains scale free and the Bianchi dynamics also do not change, one would expect that the master variable also obey a scaling ansatz on the brane $\Omega|_{y=0} = Aa^p$, where A and p are constants. Likewise, during such epochs we expect the master variable in the bulk to reach a stable solution in the variable x, the distance in units of the causal horizon, for a given wave number k/aH.

We therefore propose a new ansatz for the solution to the master Eq. (27):

$$\Omega = A(p)a^p G(x). \tag{42}$$

With this assumption, the master Eq. (27) becomes the ordinary differential equation

$$\begin{aligned} &\frac{\mathrm{d}^2G}{\mathrm{d}x^2} + \left(\frac{4h^2 + h(2p-1) + 2h'}{2h(1 + x\xi(1+2h))} - \frac{1+2p}{2(1+x\xi)} - \frac{h + h^2 + h'}{h(1 + x\xi(1+h))}\right)\xi\frac{\mathrm{d}G}{\mathrm{d}x} \\ &\quad + \left(\frac{hp(2p-5) - 2h'p}{4h^2(1+x\xi)} + \frac{3p}{2(1+x\xi)^2} - \frac{(1+2h)(h(4h+2p-1) + 2h')p}{4h^2(1+x\xi(1+2h))} + \frac{(1+h)(h+h^2+h')p}{h^2(1+(1+h)x\xi)} \right.\\ &\quad \left. - \left(\frac{k}{aH}\right)^2 \frac{(1+(1+h)x\xi)^2}{(1+x\xi)^3(1+x\xi(1+2h))}\right)\xi^2 G = 0, \end{aligned} \tag{43}$$

where the primes denote differentiation with respect to $\ln a$—the new time coordinate which will be used henceforth—and $h \equiv H'/H$. In deriving the above, we have neglected time derivatives of p and ξ: the scaling ansatz is not expected to be valid when p is not a constant, i.e. during times when the cosmology is undergoing a change from the domination of one fluid to another. In addition, strictly speaking, G is actually a function of both x and k/aH, even in the scaling limit. We have therefore also assumed that k/aH is a constant, which is valid in the $k = 0$ limit. As we explain later, as k/aH approaches unity, where its time derivatives might impact the solution significantly, the character of Eq. (43) changes and terms which do not involve the derivatives of k/aH dominate.

Note that, provided w is constant, one of the denominators in Eq. (43) can be reexpressed as

$$1 + x\xi(1+2h) = 1 - x. \tag{44}$$

Thus the equation has a regular singular point $x = 1$, exactly at the causal horizon. This is not a coordinate singularity (all the entries of the metric are regular there), but is a reflection of the junction between perturbed and unperturbed space-times.

Supplying H, p as a function of the scale factor is enough to solve this ordinary differential equation as a boundary-value problem, requiring that $G(x)$ be 1 on the brane and 0 at $x = 1$. This in turn gives the value of R as

$$R = \frac{1}{\xi}\frac{\frac{\mathrm{d}G}{\mathrm{d}x}|_{x=0}}{G|_{x=0}}, \tag{45}$$

and closes the evolution equations for the perturbations on the brane.

We will henceforth refer to this solution as the dynamical scaling or just scaling solution and use the acronym "DS."

C. Iterative solution

In practice, one does not know the scaling index p *a priori* and, moreover, it can change during the evolution of a k mode as the master variable leaves one scaling regime and enters another. We therefore solve for p iteratively by demanding consistency with the Bianchi identity.

To determine the zeroth-order solution for p, we substitute the ansatz Eq. (42) into Eq. (31) to obtain

$$\begin{aligned} A(p) &= \frac{2a^{3-p}}{k^2K(H)H^2[J(H) + F(H)(k/aH)^2]}\mu^2\rho\Delta, \\ J(H) &= p[p + h - 3F(H)] + \frac{1}{K(H)Hr_c} + \frac{2Hr_c - 1}{Hr_c}R. \end{aligned} \tag{46}$$

Before the acceleration epoch, when the expansion is dominated by a single fluid, $J(H)$ and $F(H)$ are constant, while $K(H)$ is a simple power law in a. We therefore set

$p = p^{(0)}$, the zeroth-order solution for $k/aH \ll 1$

$$p^{(0)} = 3 + \frac{\mathrm{d}\ln[\rho\Delta^{(0)}/(K(H)H^2)]}{\mathrm{d}\ln a}. \qquad (47)$$

Note that this definition allows transitions between scaling regimes where w changes. We shall see that the solutions for R become independent of p for $k/aH \gg 1$ and so we use this as the zeroth-order solution for all modes, both superhorizon and subhorizon.

Finally we need a zeroth-order solution for $\Delta^{(0)}$. Modes of interest to large-scale cosmological tests are superhorizon during radiation domination and enter the horizon either during matter domination or the current acceleration epoch. Before the acceleration epoch and in the absence of anisotropic stress, these modes obey [24]

$$\Delta^{(0)} \propto \frac{D^3 + \frac{2}{9}D^2 - \frac{8}{9}D - \frac{16}{9} + \frac{16}{9}\sqrt{D+1}}{D(D+1)}, \qquad (48)$$

where $D \equiv a/a_{\mathrm{eq}}$. In order to obtain the zeroth-order solution for $p^{(0)}$ we assume that the growth prescribed by Eq. (48) continues until today (see Fig. 1).

Given this, we solve the master equation (43) to obtain the off-brane gradient, R, and then dynamically solve the Bianchi identity Eq. (31) coupled to the cosmology, for the particular mode. Once the dynamic evolution for the first-order solution $\Omega^{(1)}$ is obtained, we can iteratively improve our estimation of p by numerically calculating

$$p^{(i)} = \frac{\mathrm{d}\ln\Omega^{(i)}}{\mathrm{d}\ln a} \qquad (49)$$

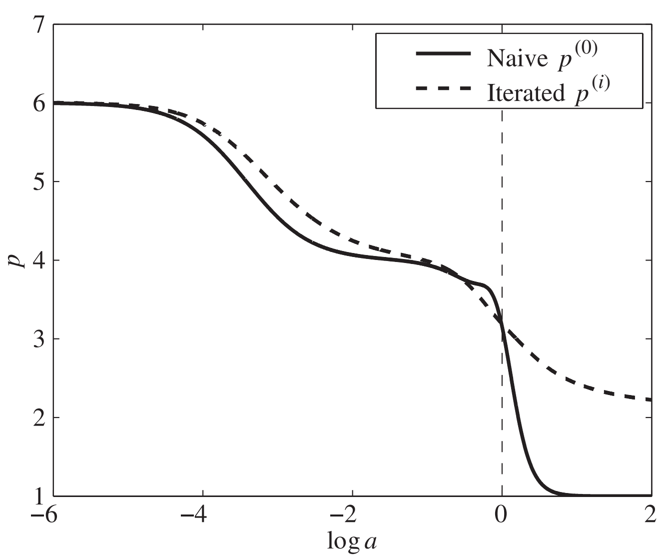

FIG. 1. Value of exponent p in the scaling solution $\Omega|_{y=0} \propto a^p$ for superhorizon modes, plotted as a function of the scale factor. The $p^{(0)}$ calculation assumes that the density perturbation Δ follows Eq. (48) at all times. The $p^{(i)}$ result is the output of an iterative process, where p is used to calculate R and hence the evolution of Ω which is in turn used to derive a correction to p.

and repeating the above prescription. We find in the next section that this procedure converges quickly and alters the value of p only when p is not a constant, as expected. We display the effects of the iteration on p in Fig. 1.

IV. COSMOLOGICAL IMPLICATIONS

A. Limiting cases and numerical solutions

The evolution of the master variable exhibits several distinct phases that are distinguished by the on-brane scaling evolution: that during radiation domination, matter domination, and the de Sitter acceleration phase, both inside and outside the horizon. The scaling of the Bianchi identity Eq. (31) determines the value of p at a particular scale factor, while the solution to the master equation Eq. (43) determines the off-brane gradient R. To understand better the nature of the solution and how it impacts perturbation evolution, we will compare the analytic expectations to the full numerical results in the various phases.

1. Superhorizon modes

During radiation domination, $Hr_c \gg 1$, $H'/H = -2 + a/2a_{\mathrm{eq}}$, and $F(H) = 1/3$. The particular combination in the denominator of $K(H)$ causes the first-order contribution of H'/H to cancel, leaving us with $K(H) \propto a^{-1}$. The Bianchi identity Eq. (31) for superhorizon modes then dictates that

$$A \propto a^{4-p}\Delta. \qquad (50)$$

Since $\Delta \propto a^2$ during radiation domination, this gives $p = 6$ for superhorizon modes. Inserted back into the master equation under the scaling ansatz Eq. (43), this value of p implies

$$R = -3, \qquad (k/aH \ll 1, \text{ radiation domination}). \qquad (51)$$

With R determined, the equations of motion for the perturbations on the brane are closed.

This analytic expectation also serves as the initial conditions for the numerical scaling solution. In practice, we begin the integration at $a = 10^{-6}$, when all modes of interest are outside the horizon. The numerical solution for p is shown in Fig. 1 and for R in Fig. 2. Note that, in the radiation-dominated era, their values stay stable at the analytic prediction for all iterations of the solution.

The large-scale modes of interest remain outside the horizon during the whole radiation-dominated epoch. In the matter-dominated epoch, the evolution outside the horizon can be obtained by noting $F(H) = 1/2$, $K(H) = 4$, $H'/H = -3/2$, and $Hr_c \gg 1$. The Bianchi identity dictates that

$$A \propto a^{3-p}\Delta. \qquad (52)$$

Given that $\Delta \propto a$, $p = 4$. Since the matter-dominated solution is of particular interest, we explicitly give the master equation under the scaling ansatz

$$\frac{d^2G}{dx^2} + \left(\frac{7-2p}{4(x-1)} - \frac{1+2p}{2(x+2)} - \frac{1}{x-4}\right)\frac{dG}{dx} + \left(\frac{p(2p-7)}{12(x-1)} + \frac{p(5-2p)}{12(x+2)} + \frac{p}{6(x-4)} + \frac{3p}{2(x+2)^2} + \left(\frac{k}{aH}\right)^2 \frac{(x-4)^2}{8(x-1)(x+2)^3}\right)G = 0. \quad (53)$$

This is solved as a boundary-value problem with boundary conditions $G(0) = 1$ and $G(1) = 0$. The form of the numerical solution to this equation is shown in Fig. 3. In the large-scale limit, the gradient reaches

$$R = -1, \qquad (k/aH \ll 1, \text{ matter domination}). \quad (54)$$

In the numerical solution of Figs. 1 and 2, these values of $p = 4$ and $R = -1$ are achieved gradually as the expansion becomes matter dominated. The iteration of the numerical solution in fact further smooths the transition until a stable form is achieved as would be expected. Note also that R is very insensitive to k/aH, provided it be less than 1 such that the mode is larger than the horizon.

2. Subhorizon modes

The modes of interest cross the horizon either during matter domination or the acceleration epoch. For large values of k/aH, the final term of the master equation (43) dominates over other parts of the coefficients of $G(x)$ and $G'(x)$. As evidenced in Fig. 3, in this regime, the solution does not penetrate very far into the bulk. We can thus expand the master equation around $x = 0$, reducing it to

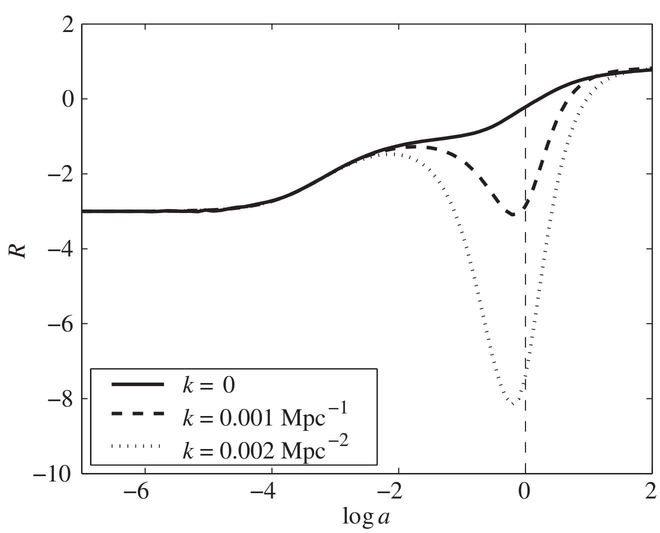

FIG. 2. Evolution of ratio of off-brane gradient to master variable, R, as defined in Eq. (32), for a selection of modes. On superhorizon scales, R is constant whenever p is a constant. Once the mode enters the horizon, it rapidly approaches $R = -k/aH$. As the Universe enters the de Sitter phase, the modes again leave the horizon and R asymptotes to 1.

$$\frac{d^2G}{dx^2} - \xi^2(1-2\xi x)\left(\frac{k}{aH}\right)^2 G = 0. \quad (55)$$

This matches the quasistatic approximation up to first order in x. Therefore, for $k/aH \gg 1$, the relation giving the gradient on the brane is exactly as in the QS approximation [11], with no dependence on the value of p:

$$R = -\frac{k}{aH} \qquad (k/aH \gg 1, \text{ matter/acceleration}). \quad (56)$$

The numerical solution for R evolved through horizon crossing is shown in Fig. 2. It reaches this scaling shortly after horizon crossing.

Despite this independence of p, the master variable does achieve a scaling form during matter domination. The Bianchi identity (31) can be reduced to the QS form, Eq. (36), and implies

$$A \propto a^{2-p}\Delta, \quad (57)$$

and hence $p = 3$.

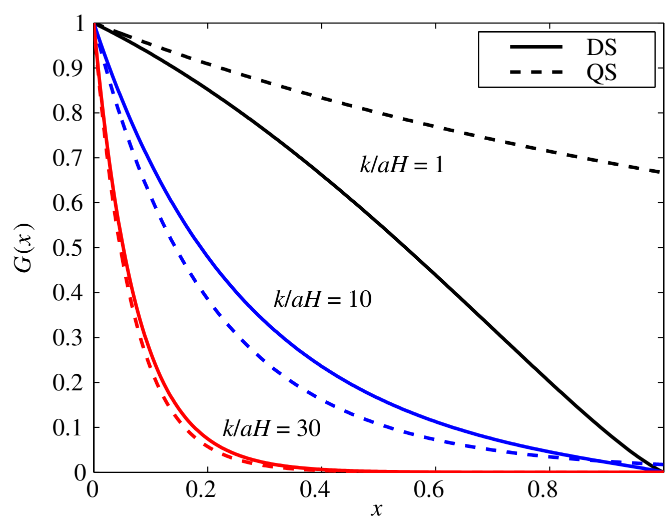

FIG. 3 (color online). Off-brane profile for $G(x)$ obtained by solving Eq. (43) during matter domination ($\log a = -2$), compared to off-brane profiles for the quasistatic (QS) solution. For high k/aH the profiles are very narrow and effectively independent of the position of the causal horizon: they penetrate very little into the bulk and the behavior of the solution is practically independent of the value of p. In this regime, the QS solution is practically coincident with the scaling solution. For modes with low k/aH, the solution is nonzero in the whole interval $x \in [0, 1)$ and therefore it depends strongly on the value of p. QS severely underestimates the gradient of the profile in this regime.

In general then, R is a function of both k/aH and $p(a)$ and therefore each mode needs to be followed separately through its evolution both outside and inside the horizon. However, only outside the horizon does the value of the off-brane gradient actually affect the evolution on the brane, since for high k/aH it is subdominant in the Bianchi identity, Eq. (31).

3. Asymptotic de Sitter phase

At late times, the DGP cosmology enters the self-accelerated de Sitter phase. During this time, all modes exit the horizon while the causal horizon in the bulk, ξ, grows rapidly toward infinity. This allows us to concentrate on the $k \ll aH$ limit. Deep into the de Sitter phase of the expansion, $Hr_c = 1$, $K(H) = F(H) = 1$. The master equation (27) can now be rewritten (in the scaling approximation) as

$$\frac{\partial^2\Omega}{\partial y^2} - \frac{2}{1+Hy}\frac{\partial\Omega}{\partial y} - \frac{p(p-3)}{(1+Hy)^2}\Omega = 0. \tag{58}$$

This equation has an analytic solution. Assuming that the perturbations vanish at the causal horizon, $\Omega = 0$ at $y = \xi/H$, and that $p \neq 3/2$, we can find the off-brane gradient:

$$R = \frac{1}{2}(3 - |2p-3|) - \frac{|2p-3|}{(1+\xi)^{|2p-3|} - 1}. \tag{59}$$

This can now be combined with the Bianchi identity

$$(p^2 - 3p + 1 + R)\Omega = \frac{6H_0^2 r_c^2 \Omega_m \Delta}{k^2}. \tag{60}$$

It can be shown that Δ becomes a constant during the acceleration era.

In the limit where $r_c^2\Delta/\Omega \to 0$ and $\xi \to \infty$, i.e. at very late times, the solution to the above two equations combined is $R = 1$ and $p = 2$ or $p = 1$, where we would expect the fastest-growing mode, $p = 2$, to dominate. We plot the numerical solutions in Figs. 1 and 2. Note that iteration of the solution here is crucial but $p = 2$, $R = 1$ is approached at late times.

The numerical results only converge to $p = 2$, $R = 1$ very slowly. This also can be understood analytically. There is a slight correction to this result arising as a result of ξ being finite, but it is already insignificant by $\log a \sim 2$. A more significant correction to the scaling behavior arises as a result of Δ not being zero. Its size can be estimated by assuming that $p = 2 + \epsilon$. Substituting this into both Eq. (59) and (60) we obtain

$$\epsilon = 1 - \frac{1}{2}\sqrt{4 + \frac{6H_0^2 r_c^2 \Omega_m \Delta}{\Omega k^2}}. \tag{61}$$

Indeed, as Ω grows, the correction decreases until $p = 2$ is achieved. In particular, the continued growth of Ω implies that the anisotropy at the largest scales will continue growing as

$$\Phi + \Psi \propto \frac{\pi_E}{a} \propto a^{1+\epsilon}. \tag{62}$$

4. Superhorizon metric evolution

Deriving an evolution equation for the metric highlights the fact that the metric evolution is not sensitive to the DGP modification until the acceleration epoch. Even in the acceleration epoch, the metric obeys a simple equation of motion on superhorizon scales.

By appropriately rearranging the linearized Einstein equations and the Bianchi identity, we can obtain a single differential equation for the gravitational potentials

$$\begin{aligned}\bigg(\Phi'' - \Psi' - \frac{H''}{H'}\Phi' + \left(\frac{H''}{H'} - \frac{H'}{H}\right)\Psi + \left(\frac{k}{aH}\right)^2\bigg[\left(c_s^2 + \frac{H'}{3H}\frac{1}{2Hr_c - 1}\right)\Phi + \frac{H'}{9(1+w)H}\frac{2Hr_c - 1}{Hr_c - 1}(\Phi+\Psi)\bigg]\bigg)(1+w) \\ = \frac{H'}{H}\frac{1}{2Hr_c}\left[(w - c_s^2)\Delta_E + \left(\frac{1}{3} - w + \frac{H'}{3H}\right)\delta_E\right].\end{aligned} \tag{63}$$

In the large-scale limit, $k/aH \to 0$, $\delta_E \approx 0$; if we also assume that $c_s^2 = w$ or that $\Delta_E = \mathcal{O}(k/aH)^2\Phi$ from the Poisson equation, we obtain

$$\Phi'' - \Psi' - \frac{H''}{H'}\Phi' + \left(\frac{H''}{H'} - \frac{H'}{H}\right)\Psi = 0. \tag{64}$$

Note that this equation for the metric perturbations is the same as that found in general relativity [see e.g. [25] Eq. (50)]. Therefore, before the DGP modifications change the expansion rate or generate anisotropic stress, the evolution of the metric is identical to general relativity. Furthermore, as pointed out by Bertschinger [26] in the absence of anisotropic stress, $\Psi = -\Phi$ and this equation yields solutions that depend only on the expansion history through H even in the acceleration epoch.

In DGP gravity, however, the anisotropy at the largest scales is never negligible and, in fact, grows at late times, as discussed in Sec. IVA 3. We can rewrite Eq. (64) by defining

$$\Phi_+ = \tfrac{1}{2}(\Phi + \Psi), \qquad \Phi_- = \tfrac{1}{2}(\Phi - \Psi). \tag{65}$$

In the de Sitter era, when $H'/H = 0$ and $H''/H' = -3$, and assuming that the two new variables obey a scaling solution with $\Phi_+ = A_+ a^{p_+}$ and $\Phi_- = A_- a^{p_-}$, Eq. (64) becomes

$$A_+ a^{p_+}(p_+^2 + 2p_+ - 3) + A_- a^{p_-}(p_-^2 + 4p_- + 3) = 0. \tag{66}$$

One would expect that Φ_+ would grow with exponent $p_+ = 1$, while Φ_- would decay away with exponent $p_- = -1$. However, because of the need to preserve the Bianchi identity with a nonzero Δ, as discussed in Sec. IV A 3, the scaling solution is slightly violated. We find that $p_+ = 1 + \epsilon$ while $p_- = 1$. This leads to the relation

$$\frac{A_-}{A_+} = -\frac{\epsilon a^{\epsilon}}{2}, \tag{67}$$

where $\epsilon \to 0$ monotonically. At late times, our solution tends to a regime where the ratio $\Phi_-/\Phi_+ \to 0$, with $\Phi_+ \propto a$.

It is interesting to note that in this opposite limit to that studied in [26] where $\Phi_- \ll \Phi_+$ Eq. (64) also becomes closed and has solutions that depend only on the expansion history through H.

In fact, Eq. (64) is equivalent to the statement that the Bardeen curvature, $\zeta \equiv \Phi + Hq$, is conserved during the de Sitter era, once the mode leaves the horizon. This is also true of the comoving density perturbation Δ, which saturates to a constant during the de Sitter era. However, the comoving and longitudinal hypersurfaces warp with a shift Hq that grows without bound.

B. Quasistatic vs dynamical-scaling solutions

It is useful to summarize the differences between the quasistatic (QS) and dynamical-scaling (DS) solutions uncovered in the previous section.

Beginning at the initial conditions in the radiation-dominated era, the superhorizon value of the master variable is highly suppressed with $\Omega_{\rm DS}/\Omega_{\rm QS} = \mathcal{O}((k/aH)^2)$ (see Fig. 4). As the mode enters the horizon during matter domination, the DS solution for Ω grows rapidly and then executes damped oscillations around the QS solution. This can be understood analytically since the Bianchi identity takes the form of a damped oscillator in Ω/a^2 that is driven by Δ. During the time when Ω significantly deviates from the QS solution, the Weyl corrections to the Poisson equation (16) are suppressed, since $Hr_c \gg 1$, and there is no additional correction to the gravitational potentials over and above that of QS (see Fig. 7).

We find that the results of the scaling solution match the quasistatic results for all modes that enter the horizon well within the matter-dominated epoch $k > 0.01\ \mathrm{Mpc}^{-1}$ (see Figs. 4–7). For larger scales, this is not so: as shown in Fig. 4, Ω now only decays toward the QS solution, rather than oscillating around it. Since at late times the Weyl perturbations are no longer suppressed, this now makes a significant contribution to the Poisson equation, resulting in additional decay of the potentials. As shown in Fig. 7, there is a 15% deviation from QS in $\Phi - \Psi$ at the scale $k = 0.001\ \mathrm{Mpc}^{-1}$ for the chosen sets of cosmological parameters. The direction of this effect agrees with estimates made by Lue in [27] and is such that the scaling solution is an even worse fit to CMB anisotropy data than the quasistatic (see [15] for a discussion).

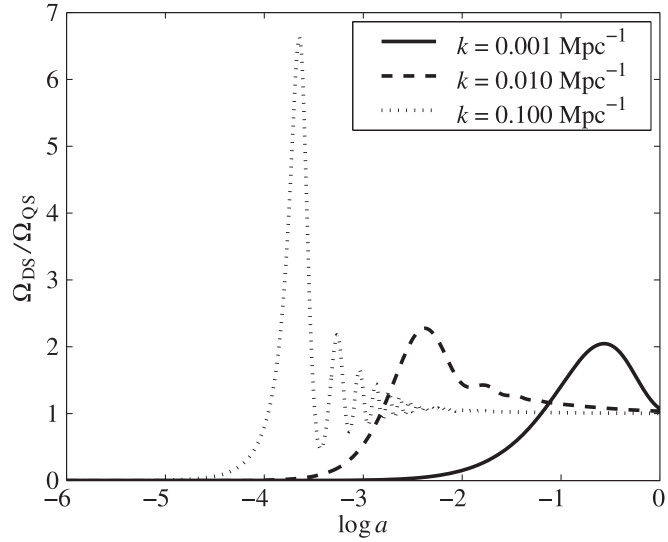

FIG. 4. Ratio of master variable Ω for dynamic scaling and quasistatic solutions. In QS, Ω responds instantaneously to changes in Δ. Fully dynamic solution to the Bianchi identity (31) requires time to respond and eventually decays to the QS solution. Initializing the calculation at earlier times changes neither the scale factor at which Ω responds nor its value today. Rapid growth occurs during the time before horizon crossing.

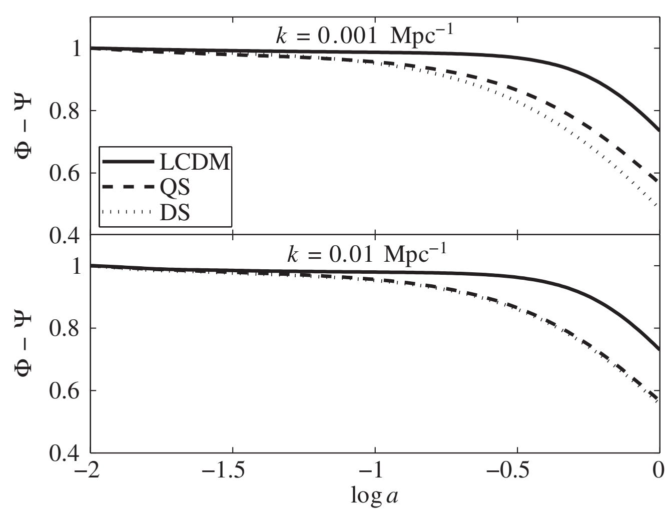

FIG. 5. Evolution of the principle gravitational observable $\Phi - \Psi$ for concordance ΛCDM, the quasistatic solution, and our new dynamical scaling solution. For scales $k \gtrsim 0.01\ \mathrm{Mpc}^{-1}$ the scaling and QS solutions do not differ appreciably: the decay in the potentials is a little faster than ΛCDM as a result of slower growth of density contrast. At larger scales, the DS solution exhibits significant additional decay owing to the different value of Ω at late times, as exhibited in Fig. 4. All potentials normalized to 1 at $\log a = -2$.

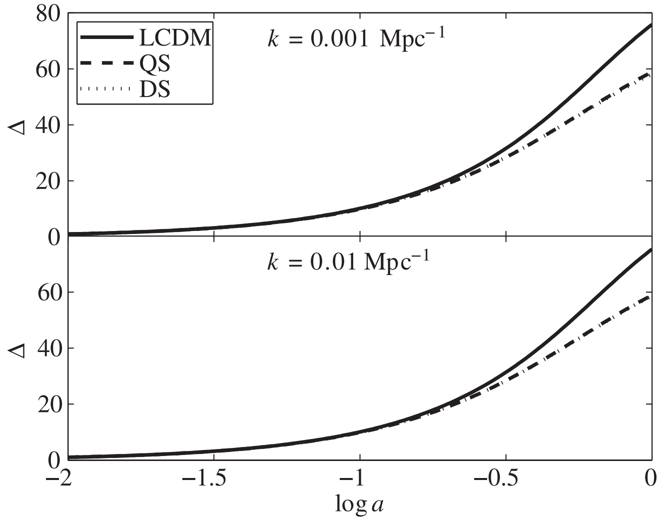

FIG. 6. Evolution of the comoving density contrast Δ for concordance ΛCDM, QS and our DS solution. The growth function in DGP is suppressed compared to ΛCDM, even in the case of a flat cosmology. There is no significant difference between the scaling ansatz and the quasistatic solution, with DS departing at most by 2% from QS at the largest scales. All quantities are normalized to 1 at $\log a = -2$.

On the other hand, up to the present time, the QS solution for comoving density perturbations Δ is a very good approximation for the DS solution at all scales. The additional suppression is of the order of 2% for $k = 0.001$ Mpc^{-1}.

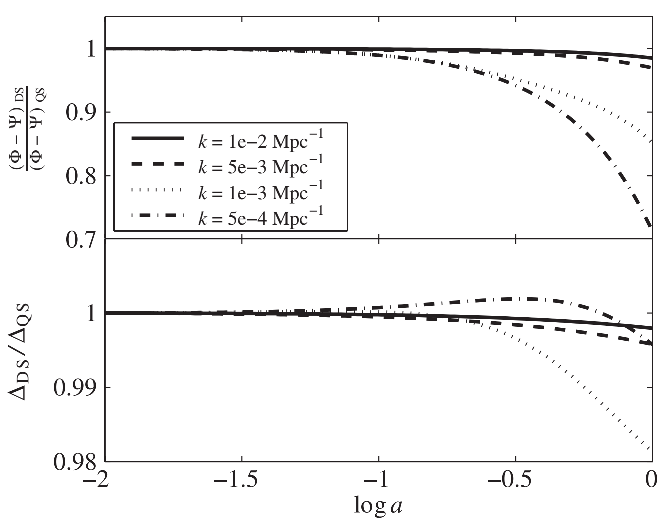

FIG. 7. Comparison of results of QS and DS solutions: Upper panel presents the ratio of $\Phi - \Psi$ in the two approximations. For modes with $k > 0.01$ Mpc^{-1} the two solutions differ by less than 2%. The difference is much more pronounced for larger scales where Ω has not decayed to the QS value, resulting in additional decays of up to 15%. Lower panel presents the ratio of comoving density perturbations in the two solutions: Δ is affected much less, with approximately a 2% deviation from QS at the largest scale, $k = 0.001$ Mpc^{-1}.

V. DISCUSSION

We have introduced a new scaling ansatz which allows solutions to linear perturbations in the DGP model on all scales less than the cross-over scale r_c up to the present epoch. The equations of motion for linear perturbations on the brane require knowledge of the gradient of the so-called master variable into the bulk. The master variable obeys a master equation in the bulk. To solve the master equation, it is sufficient to have two boundary conditions, one on the brane and the other in the bulk.

Our scaling solution begins with an ansatz for the brane boundary condition: that the evolution of the master variable is scale free on the brane. The second boundary condition is that the master variable vanishes at the causal horizon in the bulk. With these two boundary conditions, we solve the master equation to determine the gradient. With the gradient known, we can then replace the scale-free ansatz with the dynamical solution and iterate the solution until convergence.

We find that the quasistatic (QS) solution of [11] is rapidly approached once the perturbation crosses the horizon. Before horizon crossing there are strong deviations from the quasistatic solution. For modes that crossed the horizon only recently during the acceleration epoch, we find that the metric perturbation $\Phi - \Psi$ decays more rapidly that the QS solution. The QS solution itself has a stronger decay than the ΛCDM model. The extra decay compared with ΛCDM is extremely robust to changing the gradient of the master variable into the bulk, the one variable that is required to close the equations of motion on the brane. We consider the observational consequences of these results in a companion paper [15].

ACKNOWLEDGMENTS

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APPENDIX: SCALING ANSATZ ROBUSTNESS

Assuming that the scaling ansatz is appropriate, the solution presented in this paper is correct at all scales. However, the scale-free assumption for the solution in the bulk depends crucially on the existence of only one scale in the problem: the Hubble parameter. It is possible that through some additional physics in the vicinity of r_c this assumption is broken and the evolution of the bulk profile will depend on both H and r_c independently.

However, any violation of scaling will enter the equations solely through the off-brane gradient R: this is the quantity which we obtained by solving the master equation Eq. (43) using the scaling ansatz. It is important to note that, in our analysis, we have not dropped any terms either arising from the Weyl-fluid-driving Bianchi identity (31) or the master equation (27). We have assumed that the solution in the bulk depends only on yH/ξ but the form of the master equation implies that this scaling assumption should be a good one. We are only testing the robustness of our solution in order to estimate the effect of any new physics which might be important at scales around r_c and which is not embodied in the master equation already.

One way of testing the robustness of the scaling ansatz is to change the values of R at late times and investigate how much of a departure from scaling-ansatz values is necessary to significantly change the behavior of observables. We concentrate on the change to the evolution of the potential $\Phi - \Psi$, which drives the ISW effect, as we alter the off-brane gradient. Since the QS solution already has a significantly sharper decay than ΛCDM, and therefore is a worse fit to the large-angle CMB anisotropy [15], and the DS solution decays even more rapidly (see Fig. 5), we attempt to violate the scaling solution in such a way as to soften this decay. We present the modification to R in Fig. 8: we employ a linear interpolation for R between its scaling ansatz value at $\log a = -2$ and a chosen off-brane gradient value today, R_0. One should note that this breaking is rather extreme, since the scale under consideration, $k = 0.001\ \mathrm{Mpc}^{-1}$, is inside the horizon today and it should be well within the quasistatic regime at the present time.

We have found that choosing negative values for R_0 strengthens the decay of $\Phi - \Psi$, with the scaling solution approximately replicated for $R_0 \approx -2$. Positive values of R_0 reduce the rapidity of the decay at late times: the QS solution is matched for $R_0 \approx 5$, which is a value much higher than ever achieved by the scaling solution (see Fig. 9). In order to achieve the low levels of decay exhib-

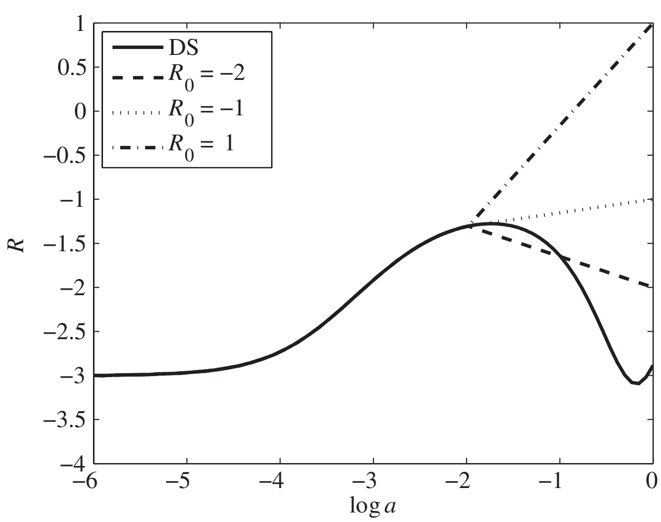

FIG. 8. Off-brane gradient for $k = 0.001\ \mathrm{Mpc}^{-1}$ for the scaling solution and the scaling-violating scenarios employed in robustness testing. Since any scaling violation is only likely to occur at scale factor close to $Hr_c \sim 1$, we modify the gradient starting at $\log a = -2$. A value of the gradient today, R_0, is chosen and the gradient is interpolated linearly between these two scale factors. In this modification, we disregard the fact that the mode enters the horizon at late times.

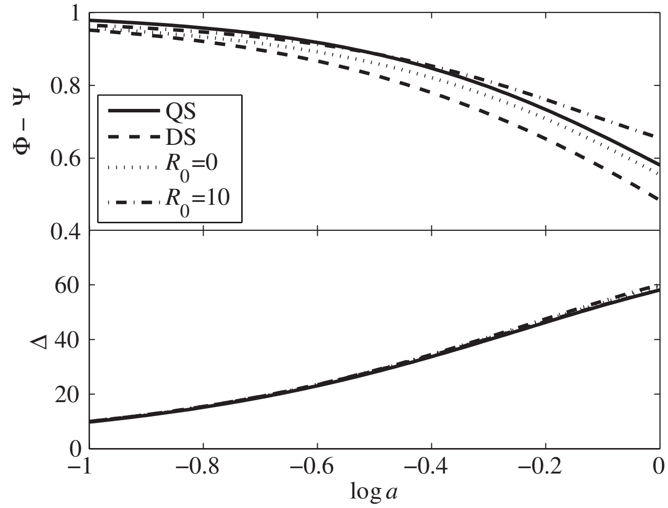

FIG. 9. Evolution of $\Phi - \Psi$ and Δ for mode $k = 0.001\ \mathrm{Mpc}^{-1}$ for a selection of scaling-violating scenarios. Increasing R_0 brings the solution closer to that of QS, and, for very large values, reduces the decay of $\Phi - \Psi$ to that of ΛCDM. The effect of changing R_0 on Δ is much smaller, the quantity remains insensitive to the precise details of the scenario.

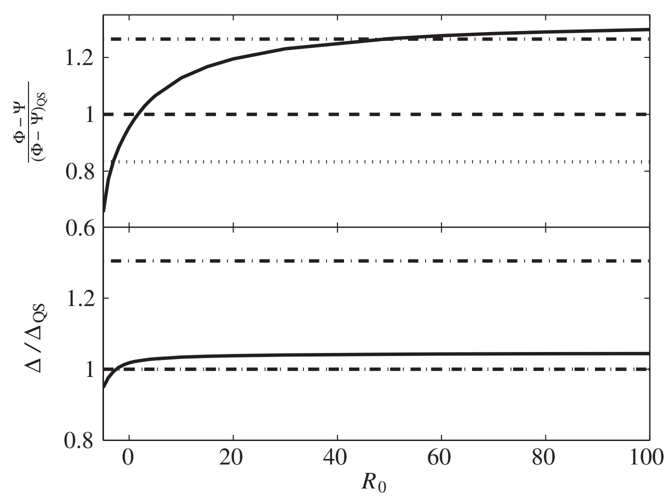

FIG. 10. Ratio of $\Phi - \Psi$ and Δ for scaling and scaling-violating scenarios to their values in QS for mode $k = 0.001\ \mathrm{Mpc}^{-1}$. Dashed line represents the final value of the quantity in QS, dotted line in DS, and dotted-dashed line in ΛCDM. The solution is very sensitive to negative values of R_0, but choosing such scenarios only increases the decay, strengthening the ISW effect. Positive values of R_0 bring the evolution of the observables closer to that of the QS solution and, for very large values, achieve decays as low as those of ΛCDM. Δ is quite insensitive to the choice of R. All quantities were normalized to the same value at $\log a = -2$.

ited by ΛCDM, R_0 needs to be set in the vicinity of 50 (see Fig. 10).

The effect is different for Δ: here, negative values of R_0 also decrease the growth rate even further beneath that of DS; however, Δ asymptotes to a value approximately 2% beneath that of the QS solution as R_0 is sent to infinity.

The above considerations show that the new physics required around r_c would have to violate the scaling behavior rather strongly in order to give an ISW effect comparable to that of ΛCDM.

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