

The Angular Trispectra of CMB Temperature and Polarization

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We develop the formalism necessary to study four-point functions of the cosmic microwave background (CMB) temperature and polarization fields. We determine the general form of CMB trispectra, with the constraints imposed by the assumption of statistical isotropy of the CMB fields, and derive expressions for their estimators, as well as their Gaussian noise properties. We apply these techniques to initial non-Gaussianity of a form motivated by inflationary models. Due to the large number of four-point configurations, the sensitivity of the trispectra to initial non-Gaussianity approaches that of the temperature bispectrum at high multipole moment. These trispectra techniques will also be useful in the study of secondary anisotropies induced for example by the gravitational lensing of the CMB by the large scale structure of the universe.

I. INTRODUCTION

Beyond the power spectra of the cosmic microwave background (CMB) lies the relatively unexplored territory of non-Gaussian statistics. Studies of its non-Gaussianity hold the potential to reveal physics at the two ends of time. Non-Gaussianity in the primary anisotropies from the recombination epoch can test the inflationary model of the origin of fluctuations (e.g. [1, 2, 3, 4, 5]). Non-Gaussianity in the secondary anisotropies, arising during the transit of a CMB photon through the large-scale structure of the universe, probes nature of the dark energy and dark matter (e.g. [6, 7, 8, 9, 10]).

The primary challenge facing non-Gaussian studies of the CMB is the selection of an appropriate statistic. The term “non-Gaussianity” tells us what the distribution is not, not what it is. Like the power spectra, the higher-point correlations of the multipole moments of CMB fields provide a set of statistics with definitive predictions in the cases of cosmological interest. Unlike the power spectra, there are a large number of potential observables, associated with the distinct configurations of the points, requiring the development of new techniques for their prediction and estimation. In particular, it is important to identify the symmetry properties of the spectra to build optimal statistics for the detection of non-Gaussianity.

Non-Gaussian signatures in the three-point function or bispectrum of the temperature distribution [2, 8, 11, 12, 13] and polarization [14] as well as techniques for their extraction [15, 16, 17, 18, 19, 20] have been studied extensively in the past few years. The four-point function or trispectrum has recently received much attention due to its use in the study of the gravitational lensing of the CMB [6, 9, 21, 22]. Techniques for measuring certain components have been tested on the COBE data [23, 24]. Still, a complete treatment incorporating the full symmetry properties of the temperature and polarization fields has been lacking in the literature.

In this paper, we complete the formalism established for the temperature trispectrum [21]. The addition of polarization information leads to a multiplicity of trispectra corresponding to all possible combinations of three observable fields. It has been recently shown that the higher point correlations of the CMB polarization contain the majority of the information on gravitational lensing in the CMB [25]. Trispectra also quantify the non-Gaussian errors to temperature and polarization power spectra measurements.

The outline of the paper is as follows. We consider the general symmetry and noise properties of trispectra in Sec. II. As an illustration of the construction and noise properties of trispectra, we apply these techniques to initial non-Gaussianity in the curvature fluctuations of the form predicted by slow-roll inflation in Sec. III. We show that the sensitivity to initial non-Gaussianity in the trispectra can approach that in the temperature bispectrum [13]. In Appendix A, we summarize relations useful for the study of high order correlations in the polarization. In Appendix

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B, we cover details in the properties, measurement and approximation of the trispectra that may be useful for future studies.

II. FORMALISM

We begin with definitions associated with the harmonic description of the temperature and polarization fields in Sec. II A. We consider the general symmetry properties of n -point harmonic functions in Sec. II B and apply them to the trispectra in Sec. II C. Finally we derive the Gaussian noise properties of trispectra estimators are considered in Sec. II D.

A. Definitions

The temperature field $\Theta(\hat{\mathbf{n}}) \equiv \Delta T(\hat{\mathbf{n}})/T$ is decomposed into multipole moments according to

$$\Theta(\hat{\mathbf{n}}) = \sum_{lm} \Theta_l^m Y_l^m(\hat{\mathbf{n}}). \quad (1)$$

The polarization, described by the Stokes parameters $Q(\hat{\mathbf{n}})$ and $U(\hat{\mathbf{n}})$ in spherical polar coordinates, is a spin-2 field and is similarly decomposed as [26, 27]

$$(Q \pm iU)(\hat{\mathbf{n}}) = \sum_{lm} \pm A_l^m \pm 2 Y_l^m(\hat{\mathbf{n}}), \quad (2)$$

where ${}_s Y_l^m(\hat{\mathbf{n}})$ is the spin-weighted spherical harmonics whose properties are reviewed in Appendix A. Note that ${}_0 Y_l^m = Y_l^m$.

Under a parity transformation \hat{P} taking $\hat{\mathbf{n}}$ to $-\hat{\mathbf{n}}$, the spin-spherical harmonics transform as

$$\hat{P}[{}_s Y_l^m(\hat{\mathbf{n}})] = (-1)^l {}_{-s} Y_l^m(\hat{\mathbf{n}}). \quad (3)$$

and so it is convenient to define the parity eigenfunctions

$${}_s \mathcal{E}_l^m \equiv \frac{{}_s Y_l^m + {}_{-s} Y_l^m}{2}, \quad (4)$$

$${}_s \mathcal{O}_l^m \equiv \frac{{}_s Y_l^m - {}_{-s} Y_l^m}{2i}, \quad (5)$$

we see that under parity, $\hat{P}[{}_s \mathcal{E}_l^m] = (-1)^l {}_s \mathcal{E}_l^m$, whereas $\hat{P}[{}_s \mathcal{O}_l^m] = (-1)^{l+1} {}_s \mathcal{O}_l^m$. A spin zero field such as the temperature fluctuation carries only the ${}_0 \mathcal{E}_l^m = Y_l^m$ mode. The the spin-2 polarization field has two components that are distinguished by parity [26, 27]

$$\pm A_l^m = E_l^m \pm i B_l^m \quad (6)$$

called the E and the B modes. This definition differs from that in [26] by an overall minus sign, so in particular, the sign of temperature-polarization cross-correlations is reversed.

The fields $E(\hat{\mathbf{n}})$ and $B(\hat{\mathbf{n}})$ on the sky are defined as

$$E(\hat{\mathbf{n}}) = \sum_{lm} E_l^m Y_l^m(\hat{\mathbf{n}}), \quad (7a)$$

$$B(\hat{\mathbf{n}}) = \sum_{lm} B_l^m Y_l^m(\hat{\mathbf{n}}). \quad (7b)$$

The parity properties of the eigenstates ${}_s \mathcal{E}_l^m$ and ${}_s \mathcal{O}_l^m$ imply that $E(\hat{\mathbf{n}})$ will be a scalar under parity, whereas $B(\hat{\mathbf{n}})$ will be a pseudoscalar.

Lastly, requiring that the fields $\Theta(\hat{\mathbf{n}})$, $Q(\hat{\mathbf{n}})$ and $U(\hat{\mathbf{n}})$ be real imposes the constraints

$$x_l^{m*} = (-1)^m x_l^{-m}, x \in \{\Theta, E, B\} \quad (8)$$

on the multipole moments. This constraint also enforces the reality of $E(\hat{\mathbf{n}})$ and $B(\hat{\mathbf{n}})$.

B. Rotational and Parity Invariance

The harmonic transform of the n -point correlation of CMB fields $x, \dots, z \in \{\Theta, E, B\}$ defines the n -point harmonic correlation functions according to

$$\langle x(\hat{\mathbf{n}}_1) \dots z(\hat{\mathbf{n}}_n) \rangle = \sum_{l_i, m_i} \langle x_{l_1}^{m_1} \dots z_{l_n}^{m_n} \rangle Y_{l_1}^{m_1}(\hat{\mathbf{n}}_1) \dots Y_{l_n}^{m_n}(\hat{\mathbf{n}}_n). \quad (9)$$

Since the CMB fields are assumed to be statistically isotropic, the n -point correlations must be invariant under rotations. A general rotation $\hat{\mathbf{R}}$ acts on spherical harmonics as

$$\hat{\mathbf{R}}[Y_l^m(\hat{\mathbf{n}})] = \sum_{m'} D_{mm'}^l(\hat{\mathbf{R}}) Y_l^{m'}(\hat{\mathbf{n}}). \quad (10)$$

Useful properties of the rotation matrix $D_{mm'}^l$ are summarized in Appendix A. The n -point harmonic correlation must therefore obey

$$\langle x_{l_1}^{m_1} \dots z_{l_n}^{m_n} \rangle = \sum_{m'_1 \dots m'_n} \langle x_{l_1}^{m'_1} \dots z_{l_n}^{m'_n} \rangle D_{m_1 m'_1}^{l_1}(\hat{\mathbf{R}}) \dots D_{m_n m'_n}^{l_n}(\hat{\mathbf{R}}). \quad (11)$$

This invariance demands a specific form for the m -dependence as we shall see.

Under a parity transformation an n -point function containing k B -fields will transform according to

$$\hat{P}[\langle x_{l_1}^{m_1} \dots z_{l_n}^{m_n} \rangle] = (-1)^{k+\sum l_i} \langle x_{l_1}^{m_1} \dots z_{l_n}^{m_n} \rangle. \quad (12)$$

Invariance under a parity transformation therefore implies that the n -point function containing k B fields will vanish when

$$k + \sum_i l_i = \text{odd}. \quad (13)$$

C. Trispectra

The reduction of the four-point harmonic function into a rotationally invariant form follows the steps outlined in [21]. Using the Clebsch-Gordan property (A7) on equation (11) to pair (l_1, l_2) and (l_3, l_4) together and applying the orthogonality condition (A3) to the resultant pair, we reduce the function to

$$\langle w_{l_1}^{m_1} x_{l_2}^{m_2} y_{l_3}^{m_3} z_{l_4}^{m_4} \rangle \equiv \sum_{LM} (-1)^M \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} l_3 & l_4 & L \\ m_3 & m_4 & M \end{pmatrix} Q_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L). \quad (14)$$

The trispectrum $Q_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L)$ represents a configuration with sides $l_1 \dots l_4$ labeled by the fields $w \dots z$, with one diagonal of length L forming a triangle with l_1 and l_2 (figure 1).

Choosing the two other pairings (l_1, l_3) and (l_1, l_4) yield alternate representations of the trispectra. Since each representation is constructed by adding pairs of angular momenta, each representation is complete, and three representations are related through Wigner 6- j symbols via

$$Q_{x_{l_2} z_{l_4}}^{w_{l_1} y_{l_3}}(L) = \sum_{L'} (-1)^{l_2+l_3} (2L+1) \begin{Bmatrix} l_1 & l_2 & L' \\ l_4 & l_3 & L \end{Bmatrix} Q_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L') \quad (15a)$$

and

$$Q_{y_{l_3} x_{l_2}}^{w_{l_1} z_{l_4}}(L) = \sum_{L'} (-1)^{L+L'} (2L+1) \begin{Bmatrix} l_1 & l_2 & L' \\ l_3 & l_4 & L \end{Bmatrix} Q_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L'). \quad (15b)$$

The trispectrum is obtained by subtracting the unconnected or Gaussian piece from $Q_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L)$, so

$$T_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L) = Q_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L) - G_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L), \quad (16)$$

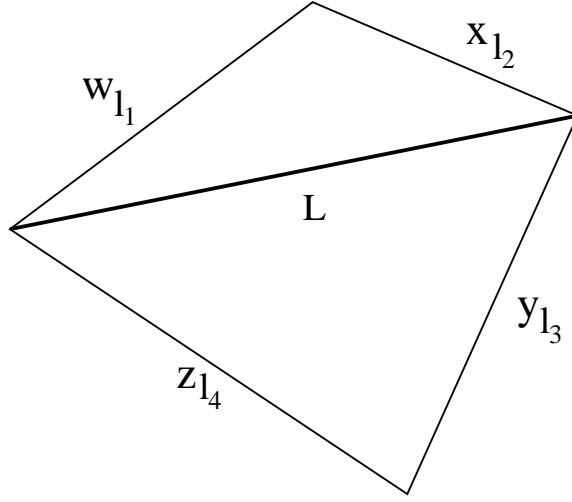


FIG. 1: Geometrical interpretation of the configuration of a trispectrum. The four-point quadrilateral in harmonic space is specified using the pairs (l_1, l_2) along with the diagonal L to define a triangle.

where the Gaussian piece is constructed from the power spectra

$$C_l^{xy} = \langle x_l^{m*} y_l^m \rangle \quad (17)$$

as

$$\begin{aligned} G_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L) &= (-1)^{l_1+l_3} \sqrt{(2l_1+1)(2l_3+1)} C_{l_1}^{wx} C_{l_3}^{yz} \delta_{l_1 l_2} \delta_{l_3 l_4} \delta_{L0} \\ &+ (2L+1) [(-1)^{l_1+l_2+L} C_{l_1}^{wy} C_{l_2}^{xz} \delta_{l_1 l_3} \delta_{l_2 l_4} + C_{l_1}^{wz} C_{l_2}^{xy} \delta_{l_1 l_4} \delta_{l_2 l_3}]. \end{aligned} \quad (18)$$

From the permutation symmetry of the trispectrum (14), additional constraints hold

$$T_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L) = (-1)^{\Sigma_U} T_{y_{l_3} z_{l_4}}^{x_{l_2} w_{l_1}}(L) = (-1)^{\Sigma_L} T_{z_{l_4} y_{l_3}}^{w_{l_1} x_{l_2}}(L) = (-1)^{\Sigma_U + \Sigma_L} T_{z_{l_4} y_{l_3}}^{x_{l_2} w_{l_1}}(L), \quad (19a)$$

where

$$\Sigma_U = L + l_1 + l_2, \quad (19b)$$

$$\Sigma_L = L + l_3 + l_4. \quad (19c)$$

The constraints (15) and (19) express redundancies in the definition of the trispectrum, where a physical configuration can be labeled by $4! = 24$ different permutations of the field labels and pairings.

In practice, the following construction guarantees that trispectra obey the constraints outlined above. Given that the connected part of the four-point function can be expanded into its three pairings as

$$\begin{aligned} \langle w_{l_1}^{m_1} x_{l_2}^{m_2} y_{l_3}^{m_3} z_{l_4}^{m_4} \rangle_c &= \sum_{LM} (-1)^M \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} l_3 & l_4 & L \\ m_3 & m_4 & M \end{pmatrix} P_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L) \\ &+ (x_{l_2} \leftrightarrow y_{l_3}) + (x_{l_2} \leftrightarrow y_{l_4}), \end{aligned} \quad (20)$$

the latter two pairings are projected onto the (l_1, l_2) basis through the recoupling relations (15) to give

$$\begin{aligned} T_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L) &= P_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L) + (2L+1) \sum_{L'} [(-1)^{l_2+l_3} \begin{Bmatrix} l_1 & l_2 & L \\ l_4 & l_3 & L' \end{Bmatrix} P_{x_{l_2} z_{l_4}}^{w_{l_1} y_{l_3}}(L') \\ &+ (-1)^{L+L'} \begin{Bmatrix} l_1 & l_2 & L \\ l_3 & l_4 & L' \end{Bmatrix} P_{y_{l_3} x_{l_2}}^{w_{l_1} z_{l_4}}(L')]. \end{aligned} \quad (21)$$

Having satisfied the recoupling relations, the remaining constraints (19) are enforced by introducing a reduced function $\mathcal{T}_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L)$, where

$$P_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L) = \mathcal{T}_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L) + (-1)^{\Sigma_U} \mathcal{T}_{y_{l_3} z_{l_4}}^{x_{l_2} w_{l_1}}(L) + (-1)^{\Sigma_L} \mathcal{T}_{z_{l_4} y_{l_3}}^{w_{l_1} x_{l_2}}(L) + (-1)^{\Sigma_U + \Sigma_L} \mathcal{T}_{z_{l_4} y_{l_3}}^{x_{l_2} w_{l_1}}(L), \quad (22)$$

with the additional constraint

$$\mathcal{T}_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L) = \mathcal{T}_{w_{l_1} x_{l_2}}^{y_{l_3} z_{l_4}}(L). \quad (23)$$

These exhaust the 4! redundancies due to permutations of the fields. Given the functional form of \mathcal{T} , the trispectrum can be constructed by permuting the fields along with their indices, as indicated above. Therefore \mathcal{T} provides the most economical description of the trispectra for a given physical effect.

It is possible to think of the trispectra configurations as being labeled by a fixed field configuration (e.g. $T_{x_{l_3} y_{l_4}}^{x_{l_1} y_{l_2}}(L)$) with an x and y always related through the diagonal L , with the indices l_i allowed to vary. The derivation of symmetry properties in such a viewpoint is straightforward, and is deferred to Appendix B 1. The above construction automatically enforces the symmetries with fixed field configurations, (B2) through (B5).

D. Noise Properties

By inverting definition (14), we obtain an estimator for a trispectrum

$$\hat{T}_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L) = \sum_{m_i, M} (2L+1)(-1)^M \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & M \end{pmatrix} \begin{pmatrix} l_3 & l_4 & L \\ m_3 & m_4 & -M \end{pmatrix} \langle w_{l_1}^{m_1} x_{l_2}^{m_2} y_{l_3}^{m_3} z_{l_4}^{m_4} \rangle - \hat{G}_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L), \quad (24)$$

where the Gaussian estimator \hat{G} is constructed using expression (18) with the power spectra replaced by their estimators. We discuss more practical forms of this estimator in Appendix B 2.

The covariance between two trispectrum estimators due to Gaussian noise then becomes

$$\frac{\langle \hat{T}_{c_{l_3} d_{l_4}}^{a_{l_1} b_{l_2}}(L) \hat{T}_{y'_{l_3} z'_{l_4}}^{w'_{l_1} x'_{l_2}}(L') \rangle}{2L+1} = \delta_{LL'} N_{34}^{12} + (2L'+1) \left[(-1)^{l_2+l_3} \begin{Bmatrix} l_1 & l_2 & L \\ l_4 & l_3 & L' \end{Bmatrix} N_{24}^{13} + (-1)^{L+L'} \begin{Bmatrix} l_1 & l_2 & L \\ l_3 & l_4 & L' \end{Bmatrix} N_{32}^{14} \right], \quad (25)$$

where no two l 's are in the primed and unprimed sets are equal, and

$$N_{kl}^{ij} = \left[\delta_{l_1 l'_i} \delta_{l_2 l'_j} C_{l_1}^{aw} C_{l_2}^{bx} + (-1)^{L'+l_1+l_2} \delta_{l_1 l'_j} \delta_{l_2 l'_i} C_{l_1}^{ax} C_{l_2}^{bw} \right] \left[\delta_{l_3 l'_k} \delta_{l_4 l'_l} C_{l_3}^{cy} C_{l_4}^{dz} + (-1)^{L'+l_3+l_4} \delta_{l_3 l'_l} \delta_{l_4 l'_k} C_{l_3}^{cz} C_{l_4}^{dy} \right] + [i \leftrightarrow k, w \leftrightarrow y][j \leftrightarrow l, x \leftrightarrow z]. \quad (26)$$

If any two l_i 's are equal, one would need to consider additional terms in the covariance arising from pairings within the primed and unprimed sets.

Using the above covariance, the total signal-to-noise ratio is then be given by

$$\left(\frac{S}{N} \right)^2 = \sum_{l_i, l'_i} \sum_{L, L'} \sum_{abcd} \sum_{wxyz} \langle \hat{T}_{c_{l_3} d_{l_4}}^{a_{l_1} b_{l_2}}(L) \rangle [\text{Cov}]^{-1} \langle \hat{T}_{y'_{l_3} z'_{l_4}}^{w'_{l_1} x'_{l_2}}(L') \rangle. \quad (27)$$

“Cov” here is the covariance in (25) viewed as a matrix and the field-type sums are over the measured fields.

This matrix will possess singular values for permutations that are equivalent. They can be eliminated by singular value decomposition or equivalently by restricting the sums to a set of inequivalent permutations. The latter is computationally more efficient and the redundancies expressed in (15) and (19) can be removed by restricting the l sums. Thus the total signal-to-noise simplifies to

$$\left(\frac{S}{N} \right)^2 = \sum_{l_1 > l_2 > l_3 > l_4} \sum_L \sum_{abcd} \sum_{wxyz} \langle \hat{T}_{c_{l_3} d_{l_4}}^{a_{l_1} b_{l_2}}(L) \rangle [\text{Cov}]^{-1} \langle \hat{T}_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L) \rangle, \quad (28)$$

where the covariance between $\hat{T}_{c_{l_3} d_{l_4}}^{a_{l_1} b_{l_2}}(L)$ and $\hat{T}_{y_{l_3} z_{l_4}}^{w_{l_1} x_{l_2}}(L)$ likewise simplifies to

$$[\text{Cov}] = (2L+1) C_{l_1}^{aw} C_{l_2}^{bx} C_{l_3}^{cy} C_{l_3}^{dz}, \quad (29)$$

so that the matrix inverse is now only over field choices for a fixed set of multipoles.

III. TRISPECTRA FROM INITIAL CONDITIONS

As an application of the formalism for describing temperature and polarization trispectra and computing the signal-to-noise ratio of their estimators, we consider here signature of non-Gaussianity that is inherent in the initial conditions. In Sec. III A-III B, we motivate a form for the trispectra based on slow-roll inflation. Although typical models predict amplitudes far below that which is potentially observable, this form is generic to local non-Gaussianity in the initial conditions. In III C, we describe the transfer of this initial non-Gaussianity to the observable temperature and polarization fields. We show in III D that the total signal-to-noise in the trispectra is comparable to that in the temperature bispectrum considered previously in the literature.

A. Inflationary Motivation

The standard inflationary paradigm is known to predict a very nearly Gaussian spectrum of initial curvature fluctuations which under linear gravitational instability theory implies a Gaussian spectrum of the CMB fluctuations. However, non-linearity corrections in inflation and gravity can produce non-Gaussian fluctuations, which may be observable in the microwave background. The imprint of such non-linearity on the bispectrum of the microwave background has been studied extensively. The theoretical predictions for the bispectrum and the related statistic of skewness has been described in [2, 3, 13, 19], and observational limits placed using existing data in [11, 18, 28]. We here extend these considerations to higher order to investigate effects on the trispectra.

Following [1], let us consider the non-Gaussianity induced in corrections to the correspondence between a Gaussian inflaton fluctuation and the Newtonian curvature Φ . Φ during matter domination can be related to the inflaton fluctuation at horizon exit according to

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{12\pi G}{5} \int_{\phi_0}^{\phi_0+\delta\phi} \left[\frac{\partial \ln H}{\partial \phi} \right]^{-1} d\phi \\ &\approx \frac{12\pi G}{5} \left[\frac{\partial \ln H}{\partial \phi} \right]^{-1} \delta\phi + \frac{6\pi G}{5} \frac{\partial}{\partial \phi} \left[\frac{\partial \ln H}{\partial \phi} \right]^{-1} \delta\phi^2 + \frac{2\pi G}{5} \frac{\partial^2}{\partial \phi^2} \left[\frac{\partial \ln H}{\partial \phi} \right]^{-1} \delta\phi^3 + \mathcal{O}(\delta\phi^4).\end{aligned}\quad (30)$$

The leading order term, which we will denote $\Phi_L(\mathbf{x})$, carries Gaussian random fluctuations from $\delta\phi$. The higher order terms can then be written as

$$\Phi(\mathbf{x}) = \Phi_L(\mathbf{x}) + f_1 (\Phi_L^2(\mathbf{x}) - \langle \Phi_L^2(\mathbf{x}) \rangle) + f_2 \Phi_L^3(\mathbf{x}) + \mathcal{O}(\Phi_L^4).\quad (31)$$

with

$$\begin{aligned}f_1 &= -\frac{5}{6} \frac{1}{8\pi G} \frac{\partial^2 \ln V}{\partial \phi^2}, \\ f_2 &= \frac{25}{54} \frac{1}{(8\pi G)^2} \left[2 \left(\frac{\partial^2 \ln V}{\partial \phi^2} \right)^2 - \frac{\partial^3 \ln V}{\partial \phi^3} \frac{\partial \ln V}{\partial \phi} \right],\end{aligned}\quad (32)$$

where $V(\phi)$ is the inflaton potential. This model generalizes the considerations of [4, 11, 13, 29, 30] to higher order. Note that our f_1 corresponds to f_{NL} in [13]. As an example consider an inflaton potential of the form $V = \lambda\phi^n$, then

$$f_1 = \frac{5n}{6} \left(\frac{M_p}{\phi} \right)^2, \quad (33)$$

$$f_2 = 0, \quad (34)$$

with $M_p^2 \equiv 1/8\pi G$ defining the reduced Planck mass. For inflation to occur, this class of models require that $\phi \approx \sqrt{120n}M_p$, so that we obtain $f_1 \sim 0.01$ as an order-of-magnitude estimate.

Although these small coupling coefficients and the observed 10^{-5} level of curvature perturbations make non-Gaussian contributions from typical inflationary models unmeasurable, the general form in Eqn. (31) is simply a perturbative expansion of the curvature fluctuations and so we leave f_1 and f_2 as free parameters and explore the extent to which they are measurable in temperature and polarization trispectra. Other sources of non-Gaussianity of this form include interaction terms in the inflaton potential [4], stochastic interactions of the long-wavelength inflaton fluctuations with the short-wavelength modes [3, 31], and some multiple field models [32].

The ansatz (31) for the curvature fluctuations imply higher order correlations in Fourier space since products become convolutions. We can decompose the contributions into linear and nonlinear parts, so that

$$\Phi(\mathbf{k}) \equiv \Phi_L(\mathbf{k}) + \Phi_A(\mathbf{k}) + \Phi_B(\mathbf{k}), \quad (35)$$

with

$$\Phi_A(\mathbf{k}) = f_1 \left[\int \frac{d^3 \mathbf{p}}{(2\pi)^3} \Phi_L(\mathbf{k} + \mathbf{p}) \Phi_L^*(\mathbf{p}) - (2\pi)^3 \delta(\mathbf{k}) \langle \Phi_L^2(\mathbf{x}) \rangle \right], \quad (36)$$

$$\Phi_B(\mathbf{k}) = f_2 \int \frac{d^3 \mathbf{p}_1}{(2\pi)^3} \frac{d^3 \mathbf{p}_2}{(2\pi)^3} \Phi_L^*(\mathbf{p}_1) \Phi_L^*(\mathbf{p}_2) \Phi_L(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{k}), \quad (37)$$

where

$$\langle \Phi_L^2(\mathbf{x}) \rangle = \int \frac{d^3 k}{(2\pi)^3} P_\Phi(\mathbf{k}), \quad (38)$$

Here $P_\Phi(\mathbf{k})$ is the power spectrum of $\Phi_L(\mathbf{k})$.

B. Curvature Trispectrum

The connected part of the four-point function $\langle \Phi(\mathbf{k}_1) \Phi(\mathbf{k}_2) \Phi(\mathbf{k}_3) \Phi(\mathbf{k}_4) \rangle$ has leading order contributions from terms of the form $\langle \Phi_A(\mathbf{k}_1) \Phi_A(\mathbf{k}_2) \Phi_L(\mathbf{k}_3) \Phi_L(\mathbf{k}_4) \rangle$ and $\langle \Phi_B(\mathbf{k}_1) \Phi_L(\mathbf{k}_2) \Phi_L(\mathbf{k}_3) \Phi_L(\mathbf{k}_4) \rangle$. To compute the curvature trispectrum, we note that the symmetries with respect to exchange of $\Phi(\mathbf{k}_i)$ and $\Phi(\mathbf{k}_j)$ in the Φ four-point function are identical to the symmetries with respect to exchange of fields in the CMB trispectrum. We therefore follow the same decomposition process by first defining

$$\langle \Phi(\mathbf{k}_1) \Phi(\mathbf{k}_2) \Phi(\mathbf{k}_3) \Phi(\mathbf{k}_4) \rangle_c = (2\pi)^3 \int d^3 K \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{K}) \delta(\mathbf{k}_3 + \mathbf{k}_4 - \mathbf{K}) T_\Phi(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4; \mathbf{K}), \quad (39)$$

and then constructing

$$\begin{aligned} T_\Phi(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4; \mathbf{K}) &= P_\Phi(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4; \mathbf{K}) + \int d^3 \mathbf{K}' [\delta(\mathbf{k}_3 - \mathbf{k}_2 - \mathbf{K} + \mathbf{K}') P_\Phi(\mathbf{k}_1, \mathbf{k}_3; \mathbf{k}_2, \mathbf{k}_4; \mathbf{K}') \\ &+ \delta(\mathbf{k}_4 - \mathbf{k}_2 - \mathbf{K} + \mathbf{K}') P_\Phi(\mathbf{k}_1, \mathbf{k}_4; \mathbf{k}_3, \mathbf{k}_2; \mathbf{K}')], \end{aligned} \quad (40a)$$

with P_Φ constructed out of a reduced trispectrum \mathcal{T}_Φ according to

$$\begin{aligned} P_\Phi(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4; \mathbf{K}) &= \mathcal{T}_\Phi(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4; \mathbf{K}) + \mathcal{T}_\Phi(\mathbf{k}_2, \mathbf{k}_1; \mathbf{k}_3, \mathbf{k}_4; \mathbf{K}) \\ &+ \mathcal{T}_\Phi(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_4, \mathbf{k}_3; \mathbf{K}) + \mathcal{T}_\Phi(\mathbf{k}_2, \mathbf{k}_1; \mathbf{k}_4, \mathbf{k}_3; \mathbf{K}). \end{aligned} \quad (40b)$$

The leading order contributions to the reduced trispectrum are

$$\mathcal{T}_{\Phi_A}(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4; \mathbf{K}) = 4f_1^2 P_\Phi(K) P_\Phi(k_1) P_\Phi(k_3), \quad (41a)$$

$$\mathcal{T}_{\Phi_B}(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4; \mathbf{K}) = f_2 [P_\Phi(k_2) P_\Phi(k_3) P_\Phi(k_4) + P_\Phi(k_1) P_\Phi(k_2) P_\Phi(k_4)]. \quad (41b)$$

The curvature trispectrum induces an angular trispectrum onto the CMB fluctuations as we shall now see.

C. Angular Trispectra

In the linear regime, curvature perturbations generate CMB fluctuations as

$$a_l^m = 4\pi(-i)^l \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \Phi(\mathbf{k}) g_{al}(k) Y_l^{m*}(\hat{\mathbf{k}}), \quad (42)$$

where a may be the temperature or E-mode multipole moment, $\Phi(\mathbf{k})$ is the primordial curvature perturbation, and $g_{al}(k)$ denotes the radiation transfer function for $a = \Theta, E$. The multipole moments a_l^m inherit their statistical

properties from $\Phi(\mathbf{k})$, so that in our case, the trispectrum is related directly to integrals of the four-point correlation function of $\Phi(\mathbf{k})$.

From expression (42), the harmonic four-point function is related to the Φ trispectrum by

$$\begin{aligned} \langle a_{l_1}^{m_1} b_{l_2}^{m_2} c_{l_3}^{m_3} d_{l_4}^{m_4} \rangle &= (4\pi)^4 (-i)^{\sum l_i} \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \cdots \frac{d^3 \mathbf{k}_4}{(2\pi)^3} \int d^3 K Y_{l_1}^{m_1*}(\hat{\mathbf{k}}_1) Y_{l_2}^{m_2*}(\hat{\mathbf{k}}_2) Y_{l_3}^{m_3*}(\hat{\mathbf{k}}_3) Y_{l_4}^{m_4*}(\hat{\mathbf{k}}_4) \\ &\times (2\pi)^3 g_{al_1}(k_1) g_{bl_2}(k_2) g_{cl_3}(k_3) g_{dl_4}(k_4) T_\Phi(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4; \mathbf{K}). \end{aligned} \quad (43)$$

The reduced trispectrum $\mathcal{T}_{c_{l_3} d_{l_4}}^{a_{l_1} b_{l_2}}(L)$ is then obtained from the reduced Φ trispectrum simply by replacing T_Φ in the above relation and performing the integrals over directions $\hat{\mathbf{k}}_i$ and $\hat{\mathbf{K}}$, so that

$$\begin{aligned} \mathcal{T}_{c_{l_3} d_{l_4}}^{a_{l_1} b_{l_2}}(L) &= \left(\frac{2}{\pi}\right)^5 \int r_1^2 dr_1 r_2^2 dr_2 (k_1^2 dk_1) \cdots (k_4^2 dk_4) K^2 dK j_L(Kr_1) j_L(Kr_2) \\ &\times [g_{al_1}(k_1) j_{l_1}(k_1 r_1)] [g_{bl_2}(k_2) j_{l_2}(k_2 r_1)] [g_{cl_3}(k_3) j_{l_3}(k_3 r_2)] [g_{dl_4}(k_4) j_{l_4}(k_4 r_2)] \\ &\times \mathcal{T}_\Phi(k_1, k_2; k_3, k_4; K) h_{l_1 L l_2} h_{l_3 L l_4}, \end{aligned} \quad (44)$$

where

$$h_{l_1 L l_2} = \sqrt{\frac{(2l_1+1)(2l_2+1)(2L+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix}, \quad (45)$$

and $j_l(x)$ are the spherical Bessel functions.

Substituting expressions (41) into the above, we find that the reduced trispectrum is given by

$$\mathcal{T}_{c_{l_3} d_{l_4}}^{a_{l_1} b_{l_2}}(L) \equiv \mathcal{T}_{A_{c_{l_3} d_{l_4}}}^{a_{l_1} b_{l_2}}(L) + \mathcal{T}_{B_{c_{l_3} d_{l_4}}}^{a_{l_1} b_{l_2}}(L), \quad (46a)$$

with

$$\mathcal{T}_{A_{c_{l_3} d_{l_4}}}^{a_{l_1} b_{l_2}}(L) = \int r_1^2 dr_1 r_2^2 dr_2 F_L(r_1, r_2) \alpha_{l_1}^a(r_1) \beta_{l_2}^b(r_1) \alpha_{l_3}^c(r_2) \beta_{l_4}^d(r_2) h_{l_1 L l_2} h_{l_3 L l_4}, \quad (46b)$$

$$\mathcal{T}_{B_{c_{l_3} d_{l_4}}}^{a_{l_1} b_{l_2}}(L) = \int r^2 dr \beta_{l_2}^b(r) \beta_{l_4}^d(r) [\mu_{l_1}^a(r) \beta_{l_3}^c(r) + \beta_{l_1}^a(r) \mu_{l_3}^c(r)] h_{l_1 L l_2} h_{l_3 L l_4}, \quad (46c)$$

and

$$F_L(r_1, r_2) = \frac{2}{\pi} \int K^2 dK P_\Phi(K) j_L(Kr_1) j_L(Kr_2), \quad (47a)$$

$$\alpha_l^a(r) = \frac{2}{\pi} \int k^2 dk (2f_1) g_{al}(k) j_l(kr), \quad (47b)$$

$$\beta_l^a(r) = \frac{2}{\pi} \int k^2 dk P_\Phi(k) g_{al}(k) j_l(kr), \quad (47c)$$

$$\mu_l^a(r) = \frac{2}{\pi} \int k^2 dk f_2 g_{al}(k) j_l(kr). \quad (47d)$$

The trispectrum is formed using equations (21) and (22).

To properly evaluate the trispectra, one must extract the radiation transfer function $g_l(k)$ numerically from an Einstein-Boltzmann solver as has been done for the temperature bispectrum [13]. This process is numerically cumbersome and we instead seek an analytic approximation of its effects.

For small multipole moments ($l \ll 100$), CMB temperature fluctuations arise mainly from the Sachs-Wolfe effect [33]. Here the radiation transfer function $g_{\Theta l}(k)$ takes on the simple form

$$g_{\Theta l}(k) = \frac{1}{3} j_l(kr_*), \quad (48)$$

with $r_* = \eta_0 - \eta_{\text{rec}}$ denoting the conformal time elapsed between recombination and the present. In this regime, $\alpha_l^\Theta(r)$ and $\mu_l^\Theta(r)$ simplify to

$$\alpha_l^\Theta(r) = \frac{2f_1}{3r_*^2} \delta(r - r_*), \quad (49)$$

$$\mu_l^\Theta(r) = \frac{f_2}{3r_*^2} \delta(r - r_*). \quad (50)$$

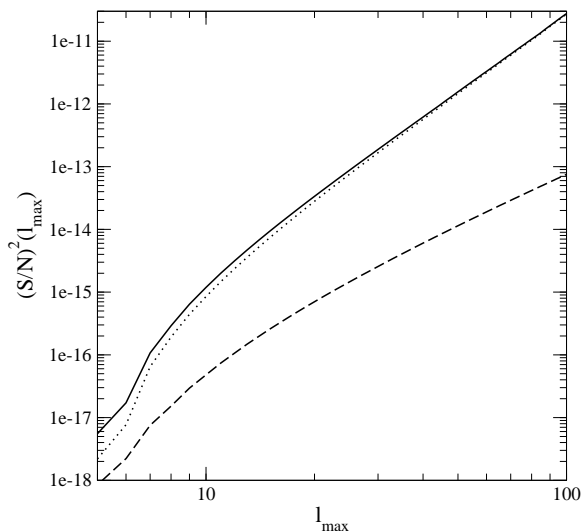


FIG. 2: Signal-to-noise ratio as a function of the maximum multipole l_{\max} in the Sachs-Wolfe approximation (eqn. 52).

Since the temperature power spectrum is given by

$$C_l^{\text{SW}} = \frac{2}{9\pi} \int k^2 dk P_{\Phi}(k) j_l^2(kr_*), \quad (51)$$

the other functions can be related to C_l^{SW} as $F_L(r_*, r_*) = 9C_l^{\text{SW}}$ and $\beta_l^{\Theta}(r_*) = 3C_l^{\text{SW}}$. The reduced trispectrum can then be expressed in terms of C_l^{SW} as

$$\mathcal{T}_{\Theta_{l_3}\Theta_{l_4}}^{\Theta_{l_1}\Theta_{l_2}}(L) = 9C_{l_2}^{\text{SW}}C_{l_4}^{\text{SW}} [4f_1^2C_L^{\text{SW}} + f_2(C_{l_1}^{\text{SW}} + C_{l_3}^{\text{SW}})] h_{l_1Ll_2}h_{l_3Ll_4}. \quad (52)$$

To estimate the trispectrum for higher multipoles note that for the f_1 term in (52), $C_{l_2}^{\text{SW}}$ and $C_{l_4}^{\text{SW}}$ appear from an integration over radiation transfer functions that is very similar in form to that of the true power spectrum $C_l^{\Theta\Theta}$. An approximation to the trispectrum induced by f_1 then becomes

$$\mathcal{T}_{\Theta_{l_3}\Theta_{l_4}}^{\Theta_{l_1}\Theta_{l_2}}(L) \approx 36h_{l_1Ll_2}h_{l_3Ll_4}f_1^2C_L^{\text{SW}}C_{l_2}^{\Theta\Theta}C_{l_4}^{\Theta\Theta} \quad (53)$$

should be valid to the extent to which the anisotropies result from slowly varying local temperature fluctuations on a thin last scattering surface. While the f_2 term does not have this simple form, we take the f_1 piece as representative.

The Newtonian curvature $\Phi(\mathbf{k})$ also acts as a source for E -mode polarization, through the anisotropy of Compton scattering which links the local quadrupoles of temperature fluctuations to local E -mode fluctuations. Although there is no equivalent to the Sachs-Wolfe approximation for the E -mode radiation transfer function, we can again take the above approximation for a slowly-varying source to obtain

$$\mathcal{T}_{E_{l_3}E_{l_4}}^{E_{l_1}E_{l_2}}(L) \approx 36h_{l_1Ll_2}h_{l_3Ll_4}f_1^2C_L^{\text{SW}}C_{l_2}^{EE}C_{l_4}^{EE}, \quad (54)$$

where C_L^{SW} is still the Sachs-Wolfe approximation to the temperature power spectrum. Thus, for $f_2 = 0$, the E -mode trispectrum should behave similarly to the temperature trispectrum. Mixed Θ and E trispectra would take on an analogous form.

D. Signal-to-Noise

We utilize the formalism described in section IID to calculate the expected signal-to-noise ratio for primordial non-Gaussianity. Setting $f_2 = 0$, we use (53) for the temperature trispectrum and compute $(S/N)^2$ to the cosmic variance limit, with Gaussian contamination from gravitational lensing. Figure 3 shows $(S/N)^2/f_1^4$ as a function of the maximum multipole moment observed l_{\max} . The tapering of $(S/N)^2$ is due to the fact that the noise contribution

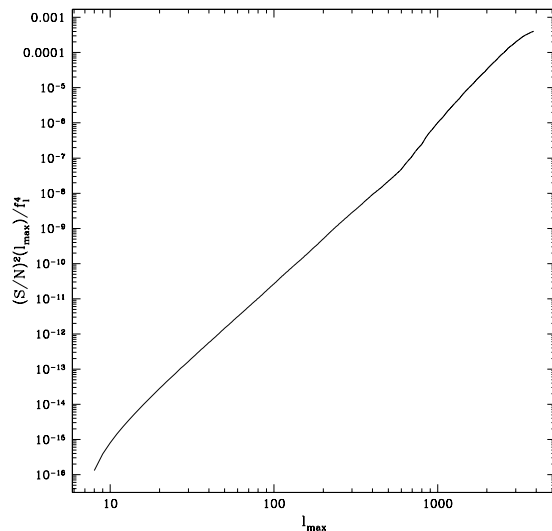


FIG. 3: $(S/N)^2/f_1^4$ vs. l_{\max} , using the approximation (53).

from lensing becomes significant at small angular scales. The figure indicates that the temperature trispectrum may be sensitive to non-Gaussianity for $f_1 \lesssim 10$, although a detailed calculation using expressions (46) involving the full radiation transfer function will be necessary to place rigorous bounds. Since polarization trispectra take a form similar to the temperature, $(S/N)^2$ can be at most enhanced by the number of independent trispectra terms. For E and Θ combinations this represents a factor of a few.

Compared with the sensitivity of the temperature bispectrum to primordial non-Gaussianity, the temperature and polarization trispectra contain a comparable amount of information [13]. The $(S/N)^2$ in the trispectra can exceed that of the bispectrum if $f_1 \gg 1$ due to the steep scaling of f_1^4 .

IV. DISCUSSION

We have introduced a complete formalism for the study of 4-point correlations in the CMB temperature and polarization fields. This formalism should be useful in future tests of the non-Gaussianity of the CMB induced in the early universe and by the evolution of structure. It is also of use in determining the non-Gaussian contributions to errors in temperature-polarization power spectra measurements.

We have applied these techniques to a particular form of trispectra motivated by inflation, generalizing previous treatments to higher order in the initial non-linearity of the curvature fluctuations. Typical slow-roll inflationary models predict an amplitude to the trispectra that is far from observable and so a detection of this type of non-Gaussianity would rule out a large class of models. We have shown that because of the large number of trispectra configurations, the sensitivity to initial non-Gaussianity in the trispectra approach that of the well-studied temperature bispectrum at high multipoles.

Trispectra from secondary anisotropies such as gravitational lensing [21] are expected to be substantially larger and should be fruitful ground for future studies. While measurement of these non-Gaussian signatures will no doubt prove challenging due to foregrounds, systematic effects and computational cost, the wealth of information potentially contained therein may justify the large effort that will be required.

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APPENDIX A: SPIN-WEIGHTED SPHERICAL HARMONICS

We summarize the conventions and properties related to the spin-weighted spherical harmonics [34]. A function ${}_s f(\theta\phi)$ on the sphere carries a spin weight s if, under a right-handed rotation of the basis $(\hat{e}_\theta, \hat{e}_\phi)$ by an angle ψ , it transforms according to ${}_s f(\theta\phi) \rightarrow e^{-is\psi} {}_s f(\theta\phi)$. For such functions, there exist a complete and orthonormal basis with spin weight s , called the spin-weighted spherical harmonics. These spin- s spherical harmonics ${}_s Y_l^m$ can be constructed from the ordinary spherical harmonics by application of the raising and lowering operators [34]. Alternately, they are given in terms of rotation matrices as

$${}_s Y_l^m(\beta\alpha) = (-1)^m \sqrt{\frac{2l+1}{4\pi}} D_{-ms}^l(\alpha\beta\gamma) e^{is\gamma}. \quad (\text{A1})$$

The Euler angles specify a rotation around the coordinate \hat{z} axis by γ , followed by a rotation by β about \hat{y} , then a rotation by α about the (original) \hat{z} axis. The rotation matrix is given explicitly by (see e.g. [35])

$$\begin{aligned} D_{-ms}^l(\alpha\beta\gamma) &= e^{-is\alpha} e^{im\gamma} \left[\frac{(l+m)!(l-m)!}{(l+s)!(l-s)!} \right]^{\frac{1}{2}} \sin^{2l}(\beta/2) \\ &\times \sum_k \binom{l-s}{k} \binom{l+s}{k+s-m} (-1)^{k+l+s} \cot^{2k+s-m}(\beta/2). \end{aligned} \quad (\text{A2})$$

Note that this convention for ${}_s Y_l^m$ differs from that presented in [34, 37] by $(-1)^m$, but corresponds to the Condon-Shortley convention for the ordinary spherical harmonics when $s = 0$ [26]. Below and throughout the paper we use the shorthand convention for the arguments of the spin-spherical harmonics and rotation matrices $\hat{\mathbf{n}} = (\theta\phi)$, $\hat{\mathbf{R}} = (\alpha\beta\gamma)$ and their differential elements $d\hat{\mathbf{n}} = d\phi d\cos\theta$, $d\hat{\mathbf{R}} = d\alpha d\cos\beta d\gamma$.

The properties of the spin-weighted spherical harmonics follow from those of the rotation matrices. In the text, we utilize four such properties: orthogonality, completeness, Clebsch-Gordan expansion, and angle addition.

Orthogonality:

$$\int d\hat{\mathbf{R}} D_{ms}^{l*}(\hat{\mathbf{R}}) D_{m's'}^l(\hat{\mathbf{R}}) = \frac{8\pi^2}{2l+1} \delta_{ll'} \delta_{mm'} \delta_{ss'} \quad (\text{A3})$$

implies

$$\int d\hat{\mathbf{n}} {}_s Y_l^{m*}(\hat{\mathbf{n}}) {}_s Y_l^{m'}(\hat{\mathbf{n}}) = \delta_{ll'} \delta_{mm'}. \quad (\text{A4})$$

Completeness:

$$\sum_{lms} D_{ms}^{l*}(\alpha\beta\gamma) D_{m's'}^l(\alpha'\beta'\gamma') = \frac{8\pi^2}{2l+1} \delta(\alpha - \alpha') \delta(\cos\beta - \cos\beta') \delta(\gamma - \gamma') \quad (\text{A5})$$

implies

$$\sum_{lm} {}_s Y_l^{m*}(\theta\phi) {}_s Y_l^m(\theta'\phi') = \delta(\phi - \phi') \delta(\cos\theta - \cos\theta'). \quad (\text{A6})$$

Clebsch-Gordon relation:

$$D_{m_1 m_1'}^{l_1}(\hat{\mathbf{R}}) D_{m_2 m_2'}^{l_2}(\hat{\mathbf{R}}) = \sum_{LMM'} (2L+1) \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ m_1' & m_2' & -M' \end{pmatrix} (-1)^{M+M'} D_{MM'}^L(\hat{\mathbf{R}}), \quad (\text{A7})$$

implies

$${}_s Y_{l_1}^{m_1}(\hat{\mathbf{n}}) {}_s Y_{l_2}^{m_2}(\hat{\mathbf{n}}) = \sqrt{(2l_1+1)(2l_2+1)} \sum_{LMS} \sqrt{\frac{2L+1}{4\pi}} \begin{pmatrix} l_1 & l_2 & L \\ -m_1 & -m_2 & M \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ s_1 & s_2 & -S \end{pmatrix} {}_s Y_L^M(\hat{\mathbf{n}}). \quad (\text{A8})$$

Auxiliary (orthogonality-Clebsch-Gordon) relation:

$$\int d\hat{\mathbf{R}} D_{m_1 m_1'}^{l_1}(\hat{\mathbf{R}}) D_{m_2 m_2'}^{l_2}(\hat{\mathbf{R}}) D_{m_3 m_3'}^{l_3}(\hat{\mathbf{R}}) = 8\pi^2 \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1' & m_2' & m_3' \end{pmatrix}, \quad (\text{A9})$$

implies

$$\int d\hat{\mathbf{n}}_{s_1} Y_{l_1}^{m_1}(\hat{\mathbf{n}})_{s_2} Y_{l_2}^{m_2}(\hat{\mathbf{n}})_{s_3} Y_{l_3}^{m_3}(\hat{\mathbf{n}}) = \frac{1}{\sqrt{4\pi}} \left[\prod_{i=1}^3 2l_i + 1 \right]^{1/2} \begin{pmatrix} l_1 & l_2 & l_3 \\ -s_1 & -s_2 & -s_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \quad (\text{A10})$$

Addition theorem:

$$D_{s_2 s_1}^{l*}(\gamma\beta - \alpha) = \sum_m D_{-m s_1}^{l*}(\phi'\theta'0) D_{-m s_2}^l(\phi\theta 0), \quad (\text{A11})$$

implies

$$\sum_m {}_{s_1} Y_l^{m*}(\theta'\phi')_{s_2} Y_l^m(\theta\phi) = \sqrt{\frac{2l+1}{4\pi}} (-1)^{s_2} {}_{s_2} Y_l^{-s_1}(\beta\alpha) e^{i s_2 \gamma}, \quad (\text{A12})$$

The relationship between the angles (shown in figure 3) is given explicitly by

$$\begin{aligned} \cot \alpha &= -\cos \theta' \cot(\phi' - \phi) + \cot \theta \frac{\sin \theta'}{\sin(\phi' - \phi)}, \\ \cos \beta &= \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi' - \phi), \\ \cot \gamma &= \cos \theta \cot(\phi' - \phi) - \cot \theta \frac{\sin \theta'}{\sin(\phi' - \phi)}. \end{aligned} \quad (\text{A13})$$

The addition relation corrects a sign error in [37] and agrees with [38], once one accounts for the differences in the

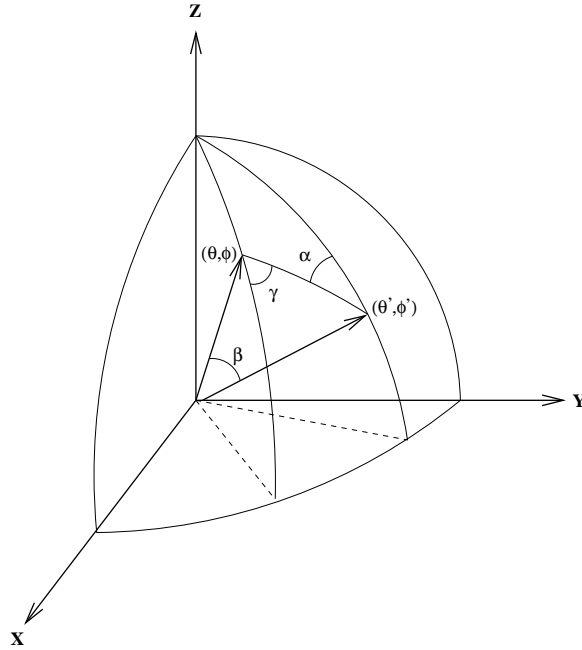


FIG. 4: Relation between Euler angles (α, β, γ) and the original rotation angles (θ, ϕ) and (θ', ϕ') . For the weighted sky maps (B8), with the identification $\hat{\mathbf{n}} \rightarrow (\theta', \phi')$, $\hat{\mathbf{q}} \rightarrow (\theta, \phi)$, so that $\phi_{\hat{\mathbf{n}}} = \alpha$, and $\hat{\mathbf{n}} \cdot \hat{\mathbf{q}} = \beta$.

phase convention for ${}_s Y_l^m$.

APPENDIX B: ADDITIONAL TRISPECTRA PROPERTIES

1. Symmetries

Here, we present the symmetry properties of trispectra when only the angular momentum labels are permuted, keeping the field labels fixed. Such a representation is redundant but can be useful if the diagonal length L is related to some physical quantity, as is true for the inflationary trispectra [see Eqn. (46)].

Using the three CMB fields $\{\Theta, E, B\}$, fifteen distinct four-point functions can be constructed, with the field contents

$$\begin{aligned}
xxxx &\in \{\Theta\Theta\Theta\Theta, EEEE, BBBB\}, \\
xxxy &\in \{\Theta\Theta\Theta E, \Theta\Theta\Theta B, EEE\Theta, EEEB, BBB\Theta, BBBE\}, \\
xyyy &\in \{\Theta\Theta EE, \Theta\Theta BB, EE BB\}, \text{ and} \\
xyyz &\in \{\Theta\Theta EB, EE\Theta B, BB\Theta E\}.
\end{aligned} \tag{B1}$$

For each case, the use of identical fields results in a restricted set of permutation symmetries.

The case for $\langle\Theta\Theta\Theta\Theta\rangle$ has been worked out in detail in [21], where the trispectrum was shown to be composed from a reduced form, thus incorporating the $4! = 24$ possible permutations of l_i 's which leave the four-point harmonic function unchanged. Following the treatment in [21], for trispectra of the form $xxxx$, permutation symmetry of l_i requires that the the trispectrum $Q_{x_{l_3}x_{l_4}}^{x_{l_1}x_{l_2}}(L) \equiv Q_{l_3l_4}^{l_1l_2}(L)$ obey the constraints

$$Q_{l_3l_4}^{l_1l_2}(L) = (-1)^{\Sigma_U} Q_{l_3l_4}^{l_2l_1}(L) = (-1)^{\Sigma_L} Q_{l_4l_3}^{l_1l_2}(L) = (-1)^{\Sigma_U + \Sigma_L} Q_{l_1l_2}^{l_3l_4}(L), \tag{B2a}$$

$$Q_{l_3l_4}^{l_1l_2}(L) = \sum_{L'} (-1)^{l_2+l_3} (2L+1) \begin{Bmatrix} l_1 & l_2 & L \\ l_4 & l_3 & L' \end{Bmatrix} Q_{l_2l_4}^{l_1l_3}(L'), \tag{B2b}$$

and

$$Q_{l_3l_4}^{l_1l_2}(L) = \sum_{L'} (-1)^{L+L'} (2L+1) \begin{Bmatrix} l_1 & l_2 & L \\ l_3 & l_4 & L' \end{Bmatrix} Q_{l_3l_2}^{l_1l_4}(L'), \tag{B2c}$$

where $\Sigma_U \equiv L + l_1 + l_2$, and $\Sigma_L \equiv L + l_3 + l_4$.

For trispectra of the form $xxxy$, the permutations (123) on the l indices are allowed, so that the constraints are given by

$$Q_{x_{l_3}y_{l_4}}^{x_{l_1}x_{l_2}}(L) = (-1)^{\Sigma_U} Q_{x_{l_3}y_{l_4}}^{x_{l_2}x_{l_1}}(L) \tag{B3a}$$

$$= \sum_{L'} (-1)^{l_2+l_3} (2L+1) \begin{Bmatrix} l_1 & l_2 & L \\ l_4 & l_3 & L' \end{Bmatrix} Q_{x_{l_2}y_{l_4}}^{x_{l_1}x_{l_3}}(L') \tag{B3b}$$

$$= \sum_{L'} (-1)^{L+L'} (2L+1) \begin{Bmatrix} l_1 & l_2 & L \\ l_3 & l_4 & L' \end{Bmatrix} Q_{x_{l_1}y_{l_4}}^{x_{l_3}x_{l_2}}(L'), \tag{B3c}$$

where the last two relations come from equation (15). These exhaust the allowed $3! = 6$ permutation symmetries.

For trispectra related to $\langle x_{l_1}^{m_1} y_{l_2}^{m_2} x_{l_3}^{m_3} y_{l_4}^{m_4} \rangle$, the permutations ($l_1 \leftrightarrow l_3$) and ($l_2 \leftrightarrow l_4$) are separately allowed, so that there should be $2 \cdot 2 = 4$ permutations to account for. In the $xyyy$ basis, the permutation symmetries imply that

$$Q_{y_{l_3}y_{l_4}}^{x_{l_1}x_{l_2}}(L) = (-1)^{\Sigma_U} Q_{y_{l_3}y_{l_4}}^{x_{l_2}x_{l_1}}(L) = (-1)^{\Sigma_L} Q_{y_{l_4}y_{l_3}}^{x_{l_1}x_{l_2}}(L) = (-1)^{\Sigma_U + \Sigma_L} Q_{y_{l_4}y_{l_3}}^{x_{l_2}x_{l_1}}(L). \tag{B4}$$

Expressions in the other bases can be derived through the use of the recoupling relations (15).

Lastly, for trispectra related to four-point functions $\langle x_{l_1}^{m_1} x_{l_2}^{m_2} y_{l_3}^{m_3} z_{l_4}^{m_4} \rangle$ the only permutation symmetry allowed is the exchange of l_1 and l_2 . Accordingly, the trispectra obey

$$Q_{y_{l_3}z_{l_4}}^{x_{l_1}x_{l_2}}(L) = (-1)^{\Sigma_U} Q_{y_{l_3}z_{l_4}}^{x_{l_2}x_{l_1}}(L), \tag{B5}$$

where, again, the recoupling relations can be used to find the symmetry constraints in the other bases.

The signal-to-noise ratio for trispectra with fixed field configurations can be obtained from expression (28) by using the recoupling relations (15) to permute the field symbols into that of the fixed field representation. The results can be simplified by using the following identities for the 6-j symbols,

$$\sum_e (2e+1) \begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} \begin{Bmatrix} a & b & e \\ c & d & g \end{Bmatrix} = \frac{\delta_{fg}}{2f+1}, \tag{B6}$$

and

$$\sum_e (-1)^{e+f+g} (2e+1) \begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} \begin{Bmatrix} a & b & e \\ d & c & g \end{Bmatrix} = \begin{Bmatrix} a & c & g \\ b & d & f \end{Bmatrix}. \tag{B7}$$

Rewriting the signal-to-noise in terms of the fixed field configurations transfers the redundancies in the field permutations into redundancies in the l configurations, so that the restrictions on the sum over l_i 's typically become relaxed.

2. Measurement

Direct measurement of the trispectrum using the estimator of equation (24) is computationally expensive due in part to the quintuple sum over m 's. Since the m -dependence of the four-point function simply reflects the rotational invariance, it is useful to find an estimator that employs these symmetries in a more efficient way. The following construction parallels that of the temperature bispectrum [17] and trispectrum [21] and takes into account the subtleties due to the spin-2 behavior of the polarization fields.

For the temperature and polarization fields, one can define a set of weighted sky maps $e_l^\alpha(\hat{q})$, where

$$e_l^\Theta(\hat{q}) = \sqrt{\frac{2l+1}{4\pi}} \int d\hat{n} \Theta(\hat{n}) P_l(\hat{n} \cdot \hat{q}), \quad (\text{B8a})$$

$$e_l^E(\hat{q}) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-2)!}{(l+2)!}} \int d\hat{n} P_l^2(\hat{n} \cdot \hat{q}) [Q(\hat{n}) \cos 2\phi_{\hat{n}} + U(\hat{n}) \sin 2\phi_{\hat{n}}], \quad (\text{B8b})$$

and

$$e_l^B(\hat{q}) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-2)!}{(l+2)!}} \int d\hat{n} P_l^2(\hat{n} \cdot \hat{q}) [U(\hat{n}) \cos 2\phi_{\hat{n}} - Q(\hat{n}) \sin 2\phi_{\hat{n}}]. \quad (\text{B8c})$$

The angle $\phi_{\hat{n}}$ is the angle between the great circles defined by (\hat{n}, \hat{z}) and (\hat{n}, \hat{q}) , and serves to transform the Stokes parameters from the spherical polar basis to the great circle basis (see figure 4). Using these maps, the quantity

$$\hat{Q}_{y_3 z_4}^{w_1 x_2}(L) \equiv \hat{T}_{y_3 z_4}^{w_1 x_2}(L) + \hat{G}_{y_3 z_4}^{w_1 x_2}(L) \quad (\text{B9})$$

can be estimated by expanding the harmonic coefficients in the direct estimator (24) back into fields, expanding the Wigner 3-j symbols using equation (A10), resulting in the expression

$$\begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_3 & l_4 & L \\ 0 & 0 & 0 \end{pmatrix} \hat{Q}_{y_3 z_4}^{w_1 x_2}(L) = (2L+1) \int \frac{d\hat{n}_1}{4\pi} \frac{d\hat{n}_2}{4\pi} P_L(\hat{n}_1 \cdot \hat{n}_2) e_{l_1}^w(\hat{n}_1) e_{l_2}^x(\hat{n}_1) e_{l_3}^y(\hat{n}_2) e_{l_4}^z(\hat{n}_2). \quad (\text{B10})$$

The 3-j symbols impose the constraint $l_1 + l_2 + L = \text{even}$, so the above expression does not allow measurement of modes with $l_1 + l_2 + L = \text{odd}$. It may be computationally advantageous to compute the double integral in (B10) as a single sum in multipole space by noting that they individually return the harmonic decomposition of a product of two e -fields [23]. One must also account for complications arising from realistic issues, including the leakage between the E and B modes due to incomplete sky coverage, for example by Monte-Carlo techniques.

3. Flat Sky Approximation

A sufficiently small patch of sky ($\theta \ll 1$) can be considered flat. In this limit it is computationally and conceptually advantageous to consider the Fourier representation of the trispectrum. Here we establish the relationship between the angular and flat-sky trispectra.

In the flat-sky approximation, the temperature and polarization fields are expanded in Fourier modes as

$$\Theta(\hat{n}) = \int \frac{d^2\mathbf{l}}{(2\pi)^2} \Theta(\mathbf{l}) e^{i\mathbf{l} \cdot \hat{n}}, \quad (\text{B11a})$$

$$\pm A(\hat{n}) = - \int \frac{d^2\mathbf{l}}{(2\pi)^2} \pm A(\mathbf{l}) e^{\pm 2i(\phi_l - \phi)} e^{i\mathbf{l} \cdot \hat{n}}, \quad (\text{B11b})$$

where ϕ_l is the azimuthal angle of \mathbf{l} , and the Stokes parameters are defined in a spherical basis. Again, the E and B modes are defined as

$$\pm A(\mathbf{l}) = E(\mathbf{l}) \pm iB(\mathbf{l}), \quad (\text{B12})$$

and the two-point angular correlation functions reduce to

$$\langle x^*(\mathbf{l}) x'(\mathbf{l}') \rangle = (2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') C_{(\mathbf{l})}^{xx'}. \quad (\text{B13})$$

The connected part of the four-point correlation function can be written as

$$\langle w(\mathbf{l}_1)x(\mathbf{l}_2)y(\mathbf{l}_3)z(\mathbf{l}_4) \rangle_c = (2\pi)^2 \delta(\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_3 + \mathbf{l}_4) T_{(y_{l_3} z_{l_4})}^{(w_{l_1} x_{l_2})}(\mathbf{l}_{12}, \mathbf{l}_{13}), \quad (\text{B14})$$

where l_{12} and l_{13} denote the lengths of the two diagonals. This can be broken up into pieces corresponding to distinct pairings, so that

$$T_{(y_{l_3} z_{l_4})}^{(w_{l_1} x_{l_2})}(\mathbf{l}_{12}, \mathbf{l}_{13}) = P_{(y_{l_3} z_{l_4})}^{(w_{l_1} x_{l_2})}(\mathbf{l}_{12}) + P_{(y_{l_2} z_{l_4})}^{(w_{l_1} x_{l_3})}(\mathbf{l}_{13}) + P_{(y_{l_3} z_{l_2})}^{(w_{l_1} x_{l_4})}(\mathbf{l}_{14}), \quad (\text{B15})$$

with l_{14} being a function of l_{12} and l_{13} . The second and third terms can be projected onto the first pairing. Denoting the trispectrum with the projected terms as $T_{(y_{l_3} z_{l_4})}^{(w_{l_1} x_{l_2})}(L)$, the four-point function becomes

$$\langle w(\mathbf{l}_1)x(\mathbf{l}_2)y(\mathbf{l}_3)z(\mathbf{l}_4) \rangle_c = (2\pi)^2 \int d^2 \mathbf{L} \delta(\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{L}) \delta(\mathbf{l}_3 + \mathbf{l}_4 - \mathbf{L}) T_{(y_{l_3} z_{l_4})}^{(w_{l_1} x_{l_2})}(L), \quad (\text{B16})$$

where we used a decomposition of the delta function

$$\delta(\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_3 + \mathbf{l}_4) = \int d^2 \mathbf{L} \delta(\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{L}) \delta(\mathbf{l}_3 + \mathbf{l}_4 - \mathbf{L}). \quad (\text{B17})$$

Separating the pairings according to the prescription in (B15), the flat-sky four-point function is separated into

$$\begin{aligned} \langle w(\mathbf{l}_1)x(\mathbf{l}_2)y(\mathbf{l}_3)z(\mathbf{l}_4) \rangle_c &= (2\pi)^2 \int d^2 \mathbf{L} \left\{ \delta(\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{L}) \delta(\mathbf{l}_3 + \mathbf{l}_4 - \mathbf{L}) P_{(y_{l_3} z_{l_4})}^{(w_{l_1} x_{l_2})}(L) \right. \\ &\quad + \delta(\mathbf{l}_1 + \mathbf{l}_3 + \mathbf{L}) \delta(\mathbf{l}_2 + \mathbf{l}_4 - \mathbf{L}) P_{(x_{l_2} z_{l_4})}^{(w_{l_1} y_{l_3})}(L) \\ &\quad \left. + \delta(\mathbf{l}_1 + \mathbf{l}_4 + \mathbf{L}) \delta(\mathbf{l}_3 + \mathbf{l}_2 - \mathbf{L}) P_{(y_{l_3} x_{l_2})}^{(w_{l_1} z_{l_4})}(L) \right\}. \end{aligned} \quad (\text{B18})$$

To relate the above expression for the four-point function in the flat sky approximation to the full-sky expression, we use the relation between the Fourier coefficients $x(\mathbf{l})$ and x_l^m

$$x(\mathbf{l}) = \sqrt{\frac{4\pi}{2l+1}} \sum_m i^m x_l^m e^{im\phi_l} \quad (\text{B19a})$$

and the inverse

$$x_l^m = \sqrt{\frac{2l+1}{4\pi}} i^{-m} \int \frac{d\phi_l}{2\pi} e^{-im\phi_l} x(\mathbf{l}), \quad (\text{B19b})$$

where $x \in \{\Theta, E, B\}$. The phase convention chosen here differs from those in [14, 36], due to differences in the choice of phase in the definition of the spin-weighted spherical harmonics.

The plane waves in the δ functions are further decomposed into spherical harmonics using the relation

$$e^{i\mathbf{l} \cdot \hat{\mathbf{n}}} \approx \sqrt{\frac{2\pi}{l}} \sum_m (-i)^m Y_l^m e^{-im\phi_l}, \quad (\text{B20})$$

where the approximation is valid for small angles (or large multipoles l).

If the four-point function has an even net parity, then $P_{(y_{l_3} z_{l_4})}^{(w_{l_1} x_{l_2})}(L)$ is independent of the orientation of the quadrilaterals, so that the integrals over the azimuthal angles ϕ_l can be performed, and we obtain the desired relation

$$P_{(y_{l_3} z_{l_4})}^{(w_{l_1} x_{l_2})}(L) \approx \frac{2L+1}{4\pi} \sqrt{(2l_1+1) \dots (2l_4+1)} \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_3 & l_4 & L \\ 0 & 0 & 0 \end{pmatrix} P_{(y_{l_3} z_{l_4})}^{(w_{l_1} x_{l_2})}(L). \quad (\text{B21})$$

For odd net parity, the Wigner 3-j symbols should be reinterpreted as their analytic continuation (see [14], Eqn. (B1) and Appendix C3).

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