Planck Power Spectrum
B-modes: Auto & Cross

\[ \ell (\ell + 1) \frac{C_{\ell}^{BB_{\alpha \beta}}}{2\pi} \]

- Planck 2014
- SPT 2013
- BICEP2 2014
- POLARBEAR 2014

\([\mu K^2]\)
CMB Blackbody

- COBE FIRAS revealed a blackbody spectrum at $T = 2.725\,\text{K}$ (or cosmological density $\Omega_\gamma h^2 = 2.471 \times 10^{-5}$).
CMB Blackbody

- CMB is a (nearly) perfect blackbody characterized by a phase space distribution function

\[ f = \frac{1}{e^{E/T} - 1} \]

where the temperature \( T(x, \hat{n}, t) \) is observed at our position \( x = 0 \) and time \( t_0 \) to be nearly isotropic with a mean temperature of \( \bar{T} = 2.725 \text{K} \)

- Our observable then is the temperature anisotropy

\[ \Theta(\hat{n}) \equiv \frac{T(0, \hat{n}, t_0) - \bar{T}}{\bar{T}} \]

- Given that physical processes essentially put a band limit on this function it is useful to decompose it into a complete set of harmonic coefficients
Spherical Harmonics

- Laplace Eigenfunctions
  \[ \nabla^2 Y_{\ell}^m = -[l(l + 1)] Y_{\ell}^m \]

- Orthogonal and complete
  \[ \int d\hat{n} Y_{\ell}^{m*}(\hat{n}) Y_{\ell}^m(\hat{n}) = \delta_{\ell\ell'} \delta_{mm'} \]
  \[ \sum_{\ell m} Y_{\ell}^{m*}(\hat{n}) Y_{\ell}^m(\hat{n}') = \delta(\phi - \phi') \delta(\cos \theta - \cos \theta') \]

  Generalizable to tensors on the sphere (polarization), modes on a curved FRW metric

- Conjugation
  \[ Y_{\ell}^{m*} = (-1)^m Y_{\ell}^{-m} \]
Multipole Moments

- Decompose into multipole moments

\[ \Theta(\hat{n}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell}^{m}(\hat{n}) \]

- So \( \Theta_{\ell m} \) is complex but \( \Theta(\hat{n}) \) real:

\[ \Theta^*(\hat{n}) = \sum_{\ell m} \Theta_{\ell m}^* Y_{\ell}^{-m}(\hat{n}) \]

\[ = \sum_{\ell m} \Theta_{\ell m}^* (-1)^m Y_{\ell}^{-m}(\hat{n}) \]

\[ = \Theta(\hat{n}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell}^{m}(\hat{n}) = \sum_{\ell -m} \Theta_{\ell -m} Y_{\ell}^{-m}(\hat{n}) \]

so \( m \) and \(-m\) are not independent

\[ \Theta_{\ell m}^* = (-1)^m \Theta_{\ell -m} \]
**$N$-pt correlation**

- Since the fluctuations are random and zero mean we are interested in characterizing the $N$-point correlation

\[
\langle \Theta(\hat{n}_1) \ldots \Theta(\hat{n}_n) \rangle = \sum_{\ell_1 \ldots \ell_n} \sum_{m_1 \ldots m_n} \langle \Theta_{\ell_1 m_1} \ldots \Theta_{\ell_n m_n} \rangle Y_{\ell_1}^{m_1}(\hat{n}_1) \ldots Y_{\ell_n}^{m_n}(\hat{n}_n)
\]

- Statistical isotropy implies that we should get the same result in a rotated frame

\[
R[Y_{\ell}^{m}(\hat{n})] = \sum_{m'} D_{m'm}(\alpha, \beta, \gamma) Y_{\ell}^{m'}(\hat{n})
\]

where $\alpha$, $\beta$ and $\gamma$ are the Euler angles of the rotation and $D$ is the Wigner function (note $Y_{\ell}^{m}$ is a $D$ function)

\[
\langle \Theta_{\ell_1 m_1} \ldots \Theta_{\ell_n m_n} \rangle = \sum_{m'_1 \ldots m'_n} \langle \Theta_{\ell_1 m'_1} \ldots \Theta_{\ell_n m'_n} \rangle D_{m_1 m'_{1}}^{\ell_1} \ldots D_{m_n m'_{n}}^{\ell_n}
\]
For any $N$-point function, combine rotation matrices (group multiplication; angular momentum addition) and orthogonality:

$$
\sum_m (-1)^{m_2-m} D_{m_1m}^\ell_1 D_{-m_2-m}^\ell_1 = \delta_{m_1m_2}
$$

The simplest case is the 2pt function:

$$
\langle \Theta_{\ell_1 m_1} \Theta_{\ell_2 m_2} \rangle = \delta_{\ell_1 \ell_2} \delta_{m_1-m_2} (-1)^{m_1} C_{\ell_1}
$$

where $C_{\ell}$ is the power spectrum. Check

$$
= \sum_{m_1' m_2'} \delta_{\ell_1 \ell_2} \delta_{m_1' - m_2'} (-1)^{m_1'} C_{\ell_1} D_{m_1 m_1'}^\ell_1 D_{m_2 m_2'}^\ell_2
$$

$$
= \delta_{\ell_1 \ell_2} C_{\ell_1} \sum_{m_1'} (-1)^{m_1'} D_{m_1 m_1'}^{\ell_1} D_{m_2 m_1'}^{\ell_2} = \delta_{\ell_1 \ell_2} \delta_{m_1-m_2} (-1)^{m_1} C_{\ell_1}
$$
\textbf{N-pt correlation}

• Using the reality of the field

\[ \langle \Theta^*_\ell_1 m_1 \Theta_{\ell_2 m_2} \rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} C_{\ell_1}. \]

• If the statistics were Gaussian then all the $N$-point functions would be defined in terms of the products of two-point contractions, e.g.

\[ \langle \Theta_{\ell_1 m_1} \Theta_{\ell_2 m_2} \Theta_{\ell_3 m_3} \Theta_{\ell_4 m_4} \rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} \delta_{\ell_3 \ell_4} \delta_{m_3 m_4} C_{\ell_1} C_{\ell_3} + \text{perm}. \]

• More generally we can define the isotropy condition beyond Gaussianity, e.g. the bispectrum

\[ \langle \Theta_{\ell_1 m_1} \ldots \Theta_{\ell_3 m_3} \rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1 \ell_2 \ell_3}. \]
CMB Temperature Fluctuations

- Angular Power Spectrum

Angular Scale

Multiplicity moment ($l$)

$l(l+1)C_{l}/2\pi(\mu{K}^2)$

Multipole moment ($l$)

Angular Scale

Model

WMAP

CBI

ACBAR

Multipole moment ($l$)
Why $\ell^2 C_\ell/2\pi$?

- Variance of the temperature fluctuation field

\[
\langle \Theta(\hat{n})\Theta(\hat{n}) \rangle = \sum_{\ell m} \sum_{\ell' m'} \langle \Theta_{\ell m} \Theta^*_{\ell' m'} \rangle Y^m_\ell (\hat{n}) Y^{m*}_{\ell'} (\hat{n})
\]

\[
= \sum_\ell C_\ell \sum_m Y^m_\ell (\hat{n}) Y^{m*}_\ell (\hat{n})
\]

\[
= \sum_\ell \frac{2\ell + 1}{4\pi} C_\ell
\]

via the angle addition formula for spherical harmonics

- For some range $\Delta \ell \approx \ell$ the contribution to the variance is

\[
\langle \Theta(\hat{n})\Theta(\hat{n}) \rangle_{\ell \pm \Delta \ell/2} \approx \Delta \ell \frac{2\ell + 1}{4\pi} C_\ell \approx \frac{\ell^2}{2\pi} C_\ell
\]

- Conventional to use $\ell(\ell + 1)/2\pi$ for reasons below
Cosmic Variance

- We only have access to our sky, not the ensemble average.
- There are $2\ell + 1$ $m$-modes of given $\ell$ mode, so average:

$$\hat{C}_\ell = \frac{1}{2\ell + 1} \sum_m \Theta^*_\ell m \Theta_{\ell m}$$

- $\langle \hat{C}_\ell \rangle = C_\ell$ but now there is a cosmic variance:

$$\sigma^2_{C_\ell} = \frac{\langle (\hat{C}_\ell - C_\ell)(\hat{C}_\ell - C_\ell) \rangle}{C^2_\ell} = \frac{\langle \hat{C}_\ell \hat{C}_\ell \rangle - C^2_\ell}{C^2_\ell}$$

- For Gaussian statistics:

$$\sigma^2_{C_\ell} = \frac{1}{(2\ell + 1)^2 C^2_\ell} \langle \sum_{mm'} \Theta^*_\ell m \Theta_{\ell m} \Theta^*_\ell m' \Theta_{\ell m'} \rangle - 1$$

$$= \frac{1}{(2\ell + 1)^2} \sum_{mm'} (\delta_{mm'} + \delta_{m-m'}) = \frac{2}{2\ell + 1}$$
Cosmic Variance

- Note that the distribution of $\hat{C}_\ell$ is that of a sum of squares of Gaussian variates
- Distributed as a $\chi^2$ of $2\ell + 1$ degrees of freedom
- Approaches a Gaussian for $2\ell + 1 \to \infty$ (central limit theorem)
- Anomalously low quadrupole is not that unlikely
- $\sigma_{C_\ell}$ is a useful quantification of errors at high $\ell$
- Suppose $C_\ell$ depends on a set of cosmological parameters $c_i$ then we can estimate errors of $c_i$ measurements by error propagation

$$F_{ij} = \text{Cov}^{-1}(c_i, c_j) = \sum_{\ell \ell'} \frac{\partial C_\ell}{\partial c_i} \text{Cov}^{-1}(C_\ell, C_{\ell'}) \frac{\partial C_{\ell'}}{\partial c_j}$$

$$= \sum_{\ell} \frac{(2\ell + 1)}{2C_\ell^2} \frac{\partial C_\ell}{\partial c_i} \frac{\partial C_\ell}{\partial c_j}$$
Idealized Statistical Errors

- Take a noisy estimator of the multipoles in the map

\[ \hat{\Theta}_{\ell m} = \Theta_{\ell m} + N_{\ell m} \]

and take the noise to be statistically isotropic

\[ \langle N_{\ell m}^* N_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_{\ell}^{NN} \]

- Construct an unbiased estimator of the power spectrum \( \langle \hat{C}_\ell \rangle = C_\ell \)

\[ \hat{C}_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} \hat{\Theta}_{\ell m}^* \hat{\Theta}_{\ell m} - C_{\ell}^{NN} \]

- Covariance in estimator

\[ \text{Cov}(C_\ell, C_{\ell'}) = \frac{2}{2\ell + 1} (C_\ell + C_{\ell}^{NN})^2 \delta_{\ell \ell'} \]
Incomplete Sky

- On a small section of sky, the number of independent modes of a given $\ell$ is no longer $2\ell + 1$

- As in Fourier analysis, there are two limitations: the lowest $\ell$ mode that can be measured is the wavelength that fits in angular patch $\theta$

$$\ell_{\text{min}} = \frac{2\pi}{\theta};$$

modes separated by $\Delta \ell < \ell_{\text{min}}$ cannot be measured independently

- Estimates of $C_\ell$ covary on a scale imposed by $\Delta \ell < \ell_{\text{min}}$

- Crude approximation: account only for the loss of independent modes by rescaling the errors rather than introducing covariance

$$\text{Cov}(C_\ell, C_{\ell'}) = \frac{2}{(2\ell + 1)f_{\text{sky}}}(C_\ell + C^{\text{NN}}_{\ell})^2 \delta_{\ell\ell'}$$
Stokes Parameters

- Specific intensity is related to quadratic combinations of the electric field.
- Define the intensity matrix (time averaged over oscillations) \( \langle \mathbf{E} \mathbf{E}^\dagger \rangle \)
- Hermitian matrix can be decomposed into Pauli matrices

\[
\mathbf{P} = \langle \mathbf{E} \mathbf{E}^\dagger \rangle = \frac{1}{2} \left( I \sigma_0 + Q \sigma_3 + U \sigma_1 - V \sigma_2 \right),
\]

where

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

- Stokes parameters recovered as \( \text{Tr}(\sigma_i \mathbf{P}) \)
- Choose units of temperature for Stokes parameters \( I \rightarrow \Theta \)
Stokes Parameters

- Consider a general plane wave solution

\[ \mathbf{E}(t, z) = E_1(t, z)\hat{e}_1 + E_2(t, z)\hat{e}_2 \]

\[ E_1(t, z) = A_1 e^{i\phi_1} e^{i(kz-\omega t)} \]

\[ E_2(t, z) = A_2 e^{i\phi_2} e^{i(kz-\omega t)} \]

- Explicitly:

\[ I = \langle E_1 E_1^* + E_2 E_2^* \rangle = A_1^2 + A_2^2 \]

\[ Q = \langle E_1 E_1^* - E_2 E_2^* \rangle = A_1^2 - A_2^2 \]

\[ U = \langle E_1 E_2^* + E_2 E_1^* \rangle = 2A_1 A_2 \cos(\phi_2 - \phi_1) \]

\[ V = -i \langle E_1 E_2^* - E_2 E_1^* \rangle = 2A_1 A_2 \sin(\phi_2 - \phi_1) \]

so that the Stokes parameters define the state up to an unobservable overall phase of the wave.
Detection

- This suggests that abstractly there are two different ways to detect polarization: separate and difference orthogonal modes (bolometers $I, Q$) or correlate the separated components ($U, V$).

- In the correlator example the natural output would be $U$ but one can recover $V$ by introducing a phase lag $\phi = \pi/2$ on one arm, and $Q$ by having the OMT pick out directions rotated by $\pi/4$.

- Likewise, in the bolometer example, one can rotate the polarizer and also introduce a coherent front end to change $V$ to $U$. 
Detection

- Techniques also differ in the systematics that can convert unpolarized sky to fake polarization
- Differencing detectors are sensitive to relative gain fluctuations
- Correlation detectors are sensitive to cross coupling between the arms
- More generally, the intended block diagram and systematic problems map components of the polarization matrix onto others and are kept track of through “Jones” or instrumental response matrices $E_{\text{det}} = J E_{\text{in}}$

$$P_{\text{det}} = J P_{\text{in}} J^\dagger$$

where the end result is either a differencing or a correlation of the $P_{\text{det}}$. 
Polarization

- Radiation field involves a directed quantity, the electric field vector, which defines the polarization.
- Consider a general plane wave solution:

\[ E(t, z) = E_1(t, z)\hat{e}_1 + E_2(t, z)\hat{e}_2 \]

\[ E_1(t, z) = \text{Re}A_1 e^{i\phi_1} e^{i(kz - \omega t)} \]

\[ E_2(t, z) = \text{Re}A_2 e^{i\phi_2} e^{i(kz - \omega t)} \]

or at \( z = 0 \) the field vector traces out an ellipse:

\[ E(t, 0) = A_1 \cos(\omega t - \phi_1)\hat{e}_1 + A_2 \cos(\omega t - \phi_2)\hat{e}_2 \]

with principal axes defined by:

\[ E(t, 0) = A'_1 \cos(\omega t)\hat{e}'_1 - A'_2 \sin(\omega t)\hat{e}'_2 \]

so as to trace out a clockwise rotation for \( A'_1, A'_2 > 0 \).
Polarization

- Define polarization angle

\[ \hat{e}_1' = \cos \chi \hat{e}_1 + \sin \chi \hat{e}_2 \]
\[ \hat{e}_2' = -\sin \chi \hat{e}_1 + \cos \chi \hat{e}_2 \]

- Match

\[ E(t, 0) = A'_1 \cos \omega t [\cos \chi \hat{e}_1 + \sin \chi \hat{e}_2] \]
\[ \quad - A'_2 \cos \omega t [-\sin \chi \hat{e}_1 + \cos \chi \hat{e}_2] \]
\[ = A_1 [\cos \phi_1 \cos \omega t + \sin \phi_1 \sin \omega t] \hat{e}_1 \]
\[ \quad + A_2 [\cos \phi_2 \cos \omega t + \sin \phi_2 \sin \omega t] \hat{e}_2 \]
Polarization

- Define relative strength of two principal states

\[ A'_1 = E_0 \cos \beta \quad A'_2 = E_0 \sin \beta \]

- Characterize the polarization by two angles

\[ A_1 \cos \phi_1 = E_0 \cos \beta \cos \chi, \quad A_1 \sin \phi_1 = E_0 \sin \beta \sin \chi, \]
\[ A_2 \cos \phi_2 = E_0 \cos \beta \sin \chi, \quad A_2 \sin \phi_2 = -E_0 \sin \beta \cos \chi \]

Or Stokes parameters by

\[ I = E_0^2, \quad Q = E_0^2 \cos 2\beta \cos 2\chi \]
\[ U = E_0^2 \cos 2\beta \sin 2\chi, \quad V = E_0^2 \sin 2\beta \]

- So \( I^2 = Q^2 + U^2 + V^2 \), double angles reflect the spin 2 field or headless vector nature of polarization
Polarization

Special cases

- If $\beta = 0, \pi/2, \pi$ then only one principal axis, ellipse collapses to a line and $V = 0 \rightarrow$ linear polarization oriented at angle $\chi$
  
  If $\chi = 0, \pi/2, \pi$ then $I = \pm Q$ and $U = 0$
  
  If $\chi = \pi/4, 3\pi/4, ...$ then $I = \pm U$ and $Q = 0$ - so $U$ is $Q$ in a frame rotated by 45 degrees

- If $\beta = \pi/4, 3\pi/4$, then principal components have equal strength and $E$ field rotates on a circle: $I = \pm V$ and $Q = U = 0 \rightarrow$ circular polarization

- $U/Q = \tan 2\chi$ defines angle of linear polarization and $V/I = \sin 2\beta$ defines degree of circular polarization
Natural Light

- A monochromatic plane wave is completely polarized
  \[ I^2 = Q^2 + U^2 + V^2 \]

- Polarization matrix is like a density matrix in quantum mechanics and allows for pure (coherent) states and mixed states

- Suppose the total \( E_{\text{tot}} \) field is composed of different (frequency) components

  \[
  E_{\text{tot}} = \sum_i E_i
  \]

- Then components decorrelate in time average

  \[
  \left\langle E_{\text{tot}} E_{\text{tot}}^\dagger \right\rangle = \sum_{ij} \left\langle E_i E_j^\dagger \right\rangle = \sum_i \left\langle E_i E_i^\dagger \right\rangle
  \]
So Stokes parameters of incoherent contributions add

\[ I = \sum_i I_i \quad Q = \sum_i Q_i \quad U = \sum_i U_i \quad V = \sum_i V_i \]

and since individual \( Q \), \( U \) and \( V \) can have either sign:
\[ I^2 \geq Q^2 + U^2 + V^2, \] all 4 Stokes parameters needed
Linear Polarization

- \( Q \propto \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle, \quad U \propto \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle. \)

- Counterclockwise rotation of axes by \( \theta = 45^\circ \)

\[
E_1 = \left( E_1' - E_2' \right) / \sqrt{2}, \quad E_2 = \left( E_1' + E_2' \right) / \sqrt{2}
\]

- \( U \propto \langle E_1' E_1'^* \rangle - \langle E_2' E_2'^* \rangle, \) difference of intensities at 45° or \( Q' \)

- More generally, \( P \) transforms as a tensor under rotations and

\[
Q' = \cos(2\theta)Q + \sin(2\theta)U
\]

\[
U' = -\sin(2\theta)Q + \cos(2\theta)U
\]

or

\[
Q' \pm iU' = e^{\pm 2i\theta} [Q \pm iU]
\]

acquires a phase under rotation and is a spin \( \pm 2 \) object
Coordinate Independent Representation

- Two directions: orientation of polarization and change in amplitude, i.e. $Q$ and $U$ in the basis of the Fourier wavevector (pointing with angle $\phi_l$) for small sections of sky are called $E$ and $B$ components

$$E(l) \pm iB(l) = -\int d\hat{n}[Q'(\hat{n}) \pm iU'(\hat{n})]e^{-il\cdot\hat{n}}$$

$$= -e^{\pm 2i\phi_l} \int d\hat{n}[Q(\hat{n}) \pm iU(\hat{n})]e^{-il\cdot\hat{n}}$$

- For the $B$-mode to not vanish, the polarization must point in a direction not related to the wavevector - not possible for density fluctuations in linear theory

- Generalize to all-sky: eigenmodes of Laplace operator of tensor
Spin Harmonics

- Laplace Eigenfunctions

\[ \nabla^2_{\pm 2} Y_{\ell m}[\sigma_3 \mp i\sigma_1] = -[l(l + 1) - 4]_{\pm 2} Y_{\ell m}[\sigma_3 \mp i\sigma_1] \]

- Spin \( s \) spherical harmonics: orthogonal and complete

\[ \int d\hat{n}_s Y^*_{\ell m}(\hat{n})_s Y_{\ell m}(\hat{n}) = \delta_{\ell \ell'} \delta_{mm'} \]

\[ \sum_{\ell m} s Y^*_{\ell m}(\hat{n})_s Y_{\ell m}(\hat{n}') = \delta(\phi - \phi') \delta(\cos \theta - \cos \theta') \]

where the ordinary spherical harmonics are \( Y_{\ell m} = 0 Y_{\ell m} \)

- Given in terms of the rotation matrix

\[ s Y_{\ell m}(\beta\alpha) = (-1)^m \sqrt{\frac{2\ell + 1}{4\pi}} D^\ell_{-ms}(\alpha\beta0) \]
Statistical Representation

- All-sky decomposition

\[ [Q(\hat{n}) \pm iU(\hat{n})] = \sum_{\ell m} [E_{\ell m} \pm iB_{\ell m}] \pm 2Y_{\ell m}(\hat{n}) \]

- Power spectra

\[ \langle E^*_{\ell m} E_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{mm'} C^{EE}_\ell \]
\[ \langle B^*_{\ell m} B_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{mm'} C^{BB}_\ell \]

- Cross correlation

\[ \langle \Theta^*_{\ell m} E_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{mm'} C^{\Theta E}_\ell \]

others vanish if parity is conserved
Inhomogeneity vs Anisotropy

• $\Theta$ is a function of position as well as direction but we only have access to our position

• Light travels at the speed of light so the radiation we receive in direction $\hat{n}$ was $(\eta_0 - \eta)\hat{n}$ at conformal time $\eta$

• Inhomogeneity at a distance appears as an anisotropy to the observer

• We need to transport the radiation from the initial conditions to the observer

• This is done with the Boltzmann or radiative transfer equation

• In the absence of scattering, emission or absorption the Boltzmann equation is simply

$$\frac{Df}{Dt} = 0$$
Last Scattering

- Angular distribution of radiation is the 3D temperature field projected onto a shell - surface of last scattering

- Shell radius is distance from the observer to recombination: called the last scattering surface

- Take the radiation distribution at last scattering to also be described by an isotropic temperature fluctuation field $\Theta(x)$
Integral Solution to Radiative Transfer

- Formal solution for specific intensity $I_\nu = 2\hbar \nu^3 f/c^2$

$$I_\nu(0) = I_\nu(\tau) e^{-\tau} + \int_0^\tau d\tau' S_\nu(\tau') e^{-\tau'}$$

- Specific intensity $I_\nu$ attenuated by absorption and replaced by source function, attenuated by absorption from foreground matter

- $\Theta$ satisfies the same relation for a blackbody
Angular Power Spectrum

- Take recombination to be instantaneous: \( d\tau e^{-\tau} = dD \delta(D - D_*) \)
and the source to be the local temperature inhomogeneity

\[
\Theta(\hat{n}) = \int dD \Theta(x) \delta(D - D_*)
\]

where \( D \) is the comoving distance and \( D_* \) denotes recombination.

- Describe the temperature field by its Fourier moments

\[
\Theta(x) = \int \frac{d^3k}{(2\pi)^3} \Theta(k) e^{ik \cdot x}
\]

- Note that Fourier moments \( \Theta(k) \) have units of volume \( k^{-3} \)

- 2 point statistics of the real-space field are translationally and rotationally invariant

- Described by power spectrum
Spatial Power Spectrum

- Translational invariance

\[
\langle \Theta(x')\Theta(x) \rangle = \langle \Theta(x' + d)\Theta(x + d) \rangle \\
\int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \langle \Theta^*(k')\Theta(k) \rangle e^{ik \cdot x - ik' \cdot x'} \\
= \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \langle \Theta^*(k')\Theta(k) \rangle e^{ik \cdot x - ik' \cdot x' + i(k-k') \cdot d}
\]

So two point function requires \(\delta(k - k')\); rotational invariance says coefficient depends only on magnitude of \(k\) not it’s direction

\[
\langle \Theta(k)^*\Theta(k') \rangle = (2\pi)^3 \delta(k - k') P_T(k')
\]

Note that \(\delta(k - k')\) has units of volume and so \(P_T\) must have units of volume
Dimensionless Power Spectrum

- Variance

\[ \sigma^2_\Theta \equiv \langle \Theta(\mathbf{x})\Theta(\mathbf{x}) \rangle = \int \frac{d^3 k}{(2\pi)^3} P_T(k) \]

\[ = \int \frac{k^2 dk}{2\pi^2} \int \frac{d\Omega}{4\pi} P_T(k) \]

\[ = \int d\ln k \frac{k^3}{2\pi^2} P_T(k) \]

- Define power per logarithmic interval

\[ \Delta^2_T(k) \equiv \frac{k^3 P_T(k)}{2\pi^2} \]

- This quantity is dimensionless.
Angular Power Spectrum

• Temperature field

\[ \Theta(\hat{n}) = \int \frac{d^3k}{(2\pi)^3} \Theta(k) e^{i k \cdot D_\ast \hat{n}} \]

• Multipole moments \( \Theta(\hat{n}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m} \)

• Expand out plane wave in spherical coordinates

\[ e^{i k D_\ast \cdot \hat{n}} = 4\pi \sum_{\ell m} i^\ell j_\ell(k D_\ast) Y_{\ell m}^*(k) Y_{\ell m}(\hat{n}) \]

• Angular moment

\[ \Theta_{\ell m} = \int \frac{d^3k}{(2\pi)^3} \Theta(k) 4\pi i^\ell j_\ell(k D_\ast) Y_{\ell m}^*(k) \]
Angular Power Spectrum

- Power spectrum

\[ \langle \Theta_{\ell m}^* \Theta_{\ell' m'} \rangle = \int \frac{d^3 k}{(2\pi)^3} (4\pi)^2 i^{\ell - \ell'} j_\ell(k D_*) j_{\ell'}(k D_*) Y_{\ell m}(k) Y_{\ell' m'}^*(k) P_T(k) \]

\[ = \delta_{\ell \ell'} \delta_{mm'} 4\pi \int d \ln k j_\ell^2(k D_*) \Delta_T^2(k) \]

with \( \int_0^\infty j_\ell^2(x) d \ln x = 1/(2\ell(\ell + 1)) \), slowly varying \( \Delta_T^2 \)

- Angular power spectrum:

\[ C_\ell = \frac{4\pi \Delta_T^2(\ell/D_*)}{2\ell(\ell + 1)} = \frac{2\pi}{\ell(\ell + 1)} \Delta_T^2(\ell/D_*) \]

- Not surprisingly, a relationship between \( \ell^2 C_\ell / 2\pi \) and \( \Delta_T^2 \) at \( \ell \gg 1 \).
  By convention use \( \ell(\ell + 1) \) to make relationship exact

- This is a property of a thin-shell isotropic source, now generalize.
Generalized Source

- For example, if the emission surface is moving with respect to the observer then radiation has an intrinsic dipole pattern at emission.

- More generally, we know the $Y_{\ell}^m$'s are a complete angular basis and plane waves are complete spatial basis.

- General source distribution can be decomposed into local multipole moments

$$S_{\ell}^{(m)}(-i)^{\ell} \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell}^m(\hat{n}) \exp(ik \cdot x)$$

where the prefactor is for convenience for later convenience when
we fix $\hat{z} = \hat{k}$
Generalized Source

- So general solution is for a single source shell is

\[
\Theta(\hat{n}) = \sum_{\ell m} S_{\ell}^{(m)}(-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell}^{m}(\hat{n}) \exp(i k \cdot D_{*}\hat{n})
\]

and for a source that is a function of distance

\[
\Theta(\hat{n}) = \int dDe^{-\tau} \sum_{\ell m} S_{\ell}^{(m)}(D)(-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell}^{m}(\hat{n}) \exp(i k \cdot D\hat{n})
\]

- Note that unlike the isotropic source, we have two pieces that depend on \( \hat{n} \)

- Observer sees the total angular structure

\[
Y_{\ell}^{m}(\hat{n}) e^{i k D_{*}\cdot\hat{n}} = 4\pi \sum_{\ell' m'} i^{\ell'} j^{(kD_{*})} Y_{\ell'}^{m'}(\hat{n}) Y_{\ell'}^{m'}(\hat{n}) Y_{\ell}^{m}(\hat{n})
\]
Generalized Source

- We extract the observed multipoles by the addition of angular momentum $Y_{\ell'}^{m'}(\hat{n})Y_{\ell}^{m}(\hat{n}) \rightarrow Y_{\ell}^{M}(\hat{n})$

- Radial functions become linear sums over $j_{\ell}$ with the recoupling (Clebsch-Gordan) coefficients

- These radial weight functions carry important information about how spatial fluctuations project onto angular fluctuations - or the sharpness of the angular transfer functions

- Same is true of polarization - source is Thomson scattering

- Polarization has an intrinsic quadrupolar distribution, recoupled by orbital angular momentum into fine scale polarization anisotropy

- Formal integral solution to the Boltzmann or radiative transfer equation

- Source functions also follow from the Boltzmann equation
Polarization Basis

- Define the angularly dependent Stokes perturbation

\[ \Theta(x, \hat{n}, \eta), \quad Q(x, \hat{n}, \eta), \quad U(x, \hat{n}, \eta) \]

- Decompose into normal modes: plane waves for spatial part and spherical harmonics for angular part

\[ G^m_\ell(k, x, \hat{n}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} Y^m_\ell(\hat{n}) \exp(ik \cdot x) \]

\[ \pm 2G^m_\ell(k, x, \hat{n}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} \pm 2Y^m_\ell(\hat{n}) \exp(ik \cdot x) \]

- In a spatially curved universe generalize the plane wave part
- For a single k mode, choose a coordinate system \( \hat{z} = \hat{k} \)
Normal Modes

- Temperature and polarization fields

\[ \Theta(x, \hat{n}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} \Theta^{(m)}_{\ell} G^m_{\ell} \]

\[ [Q \pm iU](x, \hat{n}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} [E^{(m)}_{\ell} \pm iB^{(m)}_{\ell}] \pm 2G^m_{\ell} \]

- For each k mode, work in coordinates where k \parallel z and so \( m = 0 \) represents scalar modes, \( m = \pm 1 \) vector modes, \( m = \pm 2 \) tensor modes, \(|m| > 2\) vanishes. Since modes add incoherently and \( Q \pm iU \) is invariant up to a phase, rotation back to a fixed coordinate system is trivial.
Liouville Equation

- In absence of scattering, the phase space distribution of photons in each polarization state $a$ is conserved along the propagation path.

- Rewrite variables in terms of the photon propagation direction $q = q \hat{n}$, so $f_a(x, \hat{n}, q, \eta)$ and

\[
\frac{D}{D\eta} f_a(x, \hat{n}, q, \eta) = 0 = \left( \frac{\partial}{\partial \eta} + \frac{d\mathbf{x}}{d\eta} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{d\hat{n}}{d\eta} \cdot \frac{\partial}{\partial \hat{n}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a
\]

- For simplicity, assume spatially flat universe $K = 0$ then $d\hat{n}/d\eta = 0$ and $d\mathbf{x} = \hat{n} d\eta$

\[
f_a + \hat{n} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0
\]

- The spatial gradient describes the conversion from inhomogeneity to anisotropy and the $\dot{q}$ term the gravitational sources.
Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies.

- Spatial gradient term hits plane wave:

\[ \hat{n} \cdot \nabla e^{ik \cdot x} = i \hat{n} \cdot k e^{ik \cdot x} = i \sqrt{\frac{4\pi}{3}} k Y^0_1(\hat{n}) e^{ik \cdot x} \]

- Dipole term adds to angular dependence through the addition of angular momentum:

\[
\sqrt{\frac{4\pi}{3}} Y^0_1 Y^m_\ell \quad = \quad \frac{\kappa^m_\ell}{\sqrt{(2\ell + 1)(2\ell - 1)}} Y^m_{\ell-1} + \frac{\kappa^m_{\ell+1}}{\sqrt{(2\ell + 1)(2\ell + 3)}} Y^m_{\ell+1}
\]

where \( \kappa^m_\ell = \sqrt{\ell^2 - m^2} \) is given by Clebsch-Gordon coefficients.
Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

\[
\dot{\Theta}_\ell^{(m)} = k \left[ \frac{k^m_\ell}{2\ell + 1} \Theta_{\ell-1}^{(m)} - \frac{k^m_{\ell+1}}{2\ell + 3} \Theta_{\ell+1}^{(m)} \right] - \dot{\tau} \Theta_{\ell}^{(m)} + S_{\ell}^{(m)}
\]

where \( S_{\ell}^{(m)} \) are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic \( \ell = 0 \) temperature perturbation will eventually become a high order anisotropy by “free streaming” or simple projection

- Original CMB codes solved the full hierarchy equations out to the \( \ell \) of interest.
Integral Solution

• Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface.

• In general, the solution describes the decomposition of the source $S_{\ell}^{(m)}$ with its local angular dependence as seen at a distance $D$.

• Proceed by decomposing the angular dependence of the plane wave

$$e^{i \mathbf{k} \cdot \mathbf{x}} = \sum_{\ell} (-i)^{\ell} \sqrt{4\pi (2\ell + 1)} j_{\ell}(kD) Y_{\ell}^{0}(\hat{n})$$

• Recouple to the local angular dependence of $G_{\ell}^{m}$

$$G_{\ell s}^{m} = \sum_{\ell} (-i)^{\ell} \sqrt{4\pi (2\ell + 1)} \alpha_{\ell s \ell}^{(m)}(kD) Y_{\ell}^{m}(\hat{n})$$
Integral Solution

- Projection kernels:

\[
\alpha_{\ell_s=0\ell}^{(m=0)} \equiv j_{\ell} \quad \alpha_{\ell_s=1\ell}^{(m=0)} \equiv j'_{\ell}
\]

- Integral solution:

\[
\frac{\Theta_{\ell}^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S_{\ell_s}^{(m)} \alpha_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))
\]

- Power spectrum:

\[
C_{\ell} = 4\pi \int \frac{dk}{k} \frac{k^3}{2\pi^2} \sum_m \frac{\langle \Theta_{\ell}^{(m)*} \Theta_{\ell}^{(m)} \rangle}{(2\ell + 1)^2}
\]

- Integration over an oscillatory radial source with finite width - suppression of wavelengths that are shorter than width leads to reduction in power by \(k \Delta\eta/\ell\) in the “Limber approximation”
Polarization Hierarchy

- In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hierarchy:

\[ \dot{E}_\ell^{(m)} = k \left[ \frac{2 \kappa_\ell^m}{2\ell - 1} E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell + 1)} B_\ell^{(m)} - \frac{2 \kappa_{\ell+1}^m}{2\ell + 3} E_{\ell+1}^{(m)} \right] - \dot{\tau} E_\ell^{(m)} + \mathcal{E}_\ell^{(m)} \]

\[ \dot{B}_\ell^{(m)} = k \left[ \frac{2 \kappa_\ell^m}{2\ell - 1} B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell + 1)} E_\ell^{(m)} - \frac{2 \kappa_{\ell+1}^m}{2\ell + 3} B_{\ell+1}^{(m)} \right] - \dot{\tau} B_\ell^{(m)} + \mathcal{B}_\ell^{(m)} \]

where \(2 \kappa_\ell^m = \sqrt{(\ell^2 - m^2)(\ell^2 - 4)/\ell^2}\) is given by the Clebsch-Gordon coefficients and \(\mathcal{E}, \mathcal{B}\) are the sources (scattering only).

- Note that for vectors and tensors \(|m| > 0\) and \(B\) modes may be generated from \(E\) modes by projection. Cosmologically \(\mathcal{B}_\ell^{(m)} = 0\)
Polarization Integral Solution

- Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

\[
\frac{E_{\ell}^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \epsilon_{\ell_s\ell}^{(m)} (k(\eta_0 - \eta))
\]

- Power spectrum \(XY = \Theta\Theta, \Theta E, EE, BB:\)

\[
C_{\ell}^{XY} = 4\pi \int \frac{dk}{k} \frac{k^3}{2\pi^2} \sum_m \frac{\langle X_{\ell}^{(m)*} Y_{\ell}^{(m)} \rangle}{(2\ell + 1)^2}
\]

- We shall see that the only sources of temperature anisotropy are \(\ell = 0, 1, 2\) and polarization anisotropy \(\ell = 2\)

- In the basis of \(\hat{z} = \hat{k}\) there are only \(m = 0, \pm 1, \pm 2\) or scalar, vector and tensor components
Polarization Sources

$l=2, m=0$

$l=2, m=1$

$l=2, m=2$
Polarization Transfer

- A polarization source function with $\ell = 2$, modulated with plane wave orbital angular momentum
- Scalars have no $B$ mode contribution, vectors mostly $B$ and tensor comparable $B$ and $E$

(a) Polarization Pattern
(b) Multipole Power
Polarization Transfer

- Radial mode functions characterize the projection from $k \rightarrow \ell$ or inhomogeneity to anisotropy

- Compared to the scalar monopole source:
  - scalar dipole source very broad
  - tensor quadrupole, sharper
  - scalar $E$ polarization, sharper
  - tensor $E$ polarization, broad
  - tensor $B$ polarization, very broad

- These properties determine whether features in the $k$-mode spectrum, e.g. acoustic oscillations, intrinsic structure, survive in the anisotropy