KIPMU

Set 3: Inflation

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Single Field Inflation

- In single field inflation, there is a single clock to determine how inflation proceeds, think of this as the value of a scalar field ϕ
- Quantum fluctuations in this field just advance or retard the clock in different regions of space
- Choosing a gauge where scalar field is unperturbed (comoving gauge to leading order) is a preferred hypersurface - called constant field or unitary gauge

$$\phi(\mathbf{x}, t_u) = \phi_0(t_u)$$

- Given that by assumption the universe is dominated by this scalar field and it is homogenous in this frame, the only thing that the action can be built out of is terms that depend on t_u
- In EFT language, write down all possible terms that is consistent with unbroken spatial diffeomorphism invariance in this slicing

Effective Field Theory

- In unitary gauge, there is only the metric to work with. In general it transforms as a tensor

$$\tilde{g}^{\mu\nu}(\tilde{t}, \tilde{x}^i) = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} g^{\alpha\beta}(t, x^i)$$

- Note: to stay close to the inflationary literature, 0 will represent coordinate rather than conformal time
- Consider the restricted set of gauge transformations that change only the spatial coordinates

$$\tilde{x}^i = x^i + L^i; \quad \tilde{t} = t$$

- Only component that is left invariant under this transformation is g^{00}; g_{00} is not invariant if L^i depends on t.
- So the most general action is the most general function of g^{00}

Effective Field Theory

- Now consider that $g_u^{00}+1$ is a small metric perturbation. A general function may be expanded around this value in a Taylor series

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}M_{\rm Pl}^2 R + \sum_{n=0}^{\infty}\frac{1}{n!}M_n^4(t_u)(g_u^{00}+1)^n\Big]$$

- Varying action with respect to $g^{\mu\nu}$ we get the Einstein equations
- Constant term gives a cosmological constant whereas the $n=1$ term gives the effective stress tensor of the field in the background

$$H^2 = -\frac{1}{3M_{\rm Pl}^2}\left[M_0^4 + 2M_1^4\right]$$

$$\dot{H} + H^2 = -\frac{1}{3M_{\rm Pl}^2}\left[M_0^4 - M_1^4\right]$$

K-inflation $P(X, \phi)$

- EFT was built to cover the case of scalar field Lagrangian that is a general function of its kinetic term and field value

$$\mathcal{L} = P(X, \phi)$$

where $2X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

- For a canonical scalar $P = X - V(\phi)$
- The connection to EFT goes through defining the parameters

$$M_n^4 = (-X)^n \frac{\partial^n P}{\partial X^n}$$

$M_0^4 = P$, $M_1^4 = -XP_{,X}$ and $n = 2$ defines the sound speed of field fluctuations

$$c_s^{-2} = 1 + \frac{2XP_{,XX}}{P_{,X}}$$

DBI

- An example coming from string inspired models is the Dirac-Born-Infeld Lagrangian

$$\mathcal{L} = \left[1 - \sqrt{1 - 2X/T(\phi)}\right] T(\phi) - V(\phi),$$

  where $T(\phi)$ is the warped brane tension and ϕ denotes the position of the brane in a higher dimension

- If $X/T \ll 1$ then $\mathcal{L} = X - V$, the same as a canonical scalar field

Effective Field Theory

- Friedmann equation thus associates $n = 0, 1$ with energy density and pressure

$$
\begin{aligned}
M_0^4 &= -(3H^2 + 2\dot{H})M_{\rm Pl}^2 \\
M_1^4 &= \dot{H} M_{\rm Pl}^2
\end{aligned}
$$

- Now we an restore time slicing invariance or temporal diffs allowing for a general change in the time coordinate

$$
t_u = t + \pi(t, x^i)
$$

- In particle physics language this is the Stuckelberg trick and π is a Stuckelberg field.

- To connect with the canonical scalar field treatment the field fluctuation

$$
\phi(t, x^i) = \phi_0(t) + \phi_1(t, x^i),
$$

Effective Field Theory

- Scalar field transforms as scalar field

$$\tilde{\phi}_1 = \phi_1 - \dot{\phi}_0 \pi$$

- To get to unitary slicing $\tilde{\phi}_1 = 0$, so in alternate slicing

$$\phi_1 = \dot{\phi}_0 \pi$$

- Likewise the curvature perturbations are related by

$$\mathcal{R} = H_L + \frac{H_T}{3} - \frac{\dot{a}}{a}\pi$$

- Transformation particularly simple from a spatially flat slicing where $H_L + H_T/3 = 0$, i.e. spatially unperturbed metric

$$\mathcal{R} = -\frac{\dot{a}}{a}\frac{\phi_1}{\dot{\phi}_0} = -\frac{\dot{a}}{a}\pi$$

- Stuckelberg field is the unitary (comoving) gauge curvature

Effective Field Theory

- Each $M_n^4(t_u = t + \pi)$ and hence carry extra Taylor expansion terms, these can be considered as inflationary "features" below and are ignored in slow-roll
- Transformation to arbitrary slicing is given by

$$g_u^{00} = \frac{\partial t_u}{\partial x^\mu}\frac{\partial t_u}{\partial x^\nu} g^{\mu\nu}$$

- In general, transformation mixes π and metric fluctuations $\delta g^{\mu\nu}$ including terms like

$$\dot\pi \delta g^{00}, \quad \delta g \dot\pi, \quad \partial_i \pi g^{0i}, \quad \partial_i \pi \partial_j \pi \delta g^{ij}$$

in the canonical linear theory calculation, the first three are $\dot A$, $\dot H_L$, kB terms after integration by parts and the last is cubic order

Effective Field Theory

- Again we make use of the fact that sub horizon scales these metric terms are subdominant
- In spatially flat gauge the domain of validity extends even through the horizon if we neglect slow roll corrections
- In this case we can ignore the terms associated with the spatial pieces of the metric and replace

$$g_u^{00} = -(1+\dot{\pi})^2 + \frac{(\partial_i \pi)^2}{a^2}$$

- Each $g_u^{00} + 1$ factor carries terms that are linear and quadratic in π

$$(g_u^{00} + 1)^n = (-\dot{\pi})^n \sum_{i=0}^{n} \frac{2^{n-i} n!}{i!(n-i)!} \Pi^i$$

Effective Field Theory

- So each M_n^4 term contributes from π^n to π^{2n}

$$\Pi = \dot\pi \left(1 - \frac{(\partial_i \pi)^2}{a^2 \dot\pi^2}\right)$$

- For example M_2

$$\begin{aligned}(g_u^{00}+1)^2 &= \dot\pi^2 \left[4 + 4\dot\pi \left(1 - \frac{(\partial_i \pi)^2}{a^2 \dot\pi^2}\right) + \dot\pi^2 \left(1 - \frac{(\partial_i \pi)^2}{a^2 \dot\pi^2}\right)^2\right] \\ &= 4(\dot\pi^2 + \dot\pi^3 - \dot\pi \frac{(\partial_i \pi)^2}{a^2}) + \ldots\end{aligned}$$

implies both a cubic and quartic Lagrangian. To cubic order

$$S_\pi = \int d^4x \sqrt{-g}\Big[- M_{\rm Pl}^2 \dot H \left(\dot\pi^2 - \frac{(\partial_i \pi)^2}{a^2}\right) + 2M_2^4(\dot\pi^2 + \dot\pi^3 - \dot\pi \frac{(\partial_i \pi)^2}{a^2}) + \ldots$$

Effective Field Theory

- Isolate the quadratic action

$$S_{\pi 2} = \int d^4x \sqrt{-g}\left[(-M_{\rm Pl}^2 \dot H + 2M_2^4)\dot\pi^2 + M_{\rm Pl}^2 \dot H \frac{(\partial_i \pi)^2}{a^2}\right]$$

and identify the sound speed from $\omega = (k/a)c_s$

$$c_s^{-2} = 1 - \frac{2M_2^4}{M_{\rm Pl}^2 \dot H}; \quad \Pi \sim \dot\pi\left(1 - \frac{1}{c_s^2}\right)$$

using $-\dot H = \epsilon H^2$

$$\begin{aligned} S_{\pi 2} &= \int dt d^3x (a^3 \epsilon H^2) M_{\rm Pl}^2 c_s^{-2}\left[\dot\pi^2 - c_s^2 \frac{(\partial_i \pi)^2}{a^2}\right] \\ &= \int d\eta d^3x \frac{z^2 H^2 M_{\rm Pl}^2}{2}\left[\left(\frac{\partial \pi}{\partial \eta}\right)^2 - c_s^2 (\partial_i \pi)^2\right] \end{aligned}$$

where $z^2 = 2a^2\epsilon/c_s^2$ is the generalization of Mukhanov z

Effective Field Theory

- So a field redefinition canonically normalizes the field

$$u = zH\pi M_{\rm Pl}$$

brings the EFT action to canonical form (assuming M_n^4 =const.)

$$\begin{aligned} S_u &= \int d\eta d^3x \left[\left(\frac{\partial u}{\partial \eta}\right)^2 - c_s^2 (\partial_i u)^2 - 2u \frac{\partial u}{\partial \eta}\frac{d \ln z}{d\eta} + u^2 \left(\frac{d\ln z}{d\eta}\right)^2 \right] \\ &= \int d\eta d^3x \left[\left(\frac{\partial u}{\partial \eta}\right)^2 - c_s^2 (\partial_i u)^2 + \frac{u^2}{z}\frac{d^2 z}{d\eta^2} \right] \end{aligned}$$

which is the generalization of the u field of canonical inflation

- The equation of motion of this quadratic Lagrangian is

$$\frac{\partial^2 u}{\partial \eta^2} + c_s^2 k^2 u - \frac{1}{z}\frac{d^2 z}{d\eta^2} u = 0$$

Effective Field Theory

- With ϵ=const. and c_s =const, $z \propto a$

$$\frac{1}{z}\frac{d^2 z}{d\eta^2} \approx 2(aH)^2$$

- Quantize this field assuming Bunch-Davies vacuum; $1/\sqrt{E}$ normalization factor goes to $1/\sqrt{kc_s}$ yielding modefunction

$$u = \frac{1}{\sqrt{2kc_s}}\left(1 + \frac{i}{ks}\right)e^{iks}$$

where the sound horizon is

$$s = \int_a^{a_{\rm end}} d\ln a \frac{c_s}{aH} \approx \frac{c_s}{aH}$$

Effective Field Theory

- Curvature fluctuations then freezeout at $ks \ll 1$ (sound horizon crossing) at a value

$$\mathcal{R} = -H\pi = -\frac{c_s}{a\sqrt{2\epsilon}}u = -\frac{c_s}{a\sqrt{2\epsilon}}\frac{1}{\sqrt{2kc_s}}\frac{iaH}{kc_s M_{\rm Pl}} \approx \frac{-iH}{2k^{3/2}\sqrt{\epsilon c_s}M_{\rm Pl}}$$

- So

$$\Delta^2_{\mathcal{R}} = \frac{k^3|\mathcal{R}|^2}{2\pi^2} = \frac{H^2}{8\pi^2\epsilon c_s M^2_{\rm Pl}}$$

- Scale invariant to the extent that time translation invariance is exact
- Tilt comes from taking

$$\frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln k} \equiv n_S - 1 = 2\frac{d\ln H}{d\ln k} - \frac{d\ln\epsilon}{d\ln k} - \frac{d\ln c_s}{d\ln k}$$

Tilt

- Evaluate at horizon crossing where fluctuation freezes $k = aH$

$$\frac{d}{d\ln k} \approx \frac{d}{d\ln a}$$

- So define additional parameters for the evolution

$$\begin{aligned}
\frac{d\ln H}{d\ln a} &= -\epsilon \\
\frac{d\ln \epsilon_H}{d\ln a} &= 2(\epsilon + \delta_1) \\
\frac{d\ln c_s}{d\ln a} &= \sigma_1
\end{aligned}$$

to obtain

$$n_S - 1 = -2\epsilon - 2(\epsilon + \delta_1) - \sigma_1 = -4\epsilon - 2\delta_1 - \sigma_1$$

Non Gaussianity

- Returning to the original π action, since M_2^4 carries cubic term this requires a non-Gaussianity

$$S_\pi = \int d^4x\sqrt{-g}\left[-\frac{M_{\rm Pl}^2\dot H}{c_s^2}\left(\dot\pi^2 + c_s^2\frac{(\partial_i\pi)^2}{a^2}\right) + M_{\rm Pl}^2\dot H\left(1-\frac{1}{c_s^2}\right)\left(\dot\pi^3 - \dot\pi\frac{(\partial_i\pi)^2}{a^2}\right)\right] + \ldots$$

- For $c_s \ll 1$, spatial gradients dominate temporal derivatives

$$\partial_0 \to \omega, \partial_i \to k, \omega = kc_s/a$$

and leading order cubic term is $\dot\pi(\partial_i\pi)^2$

- Estimate the size of the non-Gaussianity by taking the ratio of cubic to quadratic at $c_s \ll 1$

$$\frac{\dot\pi(\partial_i\pi)^2}{a^2\dot\pi^2} \sim \frac{k\pi_{\rm rms}}{c_s a} \qquad \text{where} \quad \pi_{\rm rms} = \left(\frac{k^3|\pi|^2}{2\pi^2}\right)^{1/2}$$

Non Gaussianity

- Deep within the horizon $u = 1/\sqrt{2kc_s}$ and so

$$\begin{aligned}
\frac{k\pi_{\rm rms}}{c_s a} &\sim \frac{k}{c_s a}\left(\frac{k^2}{2z^2H^2c_sM_{\rm Pl}^2}\right)^{1/2} \\
&\sim \left(\frac{kc_s}{aH}\right)^2 \frac{H}{M_{\rm Pl}\sqrt{\epsilon c_s}}\frac{1}{c_s^2} \\
&\sim \left(\frac{kc_s}{aH}\right)^2 \frac{\Delta_{\cal R}}{c_s^2} < 1
\end{aligned}$$

- Since $kc_s/aH \sim \omega/H$ is a ratio of an energy scale to Hubble, the bound determines the strong coupling scale

$$\frac{\omega_{sc}}{H} \sim \frac{c_s}{\sqrt{\Delta_{\cal R}}} \sim 10^2 c_s$$

- For $c_s < 0.01$ the strong coupling scale is near the horizon and the effective theory has broken down before freezeout

Non Gaussianity

- Now consider a less extreme c_s
- Here the effective theory becomes valid at least several efolds before horizon crossing and we can make predictions within the theory
- Not surprisingly non-Gaussianity is enhanced by these self interactions and freezeout at $kc_s \sim aH$

$$
\begin{aligned}
\frac{k\pi_{\rm rms}}{c_s a} &\sim \frac{k}{c_s a}\left(\frac{1}{\epsilon c_s M_{\rm Pl}^2}\right)^{1/2} \\
&\sim \frac{kc_s}{aH}\frac{H}{\sqrt{\epsilon c_s}M_{\rm Pl}}\frac{1}{c_s^2} \\
&\sim \frac{\Delta_{\cal R}}{c_s^2}
\end{aligned}
$$

and so bispectrum is enhanced over the naive expectation by c_s^{-2}

Non Gaussianity

- More generally, each M_n^4 sets its own strong coupling scale

$$\frac{\mathcal{L}_n}{\mathcal{L}_2} \sim 1$$

These coincide if

$$\frac{M_n^4}{M_2^4} \sim \left(\frac{1}{c_s^2}\right)^{n-2}$$

which would be the natural prediction if the M_2 strong coupling scale indicated the scale of new physics and we take all allowed operators as order unity at that scale

DBI

- Here

$$c_s(\phi, X) = \sqrt{1 - 2X/T(\phi)}$$

and

$$c_n = (-1)^n \frac{M_n^4}{M_2^4} = \frac{(2n-3)!!}{2^{n-2}} \left(\frac{1}{c_s^2} - 1 \right)^{n-2}$$

satisfying the c_s scaling of the EFT prescription

Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation $(K = 0)$, same as scalar field

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0\,.$$

- Acquires quantum fluctuations in same manner as ϕ. Lagrangian sets the normalization

$$\phi_1 \rightarrow H_T^{(\pm 2)}\sqrt{\frac{3}{16\pi G}}$$

- Scale-invariant gravitational wave amplitude converted back to $+$ and $\times$ states $H_T^{(\pm 2)} = -(h_+ \mp i h_\times)/\sqrt{6})$

$$\Delta^2_{+,\times} = 16\pi G \Delta^2_{\phi_1} = 16\pi G \frac{H^2}{(2\pi)^2}$$

Gravitational Waves

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation
- Tensor-scalar ratio - various definitions - WMAP standard is

$$r \equiv 4\frac{\Delta_+^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon c_s$$

- Tensor tilt:

$$\frac{d\ln\Delta_H^2}{d\ln k} \equiv n_T = 2\frac{d\ln H}{d\ln k} = -2\epsilon$$

Gravitational Waves

- Consistency relation between tensor-scalar ratio and tensor tilt

$$r = 16\epsilon c_s = -8 n_T c_s$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparison of tensor-scalar ratio and tensor tilt tests consistency of canonical $c_s = 1$ inflation and measures sound speed
- B modes formed as photons propagate – the spatial variation in the plane waves modulate the signal: described by Boltzmann eqn.

$$\Delta B_{\rm peak} \approx 0.024 \left(\frac{E_i}{10^{16} {\rm GeV}} \right)^2 \mu{\rm K}$$

Super Planckian Roll

- The larger ϵ is the more the field rolls in an e-fold

$$\epsilon = \frac{1}{2c_s M_{\rm Pl}^2}\left(\frac{d\phi_0}{dN}\right)^2 = \frac{r}{16c_s}$$

- Observable scales span $\Delta N \sim 5$ so

$$\Delta\phi_0 \approx 5\frac{d\phi}{dN} = 5(r/8)^{1/2}M_{\rm pl} \approx 0.6(r/0.1)^{1/2}M_{\rm pl}$$

- Field must roll a super Planckian distance over the ~ 60 efolds of inflation
- Does this make sense as an effective field theory? Lyth (1997)
- Models like Monodromy are explicitly constructed to protect such a long flat potential by invoking an underlying shift symmetry
- Like axion models, results in oscillatory potential

Generalized Slow Roll

- The slow roll derivation above assumes that the quantities ϵ, δ_1 σ_1 are nearly constant as well as small
- This is not required for having sufficient inflation, only $\epsilon \ll 1$. δ_1 and σ_1 can be transiently large
- For example in Monodromy the slow roll potential has axion-like oscillations whose frequency can be large
- Or large scale anomalies in the spectrum could indicate a transient fast roll period
- What happens when we relax these conditions?

Generalized Slow Roll

- Define for convenience $y = \sqrt{2kc_s}u$ and use the sound horizon as the temporal coordinate $x = ks$

$$s = \int_a^{a_{\rm end}} d\ln a \frac{c_s}{aH}$$

and take $' = d/d\ln s = d/d\ln x$

- Mukhanov-Sasaki equation of motion becomes

$$\frac{d^2y}{dx^2} + \left(1 - \frac{2}{x^2}\right) y = \left(\frac{f'' - 3f'}{f}\right)\frac{y}{x^2}$$

$$f^2 = 8\pi^2 M_{\rm Pl}^2 \frac{\epsilon c_s}{H^2}\left(\frac{aHs}{c_s}\right)^2 .$$

- RHS encapsulates the time variation of the slow roll parameters and setting it to zero yields the ordinary slow roll approximation

Generalized Slow Roll

- Generalized slow roll approximation exploits this fact by taking an iterative Green function approach
- LHS "homogeneous" equation is solved by

$$y_0(x) = \left(1 + \frac{i}{x}\right) e^{ix}$$

  so approximating RHS with $y \approx y_0$ yields an external source
- Solve iteratively with Green function techniques
- Leading order solution

$$\ln \Delta^2_{\mathcal{R}} = -\int_0^\infty \frac{dx}{x} W'(x) G(\ln x)$$

  where $-\int_0^\infty d\ln x W'(x) = 1$ and determines freezeout

$$W(x) = \frac{3\sin(2x)}{2x^3} - \frac{3\cos(2x)}{x^2} - \frac{3\sin(2x)}{2x}$$

Generalized Slow Roll

- Source function is

$$G(\ln x) = -2\ln f + \frac{2}{3}(\ln f)'$$

  and corrections to the slow-roll approximation follow from Taylor expanding $G(\ln x) = G(0) + G'(0)\ln x + \ldots$

- In general, slow variations freeze out at $\ln x \approx 1.06$ and variations faster than $\Delta \ln x = 1$ cause ringing or oscillations in the power spectrum

Canonical Scalar Fields

- Supplemental notes more fully working out the canonical scalar field case
- Establish the relationship between the Mukhanov-Sasaki equation and the general perturbation equations from previous set.
- In what follows $\mu = 0$ is conformal time again.

Canonical Scalar Fields

- A canonical scalar field is described by the action

$$S_\phi = \int d^4x \sqrt{-g}[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]$$

- Varying the action with respect to the metric

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\mathcal{L}_\phi$$

gives the stress-energy tensor of a scalar field

$$T^\mu{}_\nu = \nabla^\mu\phi\,\nabla_\nu\phi - \frac{1}{2}(\nabla^\alpha\phi\,\nabla_\alpha\phi + 2V)\delta^\mu{}_\nu\,.$$

- Equations of motion $\nabla^\mu T_{\mu\nu} = 0$ with closure relations for $p(\phi, \partial_\mu\phi)$, $\Pi(\phi, \partial_\mu\phi)$ or field equation $\nabla_\mu\nabla^\mu\phi = V'$ (vary with respect to ϕ)

Canonical Scalar Fields

- For the background $\langle\phi\rangle \equiv \phi_0$ (a^{-2} from conformal time)

$$\rho_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + V, \quad p_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 - V$$

- So for kinetic dominated $w_\phi = p_\phi/\rho_\phi \to 1$
- And potential dominated $w_\phi = p_\phi/\rho_\phi \to -1$
- A slowly rolling (potential dominated) scalar field accelerates expansion
- Can use general equations of motion of dictated by stress energy conservation

$$\dot{\rho}_\phi = -3(\rho_\phi + p_\phi)\frac{\dot{a}}{a}\,,$$

to obtain the equation of motion of the background field ϕ

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2 V' = 0\,,$$

Equation of Motion

- In terms of time instead of conformal time
$$\frac{d^2\phi_0}{dt^2} + 3H\frac{d\phi_0}{dt} + V' = 0$$
- Field rolls down potential hill but experiences “Hubble friction” to create slow roll. In slow roll $3H d\phi_0/dt \approx -V'$ and so kinetic energy is determined by field position $\rightarrow$ adiabatic – both kinetic and potential energy determined by single degree of freedom ϕ_0

Equation of Motion

- Likewise for the perturbations $\phi = \phi_0 + \phi_1$

$$\begin{aligned}
\delta\rho_\phi &= a^{-2}(\dot\phi_0\dot\phi_1 - \dot\phi_0^2 A) + V'\phi_1 \,, \\
\delta p_\phi &= a^{-2}(\dot\phi_0\dot\phi_1 - \dot\phi_0^2 A) - V'\phi_1 \,, \\
(\rho_\phi + p_\phi)(v_\phi - B) &= a^{-2}k\dot\phi_0\phi_1 \,, \\
p_\phi\pi_\phi &= 0 \,,
\end{aligned}$$

- For comoving slicing where $v_\phi = B$

$$\phi_1 = 0$$

and the field is spatially unperturbed - so all the dynamics are in the metric

Sound Speed

- In this slicing $\delta p_\phi = \delta\rho_\phi$ so the sound speed is $\delta p_\phi/\delta\rho_\phi = 1$.
- More generally the sound speed of the inflation is defined as the speed at which field fluctuations propagate - i.e. the kinetic piece to the energy density rather than the $V'\phi_1$ potential piece - much like in the background the $+1$ and -1 pieces of w.
- Non canonical kinetic terms– k-essence, DBI inflation – can generate $c_s \neq 1$ as do terms in the effective theory of inflation

Equation of Motion

- Scalar field fluctuations are stable inside the horizon and are a good candidate for the smooth dark energy
- Equivalently, conservation equations imply

$$\ddot{\phi}_1 = -2\frac{\dot{a}}{a}\dot{\phi}_1 - (k^2 + a^2 V'')\phi_1 + (\dot{A} - 3\dot{H}_L - kB)\dot{\phi}_0 - 2Aa^2V' \,.$$

- Alternately this follows from perturbing the Klein Gordon equation $\nabla_\mu \nabla^\mu \phi = V'$

Inflationary Perturbations

- Classical equations of motion for scalar field inflaton determine the evolution of scalar field fluctuations generated by quantum fluctuations
- Since the curvature $\mathcal{R}$ on comoving slicing is conserved in the absence of stress fluctuations (i.e. outside the apparent horizon, calculate this and we're done no matter what happens in between inflation and the late universe (reheating etc.)
- But in the comoving slicing $\phi_1 = 0$! no scalar-field perturbation
- Solution: solve the scalar field equation in the dual gauge where the curvature $H_L + H_T/3 = 0$ ("spatially flat" slicing) and transform the result to comoving slicing

Transformation to Comoving Slicing

- Scalar field transforms as scalar field

$$\tilde{\phi}_1 = \phi_1 - \dot{\phi}_0 T$$

- To get to comoving slicing $\tilde{\phi}_1 = 0$, $T = \phi_1/\dot{\phi}_0$, and $\tilde{H}_T = H_T + kL$ so

$$\mathcal{R} = H_L + \frac{H_T}{3} - \frac{\dot{a}}{a}\frac{\phi_1}{\dot{\phi}_0}$$

- Transformation particularly simple from a spatially flat slicing where $H_L + H_T/3 = 0$, i.e. spatially unperturbed metric

$$\mathcal{R} = -\frac{\dot{a}}{a}\frac{\phi_1}{\dot{\phi}_0}$$

Spatially Flat Gauge

- Spatially Flat (flat slicing, isotropic threading):

$$\begin{aligned}
\tilde{H}_L + \tilde{H}_T/3 &= \tilde{H}_T = 0 \\
A_f &= \tilde{A}\,, B_f = \tilde{B} \\
T &= \left(\frac{\dot{a}}{a}\right)^{-1}\left(H_L + \frac{1}{3}H_T\right) \\
L &= -H_T/k
\end{aligned}$$

- Einstein Poisson and Momentum

$$\begin{aligned}
-3(\frac{\dot{a}}{a})^2 A_f + \frac{\dot{a}}{a} k B_f &= 4\pi G a^2 \delta\rho_\phi\,, \\
\frac{\dot{a}}{a} A_f - \frac{K}{k^2}(k B_f) &= 4\pi G a^2 (\rho_\phi + p_\phi)(v_\phi - B_f)/k\,,
\end{aligned}$$

- Conservation

$$\ddot{\phi}_1 = -2\frac{\dot{a}}{a}\dot{\phi}_1 - (k^2 + a^2 V'')\phi_1 + (\dot{A}_f - k B_f)\dot{\phi}_0 - 2A_f a^2 V'\,.$$

Spatially Flat Gauge

- For modes where $|k^2/K| \gg 1$ we obtain

$$\frac{\dot a}{a} A_f = 4\pi G a^2 \dot\phi_0 \phi_1 \,,$$

$$\frac{\dot a}{a} k B_f = 4\pi G [\dot\phi_0 \dot\phi_1 - \dot\phi_0^2 A_f + a^2 V' \phi_1 + 3\frac{\dot a}{a}\dot\phi_0 \phi_1]$$

so combining $\dot A_f - k B_f$ eliminates the $\dot\phi_1$ term

- The metric source to the scalar field equation can be reexpressed in terms of the field perturbation and background quantities

$$(\dot A_f - k B_f)\dot\phi_0 - 2 A_f a^2 V' - a^2 V'' \phi_1 = f(\eta)\phi_1$$

- Single closed form 2nd order ODE for ϕ_1

Mukhanov-Sasaki Equation

- Equation resembles a damped oscillator equation with a particular dispersion relation

$$\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + [k^2 + f(\eta)]\phi_1$$

- $f(\eta)$ involves terms with $\dot{\phi}_0$, V', V'' implying that for a sufficiently flat potential $f(\eta)$ represents a small correction
- Transform out the background expansion $u \equiv a\phi_1$

$$\dot{u} = \dot{a}\phi + a\dot{\phi}$$

$$\ddot{u} = \ddot{a}\phi_1 + 2\dot{a}\dot{\phi}_1 + a\ddot{\phi}_1$$

$$\ddot{u} + [k^2 - \frac{\ddot{a}}{a} + f(\eta)]u = 0$$

- Note Friedmann equations say if $p = -\rho$, $\ddot{a}/a = 2(\dot{a}/a)^2$

Mukhanov-Sasaki Equation

- Using the background Einstein and scalar field equations, this source term can be expressed in a surprisingly compact form

$$\ddot{u} + [k^2 - \frac{\ddot{z}}{z}]u = 0$$

- and

$$z \equiv \frac{a\dot{\phi}_0}{\dot{a}/a}$$

- This equation is sometimes called the "Mukhanov Equation" and is both exact (in linear theory) and compact
- For large k (subhorizon), this looks like a free oscillator equation which can be quantized
- Let's examine the relationship between z and the slow roll parameters

Slow Roll Parameters

- Rewrite equations of motion in terms of slow roll parameters but do not require them to be small or constant.
- Deviation from de Sitter expansion

$$\begin{aligned}\epsilon &\equiv \frac{3}{2}(1+w_\phi) \\ &= \frac{\frac{3}{2}(d\phi_0/dt)^2/V}{1+\frac{1}{2}(d\phi_0/dt)^2/V}\end{aligned}$$

- Deviation from overdamped limit of $d^2\phi_0/dt^2 = 0$

$$\begin{aligned}\delta_1 &\equiv \frac{d^2\phi_0/dt^2}{H d\phi_0/dt}(= -\eta_H) \\ &= \frac{\ddot{\phi}_0}{\dot{\phi}_0}\left(\frac{\dot{a}}{a}\right)^{-1} - 1\end{aligned}$$

Slow Roll Parameters

- Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = 4\pi G \dot{\phi}_0^2 \epsilon^{-1}$$

$$\frac{d}{d\eta}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\dot{a}}{a}\right)^2 (1-\epsilon)$$

Take derivative of first equation, divide through by $(\dot{a}/a)^2$

$$2\frac{\dot{a}}{a}(1-\epsilon) = 2\frac{\ddot{\phi}_0}{\dot{\phi}_0} - \frac{\dot{\epsilon}}{\epsilon}$$

- Replace $\ddot{\phi}_0$ with δ_1

$$\dot{\epsilon} = 2\epsilon(\delta_1 + \epsilon)\frac{\dot{a}}{a}$$

- Evolution of ϵ is second order in parameters

Slow Roll parameters

- Returning to the Mukhanov equation

$$\ddot{u} + [k^2 + g(\eta)]u = 0$$

where

$$\begin{aligned} g(\eta) &\equiv f(\eta) + \epsilon - 2 \\ &= -\left(\frac{\dot{a}}{a}\right)^2 [2 + 3\delta_1 + 2\epsilon + (\delta_1 + \epsilon)(\delta_1 + 2\epsilon)] - \frac{\dot{a}}{a}\dot{\delta}_1 \\ &= -\frac{\ddot{z}}{z} \end{aligned}$$

and recall

$$z \equiv a \left(\frac{\dot{a}}{a}\right)^{-1} \dot{\phi}_0$$

Slow Roll Limit

- Slow roll $\epsilon \ll 1$, $\delta_1 \ll 1$, $\dot{\delta}_1 \ll \frac{\dot{a}}{a}$

$$\ddot{u} + [k^2 - 2\left(\frac{\dot{a}}{a}\right)^2]u = 0$$

or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\rm end}$$
$$\tilde{\eta} = \int_{a_{\rm end}}^{a} \frac{da}{Ha^2} \approx -\frac{1}{aH}$$

- Compact, slow-roll equation:

$$\ddot{u} + [k^2 - \frac{2}{\tilde{\eta}^2}]u = 0$$

Quantum Fluctuations

- Simple harmonic oscillator $\ll$ Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k, \tilde{\eta})\hat{a} + u^*(k, \tilde{\eta})\hat{a}^\dagger$$

where $u(k, \tilde{\eta})$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \qquad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\tilde{\eta}] = i$

$$u(k, \eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

Quantum Fluctuations

- Zero point fluctuations of ground state

$$
\begin{aligned}
\langle u^2 \rangle &= \langle 0 | u^\dagger u | 0 \rangle \\
&= \langle 0 | (u^* \hat{a}^\dagger + u \hat{a})(u \hat{a} + u^* \hat{a}^\dagger) | 0 \rangle \\
&= \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle | u(k, \tilde{\eta}) |^2 \\
&= \langle 0 | [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} | 0 \rangle | u(k, \tilde{\eta}) |^2 \\
&= | u(k, \tilde{\eta}) |^2 = \frac{1}{2k}
\end{aligned}
$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in.

Slow Roll Limit

- Classical equation of motion then has the exact solution

$$u = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\tilde{\eta}}\right) e^{-ik\tilde{\eta}}$$

- For $|k\tilde{\eta}| \ll 1$ (late times, $\gg$ Hubble length) fluctuation freezes in

$$\lim_{|k\tilde{\eta}|\to 0} u = -\frac{1}{\sqrt{2k}}\frac{i}{k\tilde{\eta}} \approx \frac{iHa}{\sqrt{2k^3}}$$

$$\phi_1 = \frac{iH}{\sqrt{2k^3}}$$

- Power spectrum of field fluctuations

$$\Delta^2_{\phi_1} = \frac{k^3|\phi_1|^2}{2\pi^2} = \frac{H^2}{(2\pi)^2}$$

Slow Roll Limit

- Recall $\mathcal{R} = -(\dot{a}/a)\phi_1/\dot{\phi}_0$ and slow roll says

$$\left(\frac{\dot{a}}{a}\right)^2 \frac{1}{\dot{\phi}_0^2} = \frac{8\pi G a^2 V}{3} \frac{3}{2a^2 V \epsilon} = \frac{4\pi G}{\epsilon}$$

  Thus the curvature power spectrum

$$\Delta_{\mathcal{R}}^2 = \frac{8\pi G}{2} \frac{H^2}{(2\pi)^2 \epsilon}$$

Tilt

- Curvature power spectrum is scale invariant to the extent that H is constant
- Scalar spectral index

$$
\begin{aligned}
\frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln k} &\equiv n_S - 1 \\
&= 2\frac{d\ln H}{d\ln k} - \frac{d\ln\epsilon}{d\ln k}
\end{aligned}
$$

- Evaluate at horizon crossing where fluctuation freezes

$$
\begin{aligned}
\frac{d\ln H}{d\ln k}\Big|_{-k\tilde\eta=1} &= \frac{k}{H}\frac{dH}{d\tilde\eta}\Big|_{-k\tilde\eta=1}\frac{d\tilde\eta}{dk}\Big|_{-k\tilde\eta=1} \\
&= \frac{k}{H}(-aH^2\epsilon)\Big|_{-k\tilde\eta=1}\frac{1}{k^2} = -\epsilon
\end{aligned}
$$

where $aH = -1/\tilde\eta = k$

Tilt

- Evolution of ϵ

$$\frac{d\ln\epsilon}{d\ln k} = -\frac{d\ln\epsilon}{d\ln\tilde{\eta}} = -2(\delta_1+\epsilon)\frac{\dot{a}}{a}\tilde{\eta} = 2(\delta_1+\epsilon)$$

- Tilt in the slow-roll approximation

$$n_S = 1 - 4\epsilon - 2\delta_1$$

Relationship to Potential

- To leading order in slow roll parameters

$$
\begin{aligned}
\epsilon &= \frac{\frac{3}{2}\dot\phi_0^2/a^2V}{1+\frac{1}{2}\dot\phi_0^2/a^2V} \\
&\approx \frac{3}{2}\dot\phi_0^2/a^2V \\
&\approx \frac{3}{2a^2V}\frac{a^4V'^2}{9(\dot a/a)^2}, \qquad (3\dot\phi_0\frac{\dot a}{a} = -a^2V') \\
&\approx \frac{1}{6}\frac{3}{8\pi G}\left(\frac{V'}{V}\right)^2, \qquad \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}a^2V \\
&\approx \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2
\end{aligned}
$$

so $\epsilon \ll 1$ is related to the first derivative of potential being small

Relationship to Potential

- And

$$\delta_1 = \frac{\ddot{\phi}_0}{\dot{\phi}_0}\left(\frac{\dot{a}}{a}\right)^{-1} - 1$$

$$(\dot{\phi}_0 \approx -a^2\left(\frac{\dot{a}}{a}\right)^{-1}\frac{V'}{3})$$

$$(\ddot{\phi}_0 \approx -\frac{a^2V'}{3}(1+\epsilon) + a^4\left(\frac{\dot{a}}{a}\right)^{-2}\frac{V'V''}{9})$$

$$\approx -\frac{1}{a^2V'/3}\left(-\frac{a^2V'}{3}(1+\epsilon) + \frac{a^2}{9}\frac{3}{8\pi G}\frac{V'V''}{V}\right) - 1 \quad \approx \epsilon - \frac{1}{8\pi G}\frac{V''}{V}$$

so δ_1 is related to second derivative of potential being small. Very flat potential.

Relationship to Potential

- Exact relations

$$\frac{1}{8\pi G}\left(\frac{V'}{V}\right)^2 = 2\epsilon\frac{(1+\delta_1/3)^2}{(1-\epsilon/3)^2}$$

$$\frac{1}{8\pi G}\frac{V''}{V} = \frac{\epsilon - \delta_1 - [\delta_1^2 - \epsilon\delta_1 - (a/\dot{a})\dot{\delta}_1]/3}{1-\epsilon_H/3}$$

  agree in the limit $\epsilon, |\delta_1| \ll 1$ and $|(a/\dot{a})\dot{\delta}_1| \ll \epsilon, |\delta_1|$

- Like the Mukhanov to slow roll simplification, identification with potential requires a constancy of δ_1 assumption