CMB Polarization Theory

Wayne Hu
Varenna, July 2017
Polarization Trinity

Isolating Three Cosmological Epochs
Polarization 4 Noble (Nobel?) Truths

And one integrated probe
Polarization 4 Noble (Nobel?) Truths

polarization is suffering... but cessation of suffering is nirvana
Polarized Landscape

Hu & Dodelson (2002)
CMB Power Spectra Measurements

See Burigana's lectures
Why is the CMB polarized?
Polarization from Thomson Scattering

- Differential cross section depends on polarization and angle

\[
\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\varepsilon}' \cdot \hat{\varepsilon}|^2 \sigma_T
\]
Polarization from Thomson Scattering

- Isotropic radiation scatters into unpolarized radiation
Polarization from Thomson Scattering

- Quadrupole anisotropies scatter into linear polarization

aligned with cold lobe
Whence Quadrupoles?

- Temperature inhomogeneities in a medium
- Photons arrive from different regions producing an anisotropy

(Scalar) Temperature Inhomogeneity
Whence Polarization Anisotropy?

- Observed photons scatter into the line of sight
- Polarization arises from the projection of the quadrupole on the transverse plane
E and B Modes
Polarization Multipoles

- Mathematically, a pattern is described by the tensor (spin-2) spherical harmonics [eigenfunctions of Laplacian on trace-free 2 tensor].
- Correspondence with scalar spherical harmonics established via Clebsch-Gordan coefficients (spin x orbital).
- Amplitude of the coefficients in the spherical harmonic expansion are the multipole moments; averaged square is the power.

![Amplitude varies along direction](image)

E-tensor harmonic

$l=2, m=0$
**E and B modes**

- **E-modes** are Stokes $Q$ polarization in wavenumber basis
- **B-modes** are Stokes $U$ polarization
E and B modes

- Superimposing wavevectors
- B-modes have handedness or odd parity
Modulation by Plane Wave

- Amplitude modulated by plane wave $\rightarrow$ higher multipole moments
- Direction determined by perturbation type $\rightarrow$ E-modes

edge on orientation dominates:
- nearly single $l$ per $k$
Polarization Peaks
A Catch-22

- Polarization is generated by scattering of anisotropic radiation
- Scattering isotropizes radiation
- Polarization only arises in \textit{optically thin conditions}: reionization and end of recombination
- Polarization fraction is at best a small fraction of the $10^{-5}$ anisotropy: $\sim 10^{-6}$ or $\mu$K in amplitude

Pros:
- Polarization
- Isolates
- Scattering
- Epoch
Acoustic Polarization

- Perfect fluid: no anisotropic stresses due to scattering isotropization; baryons and photons move as single fluid.
- Fluid imperfections are related to the mean free path of the photons in the baryons:

\[ \lambda_C = \dot{\tau}^{-1} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a \]

is the conformal opacity to Thomson scattering.
- Dissipation is related to the diffusion length: random walk approximation:

\[ \lambda_D = \sqrt{N} \lambda_C = \sqrt{\eta / \lambda_C} \lambda_C = \sqrt{\eta \lambda_C} \]

the geometric mean between the horizon and mean free path.
- \( \lambda_D / \eta_* \sim \text{few } \% \), so expect the peaks \( >3 \) to be affected by dissipation.
Viscosity & Heat Conduction

- Both fluid imperfections are related to the gradient of the velocity $k v_\gamma$ by opacity $\dot{\tau}$: slippage of fluids $v_\gamma - v_b$.

- **Viscosity** is an anisotropic stress or **quadrupole moment** formed by radiation **streaming** from hot to cold regions.
Back of the Envelope

- **Viscosity** = quadrupole anisotropy that follows the fluid velocity
  \[ \pi_\gamma \approx \frac{k}{\dot{\tau}} v_\gamma \]

- **Mean free path** related to the damping scale via the random walk
  \[ k_D = (\dot{\tau}/\eta_*)^{1/2} \rightarrow \dot{\tau} = k_D^2 \eta_* \]

- Damping scale at \( \ell \sim 1000 \) vs horizon scale at \( \ell \sim 100 \) so
  \[ k_D \eta_* \approx 10 \]

- Polarization **amplitude** rises to the damping scale to be \( \sim 10\% \) of anisotropy
  \[ \pi_\gamma \approx \frac{k}{k_D 10} v_\gamma \quad \Delta_P \approx \frac{\ell}{\ell_D 10} \Delta_T \]

- Polarization **phase** follows fluid velocity
Damping & Polarization

- Quadrupole moments:
  - **damp** acoustic oscillations from fluid viscosity
  - generates **polarization** from scattering

- Rise in polarization **power** coincides with fall in temperature power – \( l \approx 1000 \)
Acoustic Polarization

- Gradient of velocity is along direction of wavevector, so polarization is pure $E$-mode.
- Velocity is $90^\circ$ out of phase with temperature – turning points of oscillator are zero points of velocity:
  \[ \Theta + \Psi \propto \cos(ks); \quad v_\gamma \propto \sin(ks) \]
- Polarization peaks are at troughs of temperature power.
Cross Correlation

- Cross correlation of temperature and polarization

\[ (T)(\nu_\gamma) \propto \cos(ks) \sin(ks) \propto \sin(2ks) \]

- Oscillation at twice the frequency

- Correlation: radial or tangential around hot spots

- Partial correlation: easier to measure if polarization data is noisy

- Good check for systematics and foregrounds

- Comparison of temperature and polarization is proof against features in initial conditions mimicking acoustic features

- Polarization isolates scattering leading to reduced projection effects
Polarization and $H_0$

- **Shift to lower $H_0$** from changes in the shape of peaks indicating more CDM relative to radiation
- **Increased angular scale of sound horizon** compensated by larger distance to recombination through lower $H_0$
Polarization and $H_0$

- Residuals from the best fit $H_0 \sim 67\text{km/s/Mpc}$ $\Lambda$CDM solution
- High $H_0$ at $l < 1000$ driven by low $l$ anomaly
- Low $H_0$ at $l > 1000$ driven by smoother peaks from less radiation driving (and more lensing)

Aghanim et al (2016)

Obied et al (2017)
Polarization and $H_0$

- Polarization response to parameter shifts very sharp around first temperature peak: no intervening ISW sources, geometry of projection
- Powerful cross check in a different observable and scale
Transfer of Initial Power

(a) Temperature

(b) Polarization

Hu & Okamoto (2003)
Polarization and $H_0$

- TE residuals favor $H_0 \sim 67\text{km/s/Mpc}$ but at $l < 1000$
- As sensitive as all of TT
- Anomalous sensitivity from a $2\sigma$ outlier at $l \sim 165$ near the first polarization trough

Obied et al (2017)
Gravitational Waves
Gravitational Waves in Cosmology

• During deceleration epoch gravity waves are frozen outside the horizon
• Oscillate inside the horizon and decay or redshift as radiation
Quadrupoles from Gravitational Waves

- Changing transverse traceless distortion of space, aka gravitational waves, creates quadrupole CMB anisotropy.
- Gravitational waves are frozen when larger than the horizon and oscillate and decay as radiation inside horizon.
Quadrupoles from Gravitational Waves

- Transverse-traceless distortion provides temperature quadrupole
- Gravitational wave polarization picks out direction transverse to wavevector

transverse-traceless distortion
How do Scalars Differ?

- Temperature inhomogeneities in a medium
- Photons arrive from different regions producing an anisotropy

Azimuthally symmetric around wavevector
Gravitational Wave Pattern

- Projection of the quadrupole anisotropy gives polarization pattern
- Transverse polarization of gravitational waves breaks azimuthal symmetry
E and B modes

• **E-modes** are Stokes $Q$ polarization in wavenumber basis
• **B-modes** are Stokes $U$ polarization
Patterns and Perturbation Types

- Amplitude modulated by plane wave $\rightarrow$ Principle axis
- Direction determined by perturbation type $\rightarrow$ Polarization axis

Polarization Pattern

<table>
<thead>
<tr>
<th>Scalars</th>
<th>$\pi/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vectors</td>
<td>$\theta$</td>
</tr>
<tr>
<td>Tensors</td>
<td>$\pi/2$</td>
</tr>
</tbody>
</table>

Multipole Power

- $B/E=0$: Sharp transfer
- $B/E=6$
- $B/E=8/13$
Recombination B-Modes

- Rescattering of quadrupoles at recombination yield a peak in B-modes

![Graph showing power (μK^2) vs. l (labeled Temperature and until recently dominant constraint)]
Polarized Landscape

- Two scattering epochs: recombination and reionization leave two imprints on B-modes

![Graph showing the power spectrum of B-modes with labels for reionization B-bump, recombination B-peak, EE, BB, and lensing contaminant.](image-url)
Inflation
Gravitational Waves during Inflation

- During acceleration epoch gravity waves behave oppositely to deceleration epoch
- Oscillate inside the horizon and freeze when crossing horizon
Gravitational Waves

- Gravitational wave amplitude $h_{+,\times}$ satisfies same Klein-Gordon equation as scalars
- Just like inflaton $\phi$, quantum fluctuations freeze out at horizon crossing with power per $\ln k$ given by the Hubble scale $H$

$$
\Delta^2_{\delta\phi} = \frac{H^2}{(2\pi)^2}; \quad \Delta^2_{+,\times} = \frac{2}{M^2_{pl}} \frac{H^2}{(2\pi)^2}
$$

- By the Friedmann equation

$$
H^2 = \frac{\rho}{3M^2_{pl}} \approx \frac{V(\phi)}{3M^2_{pl}}
$$

Measurement of $B$-modes determines energy scale $E_i = V^{1/4}$

$$
B_{\text{peak}} \approx 0.024 \left( \frac{E_i}{10^{16}\text{GeV}} \right)^2 \mu\text{K}
$$
Scaling with Inflationary Energy Scale

- RMS B-mode signal scales with inflationary energy scale squared $E_i^2$
Contamination for Gravitational Waves

- Gravitational lensing contamination of B-modes from gravitational waves cleaned to $E_i \sim 0.3 \times 10^{16}$ GeV
Polarized Foregrounds

- Dust and synchrotron

\[ \ell (\ell + 1) \frac{C^{\ell \ell}_B}{2\pi} [\mu K^2] \]

See Burigana's lectures
Tensor-Scalar Ratio

- Unlike gravitational waves, **inflaton fluctuations** determine when inflation **ends** in a given patch, changing the scale factor or curvature.
- **Curvature power** is enhanced by the slowness of the roll.

\[
\varepsilon = \frac{\dot{\phi}^2}{2H^2M_{pl}^2}
\]

\[
\Delta^2_R = \frac{H^2}{8\pi^2M_{pl}^2\varepsilon}
\]

\[
V(\phi) = \dot{\phi}^2
\]

\[
\mathcal{R} = -\frac{\delta a}{a} = -H \frac{\delta \phi}{\phi}
\]

\[
\delta \phi \downarrow
\]

\[
\phi_{\text{end}}
\]
Tensor-Scalar Ratio

- Unlike gravitational waves, **inflaton fluctuations** determine when inflation **ends** in a given patch, changing the **scale factor** or curvature.
- Curvature power is enhanced by the slowness of the roll.

\[
\varepsilon = \frac{\dot{\phi}^2}{2H^2M_{pl}^2}, \quad \Delta_R^2 = \frac{H^2}{8\pi^2M_{pl}^2\varepsilon}
\]

\[
V(\phi) \quad r \uparrow \varepsilon, \text{ faster roll} \quad \delta \phi \downarrow \quad \delta a \quad R = -\frac{\delta a}{a} = -H\frac{\delta \phi}{\phi} \quad \text{superPlanck distance}
\]
Tensor-Scalar Ratio

- Tensor-scalar ratio \( r \)

\[
r \equiv 4 \frac{\Delta^2_+}{\Delta^2_R} = 16\epsilon
\]

- A large \( r \) implies a large \( \epsilon \) and a large roll

\[
\epsilon = \frac{1}{2M_{pl}^2} \left( \frac{d\phi}{d\ln a} \right)^2
\]

- Observable scales span \( d\ln a = d\ln k \sim 5 \) so

\[
\Delta \phi \approx 5 \frac{d\phi}{d\ln a} = 5(r/8)^{1/2}M_{pl} \approx 0.6(r/0.1)^{1/2}M_{pl}
\]

- For \( r = 0.2 \) the field must roll by at least \( M_{pl} \)

- Difficult to protect the flat potential across this large a range in field space
$n_S - r$ Plane

- **Scalar** power spectrum depends on both $H$ and $\epsilon$, so its tilt:

\[
\frac{d \ln \Delta_{\mathcal{R}}^2}{d \ln k} \equiv n_S - 1
\]

\[
= 2 \frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon}{d \ln k} = -2\epsilon - \frac{d \ln \epsilon}{d \ln k}
\]

- Measuring both $n_S - 1$ and $r$ constrain the inflationary model

- In slow roll, related to derivatives of potential

\[
\epsilon \approx \frac{M_{\text{pl}}^2}{2} \left( \frac{V'}{V} \right)^2
\]

\[
\frac{d \ln \epsilon}{d \ln k} = 4\epsilon - 2M_{\text{pl}}^2 \frac{V''}{V}
\]
$r-n_s$ Trajectories and Constraints

- Each **inflationary model** executes a trajectory in the plane
- Scale free models predict large tensors and large field excursions
$r-n_s$ Trajectories and Constraints

- Each inflationary model executes a trajectory in the plane
- Large improvements in $r$ limits from B-modes, moderate improvement in $n_s$ possible
Inflationary GW Background

- Near scale invariant spectrum gives flat $\Omega$ contributions in radiation domination
- Blue tilted spectra directly constrained

See Christensen's lectures

Gravitational Lensing
CMB Power Spectra

Angular scale $\theta$ [degrees]

$\ell(\ell + 1)/2\pi C_\ell [\mu K^2]$

Multipole number $\ell$

Temperature

E modes

Lensing B modes

CMB-S4 Forecast
Planck 2015
ACTPol
BICEP2/Keck
Polarbear
SPT(TT) / SPTpol
Gravitational Lensing

- Lensing is a surface brightness conserving remapping of source to image planes by the gradient of the projected potential

$$\phi(\hat{n}) = 2 \int \frac{dz}{H(z)} \frac{D_A(D_s - D)}{D_A(D) D_A(D_s)} \Phi(D_A\hat{n}, D) \ ,$$

such that the fields are remapped as

$$x(\hat{n}) \rightarrow x(\hat{n} + \nabla \phi) \ ,$$

where \( x \in \{T, Q, U\} \) temperature and polarization.

- Taylor expansion leads to product of fields and Fourier mode-coupling

- Appears in the power spectrum as a convolution kernel for \( T \) and \( E \) and an \( E \rightarrow B \).
Lensing of a Gaussian Random Field

- **CMB temperature and polarization** anisotropies are **Gaussian random fields** – unlike galaxy weak lensing
- **Average over many noisy images** – like galaxy weak lensing

highly exaggerated: see Burigana's talk for realism
Temperature Power Spectrum

- Lensing acts to smooth temperature (and E polarization) peaks.
- Subtle effect reaches 10% deep in the damping tail.
- Statistically detected at high significance.
Polarization Lensing
E and B modes

- **E-modes** are Stokes $Q$ polarization in wavenumber basis
- **B-modes** are Stokes $U$ polarization
Polarization Lensing

- Since $E$ and $B$ denote the relationship between the polarization amplitude and direction, warping due to lensing creates $B$-modes.
Deflection Power Spectrum

- Fundamental observable is deflection power spectrum (or convergence / $l^2$)
- Nearly entirely in linear regime
Lensed Power Spectrum Observables

- Principal components show two observables in lensed power spectra:
  - Temperature and E-polarization: deflection power at $l \sim 100$
  - B-polarization: deflection power at $l \sim 500$
- Normalized so that observables error = fractional lens power error

Smith, Hu & Kaplinghat (2006)
Principal in Practice

- Extracting principal components from LensPix simulated CMB temperature and polarization maps

Motloch & Hu (2017)
Principal in Practice

- Treating CMB maps as Gaussian leads to overly tight constraints and potentially misleading tension
Mass Reconstruction
Why Care

- Gravitational lensing sensitive to amount and hence growth of structure

- Examples: massive neutrinos - $d \ln C_{\ell}^{BB} / dm_\nu \approx -1/3 \text{eV}$, dark energy - $d \ln C_{\ell}^{BB} / dw \approx -1/8$

- Mass reconstruction measures the large scale structure on large scales and the mass profile of objects on small scales

- Large scale delensing of the gravitational wave

- Lensing by high-z dark matter halos: mass calibration of clusters and cosmography (same lens, different sources)

See Simon White's Lectures
Lensing Contamination

- Lensing acts as cosmic noise that isn't Gaussian - delensing
Quadratic Estimator

- Taylor expand mapping
  \[
  T(\hat{n}) = \tilde{T}(\hat{n} + \nabla \phi)
  = \tilde{T}(\hat{n}) + \nabla_i \phi(\hat{n}) \nabla^i \tilde{T}(\hat{n}) + \ldots
  \]

- Fourier decomposition → mode coupling of harmonics
  \[
  T(l) = \int d\hat{n} T(\hat{n}) e^{-i l \cdot \hat{n}}
  = \tilde{T}(l) - \int \frac{d^2 l_1}{(2\pi)^2} (1 - l_1) \cdot l_1 \tilde{T}(l_1) \phi(l - l_1)
  \]

- Consider fixed lens and Gaussian random CMB realizations: each pair is an estimator of the lens at \( L = l_1 + l_2 \):
  \[
  \langle T(l) T'(l') \rangle_{\text{CMB}} \approx \left[ \tilde{C}^{TT}_{l_1}(L \cdot l_1) + \tilde{C}^{TT}_{l_2}(L \cdot l_2) \right] \phi(L) \quad (l \neq -l')
  \]
Reconstruction from the CMB

- Generalize to polarization: each quadratic pair of fields estimates the lensing potential

\[ \langle x(l)x'(l') \rangle_{\text{CMB}} = f_\alpha(l, l') \phi(l + l'), \]

where \( x \in \text{temperature, polarization fields} \) and \( f_\alpha \) is a fixed weight that reflects geometry.

- Each pair forms a noisy estimate of the potential or projected mass - just like a pair of galaxy shears.

- Minimum variance weight all pairs to form an estimator of the lensing mass.

- Generalize to inhomogeneous noise, cut sky and maximum likelihood by iterating the quadratic estimator.
High Signal-to-Noise B-modes

- Cosmic variance of CMB fields sets ultimate limit for $T, E$
- $B$-polarization allows mapping to finer scales and in principle is not limited by cosmic variance of $E$

mass, temp. reconstruction, EB pol. reconstruction

100 sq. deg; 4' beam; 1$\mu$K-arcmin

Hu & Okamoto (2001)
Lensing Reconstruction

- SPT+Planck example

Lens Power Spectra

- Temperature and polarization reconstruction

\[ L^2 (L + 1)^2 C_{\phi\phi}^L / (2\pi) \times 10^{-7} \]

**Temperature**
- SPTpol
- SPT
- ACT
- Planck

**Polarization**
- SPTpol
- POLARBEAR
- BICEP/Keck
• Measuring projected matter power spectrum to cosmic variance limit across whole linear regime $0.002 < k < 0.2 \, h/\text{Mpc}$

$$\Delta P \approx -0.6 \left( \frac{m_{\text{tot}}}{\text{eV}} \right)$$

Hu & Okamoto (2001)
Delensing with External Template

- Herschel CIB data as tracer of lensing
- Predict and subtract B-mode contamination - SPT example

Manzotti et al (2017)
Delensing with External Template

- Herschel CIB data as tracer of lensing
- Predict and subtract B-mode contamination - SPT example

![Graph showing delensed CBB](image)

Manzotti et al (2017)
Consistency in Lens Observables

- Consistency between lensed CMB power spectra and lensing reconstruction critical for delensing
- Compare directly lens power spectrum information in model independent and nearly sample variance free way (consistency modes: a more precise $A_{1\text{ens}}$ test)

Delensing Goals

- **Lensing noise isn't Gaussian, may be removed to uncover** $r$
Reionization
Polarization Anisotropy

- Electron sees the temperature anisotropy on its recombination surface and scatters it into a polarization
Temperature Correlation

- Pattern correlated with the temperature anisotropy that generates it; here an $m=0$ quadrupole
Why Care?

• Early ionization would imply more exotic astrophysics (Pop-III stars) or physics (dark matter annihilation)

• Reionization screens temperature anisotropy on small scales making the true amplitude of initial fluctuations larger by $e^\tau$

• Measuring the growth of fluctuations is one of the best ways of determining the neutrino masses and the dark energy limits lensing information if not substantially better than 1%

• Offers an opportunity to study the origin of the low multipole statistical anomalies

• Presents a second, and statistically cleaner, window on gravitational waves from the early universe
Anisotropy Suppression

- A fraction $\tau$ of photons rescattered during reionization out of line of sight and replaced statistically by photon with random temperature fluctuation - suppressing anisotropy as $e^{-\tau}$
Transfer Function

- Linearized response to delta function ionization perturbation

\[ T_{\ell i} \equiv \frac{\partial \ln C_{\ell}^{EE}}{\partial x(z_i)} , \quad \delta C_{\ell}^{EE} = C_{\ell}^{EE} \sum_i T_{\ell i} \delta x(z_i) \]
Principal Components

- Eigenvectors of the Fisher Matrix

\[ F_{ij} \equiv \sum_{\ell} (\ell + 1/2) T_{\ell i} T_{\ell j} = \sum_{\mu} S_{i\mu} \sigma^{-2}_{\mu} S_{j\mu} \]

Hu & Holder (2003)
Representation in Modes

- Reproduces the power spectrum with sum over >3 modes. More generally, 5 modes suffice: e.g., total $\tau=0.1375$ vs 0.1377.

Hu & Holder (2003)
- Optical depth measurement unbiased
- Ultimate errors set by cosmic variance here 0.01
- Equivalently 1% measure of initial amplitude, impt for dark energy

Hu & Holder (2003)
Complete Planck 2015 Reionization

- 5PCs completely span $z<30$ reionization observables
- Step function models only skirt the favored regions

Heinrich, Miranda, Hu (2016)
Complete Planck 2015 Reionization

- Allows for a high redshift component of ionization

Heinrich, Miranda, Hu (2016)
Complete Planck 2015 Reionization

- Shifts optical depth higher
Total Optical Depth

• Optical depth measurement unbiased
• Ultimate errors set by cosmic variance here 0.01
• Equivalently 1% measure of initial amplitude, important for massive $\nu$

$\sigma_\mu$

$\sigma_\tau$ (cumul.)

$\tau_\mu$

Hu & Holder (2003)
The B-Bump

- Rescattering of gravitational wave anisotropy generates the B-bump
- If $r$ is near current upper limit, motivates next generation satellite
- Potentially enables test consistency test of canonical inflation
Slow Roll Consistency Relation

- Consistency relation between tensor-scalar ratio and tensor tilt $r = -8n_t$ tested by reionization
- Reionization uncertainties controlled by a complete p.c. analysis

Mortonson & Hu (2007)
Temperature and Polarization Spectra

- reionization
- gravitational waves
- gravitational lensing

$\Delta (\mu K)$ vs $l$ (multipole)

$\Theta E$, $EE$, $BB$
Summary

- **CMB polarized by Thomson scattering of quadrupole anisotropy**: isolates recombination, reionization with little projection effects in transfer.

- **Linear scalar fluctuations** generate **E-modes** where polarization direction (anti)aligned with amplitude change.

- **Linear tensor fluctuations** also generate **B-modes** where polarization direction (anti)crossed with amplitude change.

- **B-mode gravitational wave** amplitude measures the **inflation energy scale**: if observably large imply **superPlanckian roll**.

- **Beyond linear theory**, scalar fluctuations generate **B-modes**.

- **Gravitational lensing B-modes** measure amplitude of structure at $z \sim 2$, **neutrino mass** and can be quadratically reconstructed.

- **Delensing of the CMB** can enable measurements to $r \sim 10^{-3}$. 
Ciao!