Astro 448

Cosmic Microwave Background

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Astro 448

Acoustic Kinematics

Recombination

- Equilibrium number density distribution of a non-relativistic species

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T}$$

- Apply to the $e^- + p \leftrightarrow H$ system: Saha Equation

$$\frac{n_e n_p}{n_H n_b} = \frac{x_e^2}{1-x_e} = \frac{1}{n_b}\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-B/T}$$

where $B = m_e + m_p - m_H = 13.6\text{eV}$

- Naive guess of $T = B$ for recombination would put $z_* \approx 45000$.

Recombination

- But the photon-baryon ratio is very low

$$\eta_{b\gamma} \equiv n_b/n_\gamma \approx 3 \times 10^{-8}\Omega_b h^2$$

- Eliminate in favor of $\eta_{b\gamma}$ and B/T through

$$n_\gamma = 0.244T^3\,, \qquad \frac{m_e}{B} = 3.76 \times 10^4$$

- Big coefficient

$$\frac{{x_e}^2}{1-x_e} = 3.16 \times 10^{15} \left(\frac{B}{T}\right)^{3/2} e^{-B/T}$$

$T = 1/3\mathrm{eV} \rightarrow x_e = 0.7$, $T = 0.3\mathrm{eV} \rightarrow x_e = 0.2$

- Further delayed by inability to maintain equilibrium since net is through 2γ process and redshifting out of line

Thomson Scattering

- Thomson scattering of photons off of free electrons is the most important CMB process with a cross section (averaged over polarization states) of

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25}\mathrm{cm}^2$$

- Density of free electrons in a fully ionized $x_e = 1$ universe

$$n_e = (1 - Y_p/2)x_e n_b \approx 10^{-5}\Omega_b h^2(1+z)^3\mathrm{cm}^{-3}\,,$$

where $Y_p \approx 0.24$ is the Helium mass fraction, creates a high (comoving) Thomson opacity

$$\dot{\tau} \equiv n_e \sigma_T a$$

where dots are conformal time $\eta \equiv \int dt/a$ derivatives and τ is the optical depth.

Temperature Fluctuations

- Observe blackbody radiation with a temperature that differs at 10^{-5} coming from the surface of last scattering, with distribution function (specific intensity $I_\nu = 4\pi\nu^3 f(\nu)$ each polarization)

$$f(\nu) = [\exp(2\pi\nu/T(\hat{\mathbf{n}})) - 1]^{-1}$$

- Decompose the temperature perturbation in spherical harmonics

$$T(\hat{\mathbf{n}}) = \sum_{\ell m} T_{\ell m} Y_{\ell m}(\hat{\mathbf{n}})$$

- For Gaussian random fluctuations, the statistical properties of the temperature field are determined by the power spectrum

$$\langle T^*_{\ell m} T_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell$$

where the δ-function comes from statistical isotropy

Spatial vs Angular Power

- Take the radiation distribution at last scattering to also be described by an isotropic temperature field $T(\mathbf{x})$ and recombination to be instantaneous

$$T(\hat{\mathbf{n}}) = \int dD\, T(\mathbf{x})\delta(D - D_*)$$

where D is the comoving distance and D_* denotes recombination.

- Describe the temperature field by its Fourier moments

$$T(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} T(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

with a power spectrum

$$\langle T(\mathbf{k})^* T(\mathbf{k}')\rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') P_T(k)$$

Spatial vs Angular Power

- Note that the variance of the field

$$\langle T(\mathbf{x})T(\mathbf{x})\rangle = \int \frac{d^3k}{(2\pi)^3} P(k)$$
$$= \int d\ln k\, \frac{k^3 P(k)}{2\pi^2} \equiv \int d\ln k\, \Delta_T^2(k)$$

so it is more convenient to think in the log power spectrum $\Delta_T^2(k)$

- Temperature field

$$T(\hat{\mathbf{n}}) = \int \frac{d^3k}{(2\pi)^3} T(\mathbf{k}) e^{i\mathbf{k}\cdot D_*\hat{\mathbf{n}}}$$

- Expand out plane wave in spherical coordinates

$$e^{i\mathbf{k}D_*\cdot\hat{\mathbf{n}}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kD_*) Y_{\ell m}^*(\mathbf{k}) Y_{\ell m}(\hat{\mathbf{n}})$$

Spatial vs Angular Power

- Multipole moments

$$T_{\ell m} = \int \frac{d^3k}{(2\pi)^3} T(\mathbf{k}) 4\pi i^\ell j_\ell(kD_*) Y_{\ell m}(\mathbf{k})$$

- Power spectrum

$$\langle T^*_{\ell m} T_{\ell' m'} \rangle = \int \frac{d^3k}{(2\pi)^3} (4\pi)^2 (i)^{\ell-\ell'} j_\ell(kD_*) j_{\ell'}(kD_*) Y^*_{\ell m}(\mathbf{k}) Y_{\ell' m'}(\mathbf{k}) P_T(k)$$

$$= \delta_{\ell\ell'} \delta_{mm'} 4\pi \int d\ln k \, j_\ell^2(kD_*) \Delta_T^2(k)$$

with $\int_0^\infty j_\ell^2(x) d\ln x = 1/(2\ell(\ell+1))$, slowly varying Δ_T^2

$$C_\ell = \frac{4\pi \Delta_T^2(\ell/D_*)}{2\ell(\ell+1)} = \frac{2\pi}{\ell(\ell+1)} \Delta_T^2(\ell/D_*)$$

so $\ell(\ell+1)C_\ell/2\pi = \Delta_T^2$ is commonly used log power

Tight Coupling Approximation

- Near recombination $z \approx 10^3$ and $\Omega_b h^2 \approx 0.02$, the (comoving) mean free path of a photon

$$\lambda_C \equiv \frac{1}{\dot{\tau}} \sim 2.5\mathrm{Mpc}$$

  small by cosmological standards!

- On scales $\lambda \gg \lambda_C$ photons are tightly coupled to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions

- Specifically, their bulk velocities are defined by a single fluid velocity $v_\gamma = v_b$ and the photons carry no anisotropy in the rest frame of the baryons

- $\rightarrow$ No heat conduction or viscosity (anisotropic stress) in fluid

Zeroth Order Approximation

- Momentum density of a fluid is $(\rho + p)v$, where p is the pressure
- Neglect the momentum density of the baryons

$$R \equiv \frac{(\rho_b + p_b)v_b}{(\rho_\gamma + p_\gamma)v_\gamma} = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}$$
$$\approx 0.6 \left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{a}{10^{-3}}\right)$$

since $\rho_\gamma \propto T^4$ is fixed by the CMB temperature $T = 2.73(1+z)$K – OK substantially before recombination

- Neglect radiation in the expansion

$$\frac{\rho_m}{\rho_r} = 3.6 \left(\frac{\Omega_m h^2}{0.15}\right)\left(\frac{a}{10^{-3}}\right)$$

Number Continuity

- Photons are not created or destroyed. Without expansion

$$\dot{n}_\gamma + \nabla \cdot (n_\gamma \mathbf{v}_\gamma) = 0$$

but the expansion or Hubble flow causes $n_\gamma \propto a^{-3}$ or

$$\dot{n}_\gamma + 3n_\gamma \frac{\dot{a}}{a} + \nabla \cdot (n_\gamma \mathbf{v}_\gamma) = 0$$

- Linearize $\delta n_\gamma = n_\gamma - \bar{n}_\gamma$

$$(\delta n_\gamma)^\cdot = -3\delta n_\gamma \frac{\dot{a}}{a} - n_\gamma \nabla \cdot \mathbf{v}_\gamma$$

$$\left(\frac{\delta n_\gamma}{n_\gamma}\right)^\cdot = -\nabla \cdot \mathbf{v}_\gamma$$

Continuity Equation

- Number density $n_\gamma \propto T^3$ so define temperature fluctuation Θ

$$\frac{\delta n_\gamma}{n_\gamma} = 3\frac{\delta T}{T} \equiv 3\Theta$$

- Real space continuity equation

$$\dot{\Theta} = -\frac{1}{3}\nabla \cdot \mathbf{v}_\gamma$$

- Fourier space

$$\dot{\Theta} = -\frac{1}{3}i\mathbf{k} \cdot \mathbf{v}_\gamma$$

Momentum Conservation

- No expansion: $\dot{\mathbf{q}} = \mathbf{F}$
- De Broglie wavelength stretches with the expansion

$$\dot{\mathbf{q}} + \frac{\dot{a}}{a}\mathbf{q} = \mathbf{F}$$

for photons this the redshift, for non-relativistic particles expansion drag on peculiar velocities

- Collection of particles: momentum $\rightarrow$ momentum density $(\rho_\gamma + p_\gamma)\mathbf{v}_\gamma$ and force $\rightarrow$ pressure gradient

$$[(\rho_\gamma + p_\gamma)\mathbf{v}_\gamma]^{\cdot} = -4\frac{\dot{a}}{a}(\rho_\gamma + p_\gamma)\mathbf{v}_\gamma - \nabla p_\gamma$$

$$\frac{4}{3}\rho_\gamma \dot{\mathbf{v}}_\gamma = \frac{1}{3}\nabla\rho_\gamma$$

$$\dot{\mathbf{v}}_\gamma = -\nabla\Theta$$

Euler Equation

- Fourier space

$$\dot{\mathbf{v}}_\gamma = -ik\Theta$$

- Pressure gradients (any gradient of a scalar field) generates a curl-free flow
- For convenience define velocity amplitude:

$$\mathbf{v}_\gamma \equiv -iv_\gamma\hat{\mathbf{k}}$$

- Euler Equation:

$$\dot{v}_\gamma = k\Theta$$

- Continuity Equation:

$$\dot{\Theta} = -\frac{1}{3}kv_\gamma$$

Oscillator: Take One

- Combine these to form the simple harmonic oscillator equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = 0$$

where the adiabatic sound speed is defined through

$$c_s^2 \equiv \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma}$$

here $c_s^2 = 1/3$ since we are photon-dominated

- General solution:

$$\Theta(\eta) = \Theta(0)\cos(ks) + \frac{\dot{\Theta}(0)}{kc_s}\sin(ks)$$

where the sound horizon is defined as $s \equiv \int c_s d\eta$

Harmonic Extrema

- All modes are frozen in at recombination (denoted with a subscript $*$) yielding temperature perturbations of different amplitude for different modes. For the adiabatic (curvature mode) $\dot{\Theta}(0) = 0$

$$\Theta(\eta_*) = \Theta(0)\cos(ks_*)$$

- Modes caught in the extrema of their oscillation will have enhanced fluctuations

$$k_n s_* = n\pi$$

yielding a fundamental scale or frequency, related to the inverse sound horizon

$$k_A = \pi/s_*$$

and a harmonic relationship to the other extrema as $1:2:3\ldots$

Peak Location

- The fundmental physical scale is translated into a fundamental angular scale by simple projection according to the angular diameter distance D_A

$$\begin{aligned} \theta_A &= \lambda_A / D_A \\ \ell_A &= k_A D_A \end{aligned}$$

- In a flat universe, the distance is simply $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$, the horizon distance, and $k_A = \pi / s_* = \sqrt{3}\pi / \eta_*$ so

$$\theta_A \approx \frac{\eta_*}{\eta_0}$$

- In a matter-dominated universe $\eta \propto a^{1/2}$ so $\theta_A \approx 1/30 \approx 2^\circ$ or

$$\ell_A \approx 200$$

Curvature

- In a curved universe, the apparent or angular diameter distance is no longer the conformal distance $D_A = R\sin(D/R) \neq D$
- Objects in a closed universe are further than they appear! gravitational lensing of the background...
- Curvature scale of the universe must be substantially larger than current horizon
- Flat universe indicates critical density and implies missing energy given local measures of the matter density “dark energy”
- D also depends on dark energy density $\Omega_{\rm DE}$ and equation of state $w = p_{\rm DE}/\rho_{\rm DE}$.
- Expansion rate at recombination or matter-radiation ratio enters into calculation of k_A.

Doppler Effect

- Bulk motion of fluid changes the observed temperature via Doppler shifts

$$\left(\frac{\Delta T}{T}\right)_{\rm dop} = \hat{\mathbf{n}} \cdot \mathbf{v}_\gamma$$

- Averaged over directions

$$\left(\frac{\Delta T}{T}\right)_{\rm rms} = \frac{v_\gamma}{\sqrt{3}}$$

- Acoustic solution

$$\begin{aligned}\frac{v_\gamma}{\sqrt{3}} &= -\frac{\sqrt{3}}{k}\dot{\Theta} = \frac{\sqrt{3}}{k} k c_s \, \Theta(0) \sin(ks) \\ &= \Theta(0)\sin(ks)\end{aligned}$$

Doppler Peaks?

- Doppler effect for the photon dominated system is of equal amplitude and $\pi/2$ out of phase: extrema of temperature are turning points of velocity
- Effects add in quadrature:

$$\left(\frac{\Delta T}{T}\right)^2 = \Theta^2(0)[\cos^2(ks) + \sin^2(ks)] = \Theta^2(0)$$

- No peaks in k spectrum! However the Doppler effect carries an angular dependence that changes its projection on the sky $\hat{\mathbf{n}} \cdot \mathbf{v}_\gamma \propto \hat{\mathbf{n}} \cdot \hat{\mathbf{k}}$
- Coordinates where $\hat{\mathbf{z}} \parallel \hat{\mathbf{k}}$

$$Y_{10} Y_{\ell 0} \rightarrow Y_{\ell \pm 1\, 0}$$

recoupling $j_\ell' Y_{\ell 0}$: no peaks in Doppler effect

Restoring Gravity

- Take a simple photon dominated system with gravity
- Continuity altered since a gravitational potential represents a stretching of the spatial fabric that dilutes number densities – formally a spatial curvature perturbation
- Think of this as a perturbation to the scale factor $a \rightarrow a(1+\Phi)$ so that the cosmogical redshift is generalized to

$$\frac{\dot{a}}{a} \rightarrow \frac{\dot{a}}{a} + \dot{\Phi}$$

$$(\delta n_\gamma)^{\cdot} = -3\delta n_\gamma \left(\frac{\dot{a}}{a} + \dot{\Phi} \right) - n_\gamma \nabla \cdot \mathbf{v}_\gamma$$

so that the continuity equation becomes

$$\dot{\Theta} = -\frac{1}{3} k v_\gamma - \dot{\Phi}$$

Restoring Gravity

- Gravitational force in momentum conservation $\mathbf{F} = -m\nabla\Psi$ generalized to momentum density modifies the Euler equation to

$$\dot{v}_\gamma = k(\Theta + \Psi)$$

- General relativity says that Φ and Ψ are the relativistic analogues of the Newtonian potential and that $\Phi \approx -\Psi$.
- In our matter-dominated approximation, Φ represents matter density fluctuations through the cosmological Poisson equation

$$k^2\Phi = 4\pi G a^2 \rho_m \Delta_m$$

where the difference comes from the use of comoving coordinates for k (a^2 factor), the removal of the background density into the background expansion $(\rho\Delta_m)$ and finally a coordinate subtlety that enters into the definition of Δ_m

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Acoustic Dynamics

Constant Potentials

- In the matter dominated epoch potentials are constant because infall generates velocities as $v_m \sim k\eta\Psi$
- Velocity divergence generates density perturbations as $\Delta_m \sim -k\eta v_m \sim -(k\eta)^2\Psi$
- And density perturbations generate potential fluctuations as $\Phi \sim \Delta_m/(k\eta)^2 \sim -\Psi$, keeping them constant. Note that because of the expansion, density perturbations must grow to keep potentials constant.
- Here we have used the Friedman equation $H^2 = 8\pi G\rho_m/3$ and $\eta = \int d\ln a/(aH) \sim 1/(aH)$
- More generally, if stress perturbations are negligible compared with density perturbations ($\delta p \ll \delta\rho$) then potential will remain roughly constant – more specifically a variant called the Bardeen or comoving curvature ζ is constant

Oscillator: Take Two

- Combine these to form the simple harmonic oscillator equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - \ddot{\Phi}$$

- In a CDM dominated expansion $\dot{\Phi} = \dot{\Psi} = 0$. Also for photon domination $c_s^2 = 1/3$ so the oscillator equation becomes

$$\ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0$$

- Solution is just an offset version of the original

$$[\Theta + \Psi](\eta) = [\Theta + \Psi](0) \, \cos(ks)$$

- $\Theta + \Psi$ is also the observed temperature fluctuation since photons lose energy climbing out of gravitational potentials at recombination

Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or effective temperature

$$\Theta + \Psi$$

- Effective temperature oscillates around zero with amplitude given by the initial conditions
- Note: initial conditions are set when the perturbation is outside of horizon, need inflation or other modification to matter-radiation FRW universe.
- GR says that initial temperature is given by initial potential

Sachs-Wolfe Effect and the Magic 1/3

- A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]

$$\frac{\delta t}{t} = \Psi$$

- Convert this to a perturbation in the scale factor,

$$t = \int \frac{da}{aH} \propto \int \frac{da}{a\rho^{1/2}} \propto a^{3(1+w)/2}$$

where $w \equiv p/\rho$ so that during matter domination

$$\frac{\delta a}{a} = \frac{2}{3}\frac{\delta t}{t}$$

- CMB temperature is cooling as $T \propto a^{-1}$ so

$$\Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3}\Psi$$

Baryon Loading

- Baryons add extra mass to the photon-baryon fluid
- Controlling parameter is the momentum density ratio:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}}\right)$$

of order unity at recombination

- Momentum density of the joint system is conserved

$$\begin{aligned}(\rho_\gamma + p_\gamma)v_\gamma + (\rho_b + p_b)v_b &\approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma)v_\gamma \\ &= (1+R)(\rho_\gamma + p_\gamma)v_{\gamma b}\end{aligned}$$

where the controlling parameter is the momentum density ratio:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}}\right)$$

of order unity at recombination

New Euler Equation

- Momentum density ratio enters as

$$[(1+R)(\rho_\gamma + p_\gamma)\mathbf{v}_{\gamma b}]^{\cdot} = -4\frac{\dot a}{a}(1+R)(\rho_\gamma + p_\gamma)\mathbf{v}_{\gamma b} - \nabla p_\gamma - (1+R)(\rho_\gamma + p_\gamma)\nabla\Psi$$

same as before except for $(1+R)$ terms so

$$[(1+R)v_{\gamma b}]^{\cdot} = k\Theta + (1+R)k\Psi$$

- Photon continuity remains the same

$$\dot\Theta = -\frac{k}{3}v_{\gamma b} - \dot\Phi$$

- Modification of oscillator equation

$$[(1+R)\dot\Theta]^{\cdot} + \frac{1}{3}k^2\Theta = -\frac{1}{3}k^2(1+R)\Psi - [(1+R)\dot\Phi]^{\cdot}$$

Oscillator: Take Three

- Combine these to form the not-quite-so simple harmonic oscillator equation

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

where $c_s^2 \equiv \dot{p}_{\gamma b}/\dot{\rho}_{\gamma b}$

$$c_s^2 = \frac{1}{3}\frac{1}{1+R}$$

- In a CDM dominated expansion $\dot{\Phi} = \dot{\Psi} = 0$ and the adiabatic approximation $\dot{R}/R \ll \omega = kc_s$

$$[\Theta + (1+R)\Psi](\eta) = [\Theta + (1+R)\Psi](0)\cos(ks)$$

Baryon Peak Phenomenology

- Photon-baryon ratio enters in three ways
- Overall larger amplitude:

$$[\Theta + (1 + R)\Psi](0) = \frac{1}{3}(1 + 3R)\Psi(0)$$

- Even-odd peak modulation of effective temperature

$$[\Theta + \Psi]_{\rm peaks} = [\pm(1 + 3R) - 3R]\,\frac{1}{3}\Psi(0)$$

$$[\Theta + \Psi]_1 - [\Theta + \Psi]_2 = [-6R]\frac{1}{3}\Psi(0)$$

- Shifting of the sound horizon down or ℓ_A up

$$\ell_A \propto \sqrt{1 + R}$$

- Actual effects smaller since R evolves

Photon Baryon Ratio Evolution

- Oscillator equation has time evolving mass

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = 0$$

- Effective mass is is $m_{\rm eff} = 3c_s^{-2} = (1+R)$
- Adiabatic invariant

$$\frac{E}{\omega} = \frac{1}{2} m_{\rm eff}\,\omega A^2 = \frac{1}{2} 3 c_s^{-2} k c_s A^2 \propto A^2 (1+R)^{1/2} = const.$$

- Amplitude of oscillation $A \propto (1+R)^{-1/4}$ decays adiabatically as the photon-baryon ratio changes

Oscillator: Take Three and a Half

- The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\Phi)$$

  changes in the gravitational potentials alter the form of the acoustic oscillations

- If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator
- Term involving Ψ is the ordinary gravitational force
- Term involving Φ involves the $\dot{\Phi}$ term in the continuity equation as a (curvature) perturbation to the scale factor

Potential Decay

- Matter-to-radiation ratio

$$\frac{\rho_m}{\rho_r} \approx 24\Omega_m h^2 \left(\frac{a}{10^{-3}}\right)$$

of order unity at recombination in a low Ω_m universe

- Radiation is not stress free and so impedes the growth of structure

$$k^2\Phi = 4\pi G a^2 \rho_r \Delta_r$$

$\Delta_r \sim 4\Theta$ oscillates around a constant value, $\rho_r \propto a^{-4}$ so the Netwonian curvature decays.

- General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale

Radiation Driving

- Decay is timed precisely to drive the oscillator - close to fully coherent

$$[\Theta + \Psi](\eta) = [\Theta + \Psi](0) + \Delta\Psi - \Delta\Phi$$
$$= \frac{1}{3}\Psi(0) - 2\Psi(0) = \frac{5}{3}\Psi(0)$$

- $5\times$ the amplitude of the Sachs-Wolfe effect!
- Coherent approximation is exact for a photon-baryon fluid but reality is reduced to $\sim 4\times$ because of neutrino contribution to radiation
- Actual initial conditions are $\Theta + \Psi = \Psi/2$ for radiation domination but comparison to matter dominated SW correct

External Potential Approach

- Solution to homogeneous equation

$$(1+R)^{-1/4}\cos(ks)\,, \qquad (1+R)^{-1/4}\sin(ks)$$

- Give the general solution for an external potential by propagating impulsive forces

$$(1+R)^{1/4}\Theta(\eta) = \Theta(0)\cos(ks) + \frac{\sqrt{3}}{k}\left[\dot\Theta(0) + \frac{1}{4}\dot R(0)\Theta(0)\right]\sin ks$$
$$+ \frac{\sqrt{3}}{k}\int_0^\eta d\eta'(1+R')^{3/4}\sin[ks - ks']F(\eta')$$

where

$$F = -\ddot\Phi - \frac{\dot R}{1+R}\dot\Phi - \frac{k^2}{3}\Psi$$

- Useful if general form of potential evolution is known

Damping

- Tight coupling equations assume a perfect fluid: no viscosity, no heat conduction
- Fluid imperfections are related to the mean free path of the photons in the baryons

$$\lambda_C = \dot{\tau}^{-1} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a$$

is the conformal opacity to Thompson scattering

- Dissipation is related to the diffusion length: random walk approximation

$$\lambda_D = \sqrt{N}\lambda_C = \sqrt{\eta/\lambda_C}\,\lambda_C = \sqrt{\eta\lambda_C}$$

the geometric mean between the horizon and mean free path

- $\lambda_D/\eta_* \sim$ few %, so expect the peaks $:> 3$ to be affected by dissipation

Equations of Motion

- Continuity

$$\dot\Theta = -\frac{k}{3} v_\gamma - \dot\Phi\,, \qquad \dot\delta_b = -k v_b - 3\dot\Phi$$

where the photon equation remains unchanged and the baryons follow number conservation with $\rho_b = m_b n_b$

- Euler

$$\begin{aligned} \dot v_\gamma &= k(\Theta + \Psi) - \frac{k}{6}\pi_\gamma - \dot\tau(v_\gamma - v_b) \\ \dot v_b &= -\frac{\dot a}{a} v_b + k\Psi + \dot\tau(v_\gamma - v_b)/R \end{aligned}$$

where the photons gain an anisotropic stress term π_γ from radiation viscosity and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation

Viscosity

- Viscosity is generated from radiation streaming from hot to cold regions
- Expect

$$\pi_\gamma \sim v_\gamma \frac{k}{\dot\tau}$$

generated by streaming, suppressed by scattering in a wavelength of the fluctuation. Radiative transfer says

$$\pi_\gamma \approx 2 A_v v_\gamma \frac{k}{\dot\tau}$$

where $A_v = 16/15$

$$\dot v_\gamma = k(\Theta + \Psi) - \frac{k}{3} A_v \frac{k}{\dot\tau} v_\gamma$$

Oscillator: Penultimate Take

- Adiabatic approximation ($\omega \gg \dot{a}/a$)

$$\dot{\Theta} \approx -\frac{k}{3} v_\gamma$$

- Oscillator equation contains a $\dot{\Theta}$ damping term

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} A_v \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

- Heat conduction term similar in that it is proportional to v_γ and is suppressed by scattering $k/\dot{\tau}$. Expansion of Euler equations to leading order in $k\dot{\tau}$ gives

$$A_h = \frac{R^2}{1+R}$$

since the effects are only significant if the baryons are dynamically important

Oscillator: Final Take

- Final oscillator equation

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}}[A_v + A_h]\dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

- Solve in the adiabatic approximation

$$\Theta \propto \exp(i \int \omega d\eta)$$

$$-\omega^2 + \frac{k^2 c_s^2}{\dot{\tau}}(A_v + A_h) i\omega + k^2 c_s^2 = 0 \tag{1}$$

Dispersion Relation

- Solve

$$\omega^2 = k^2 c_s^2 \left[1 + i\frac{\omega}{\dot\tau}(A_v + A_h)\right]$$

$$\omega = \pm k c_s \left[1 + \frac{i}{2}\frac{\omega}{\dot\tau}(A_v + A_h)\right]$$

$$= \pm k c_s \left[1 \pm \frac{i}{2}\frac{k c_s}{\dot\tau}(A_v + A_h)\right]$$

- Exponentiate

$$\exp(i\int \omega d\eta) = e^{\pm iks} \exp[-k^2 \int d\eta \frac{1}{2}\frac{c_s^2}{\dot\tau}(A_v + A_h)]$$

$$= e^{\pm iks} \exp[-(k/k_D)^2] \quad (2)$$

- Damping is exponential under the scale k_D

Diffusion Scale

- Diffusion wavenumber

$$k_D^{-2} = \int d\eta \frac{1}{\dot{\tau}} \frac{1}{6(1+R)} \left(\frac{16}{15} + \frac{R^2}{(1+R)} \right)$$

- Limiting forms

$$\lim_{R \to 0} k_D^{-2} = \frac{1}{6} \frac{16}{15} \int d\eta \frac{1}{\dot{\tau}}$$

$$\lim_{R \to \infty} k_D^{-2} = \frac{1}{6} \int d\eta \frac{1}{\dot{\tau}}$$

- Geometric mean between horizon and mean free path as expected from a random walk

$$\lambda_D = \frac{2\pi}{k_D} \sim \frac{2\pi}{\sqrt{6}} (\eta \dot{\tau}^{-1})^{1/2}$$

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Polarization

Stokes Parameters

- Polarization state of radiation in direction $\hat{\mathbf{n}}$ described by the intensity matrix $\left\langle E_i(\hat{\mathbf{n}}) E_j^*(\hat{\mathbf{n}}) \right\rangle$, where $\mathbf{E}$ is the electric field vector and the brackets denote time averaging.
- As a hermitian matrix, it can be decomposed into the Pauli basis

$$\begin{aligned} \mathbf{P} &= C \left\langle \mathbf{E}(\hat{\mathbf{n}})\, \mathbf{E}^\dagger(\hat{\mathbf{n}}) \right\rangle \\ &= \Theta(\hat{\mathbf{n}}) \boldsymbol{\sigma}_0 + Q(\hat{\mathbf{n}})\, \boldsymbol{\sigma}_3 + U(\hat{\mathbf{n}})\, \boldsymbol{\sigma}_1 + V(\hat{\mathbf{n}})\, \boldsymbol{\sigma}_2 \,, \end{aligned}$$

where

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Stokes parameters recovered as $\mathrm{Tr}(\sigma_i \mathbf{P})/2$

Linear Polarization

- $Q \propto \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle$, $U \propto \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle$.
- Counterclockwise rotation of axes by $\theta = 45°$

$$E_1 = (E_1' - E_2')/\sqrt{2}\,, \quad E_2 = (E_1' + E_2')/\sqrt{2}$$

- $U \propto \langle E_1' E_1'^* \rangle - \langle E_2' E_2'^* \rangle$, difference of intensities at $45°$ or Q'
- More generally, $\mathbf{P}$ transforms as a tensor under rotations and

$$Q' = \cos(2\theta)Q + \sin(2\theta)U$$
$$U' = -\sin(2\theta)Q + \cos(2\theta)U$$

- or

$$Q' \pm iU' = e^{\mp 2i\theta}[Q \pm iU]$$

acquires a phase under rotation and is a spin ± 2 object

Coordinate Independent Representation

- Two directions: orientation of polarization and change in amplitude, i.e. Q and U in the basis of the Fourier wavevector for small sections of sky are called E and B components

$$
\begin{aligned}
E(\mathbf{l}) \pm iB(\mathbf{l}) &= -\int d\hat{\mathbf{n}}[Q'(\hat{\mathbf{n}}) \pm iU'(\hat{\mathbf{n}})]e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}} \\
&= -e^{\mp 2i\phi_l}\int d\hat{\mathbf{n}}[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})]e^{i\mathbf{l}\cdot\hat{\mathbf{n}}}
\end{aligned}
$$

- For the B-mode to not vanish, the polarization must point in a direction not related to the wavevector - not possible for density fluctuations in linear theory
- Generalize to all-sky: plane waves are eigenmodes of the Laplace operator on the tensor P.

Spin Harmonics

- Laplace Eigenfunctions

$$\nabla^2{}_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1] = -[l(l+1)-4]_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1]$$

- Spin s spherical harmonics: orthogonal and complete

$$\int d\hat{\mathbf{n}}\, {}_sY^*_{\ell m}(\hat{\mathbf{n}})\, {}_sY_{\ell m}(\hat{\mathbf{n}}) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\sum_{\ell m} {}_sY^*_{\ell m}(\hat{\mathbf{n}})\, {}_sY_{\ell m}(\hat{\mathbf{n}}') = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta')$$

where the ordinary spherical harmonics are $Y_{\ell m} = {}_0Y_{\ell m}$

- Given in terms of the rotation matrix

$${}_sY_{\ell m}(\beta\alpha) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell}_{-ms}(\alpha\beta 0)$$

Statistical Representation

- All-sky decomposition

$$[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})] = \sum_{\ell m} [E_{\ell m} \pm iB_{\ell m}]_{\pm 2}Y_{\ell m}(\hat{\mathbf{n}})$$

- Power spectra

$$\langle E^*_{\ell m} E_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{EE}$$

$$\langle B^*_{\ell m} B_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{BB}$$

- Cross correlation

$$\langle E^*_{\ell m} E_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{\Theta E}$$

others vanish if parity is conserved

Thomson Scattering

- Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi}|\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \, ,$$

where $\sigma_T = 8\pi\alpha^2/3m_e$ is the Thomson cross section, $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

- Summed over angle and incoming polarization

$$\sum_{i=1,2} \int d\hat{\mathbf{n}}' \frac{d\sigma}{d\Omega} = \sigma_T$$

Polarization Generation

- Heuristic: incoming radiation shakes an electron in direction of electric field vector $\hat{\mathbf{E}}'$
- Radiates photon with polarization also in direction $\hat{\mathbf{E}}'$
- But photon cannot be longitudinally polarized so that scattering into $90°$ can only pass one polarization
- Linearly polarized radiation like polarization by reflection
- Unlike reflection of sunlight, incoming radiation is nearly isotropic
- Missing linear polarization supplied by scattering from direction orthogonal to original incoming direction
- Only quadrupole anisotropy generates polarization by Thomson scattering

Acoustic Polarization

- Break down of tight-coupling leads to quadrupole anisotropy of

$$\pi_\gamma \approx \frac{k}{\dot\tau} v_\gamma$$

- Scaling $k_D = (\dot\tau/\eta_*)^{1/2} \rightarrow \dot\tau = k_D^2 \eta_*$
- Know: $k_D s_* \approx k_D \eta_* \approx 10$
- So:

$$\pi_\gamma \approx \frac{k}{k_D}\frac{1}{10} v_\gamma$$

$$\Delta_P \approx \frac{\ell}{\ell_D}\frac{1}{10}\Delta_T$$

Acoustic Polarization

- Gradient of velocity is along direction of wavevector, so polarization is pure E-mode
- Velocity is 90° out of phase with temperature – turning points of oscillator are zero points of velocity:

$$\Theta + \Psi \propto \cos(ks); \quad v_\gamma \propto \sin(ks)$$

- Polarization peaks are at troughs of temperature power

Cross Correlation

- Cross correlation of temperature and polarization

$$(\Theta + \Psi)(v_\gamma) \propto \cos(ks)\sin(ks) \propto \sin(2ks)$$

- Oscillation at twice the frequency
- Correlation: radial or tangential around hot spots
- Partial correlation: easier to measure if polarization data is noisy, harder to measure if polarization data is high S/N or if bands do not resolve oscillations
- Good check for systematics and foregrounds
- Comparison of temperature and polarization is proof against features in initial conditions mimicking acoustic features

Reionization

- Ionization depth during reionization

$$\tau(z) = \int d\eta n_e \sigma_T a = \int d\ln a \frac{n_e \sigma_T}{H(a)} \propto (\Omega_b h^2)(\Omega_m h^2)^{-1/2}(1+z)^{3/2}$$
$$= \left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{\Omega_m h^2}{0.15}\right)^{-1/2}\left(\frac{1+z}{61}\right)^{3/2}$$

- Quasars say $z_{ri} \geq 7$ so $\tau > 0.04$
- During reionization, cosmic quadrupole of $\sim 30\mu$K from the Sachs-Wolfe effect scatters into E-polarization
- Few percent optical depth leads to fraction of a μK signal
- Peaks at horizon scale at recombination: quadrupole source $j_2(kD_*)$ maximal at $kD_* \approx k\eta \approx 2$

Breaking degeneracies

- First objects, breaking degeneracy of initial amplitude vs optical depth in the peak heights

$$C_\ell \propto e^{-2\tau}$$

  only below horizon scale at reionization

- Breaks degeneracies in angular diameter distance by removing an ambiguity for ISW-dark energy measure, helps in $\Omega_{DE} - w_{DE}$ plane

Gravitational Wave

- Gravitational waves produce a quadrupolar distortion in the temperature of the CMB like effect on a ring of test particles
- Like ISW effect, source is a metric perturbation with time dependent amplitude
- After recombination, is a source of observable temperature anisotropy – but is therefore confined to low order multipoles
- Generated during inflation by quandum fluctuations

Gravitational Wave Polarization

- In the tight coupling regime, quadrupole anisotropy suppressed by scattering

$$\pi_\gamma \approx \frac{\dot{h}}{\dot{\tau}}$$

- Since gravitational waves oscillate and decay at horizon crossing, the polarization peaks at the horizon scale at recombination not the damping scale
- More distinct signature in the B-mode polarization since symmetry of plane wave is broken by the transverse nature of gravity wave polarization

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Linear Perturbation Theory

Covariant Perturbation Theory

- Covariant = takes same form in all coordinate systems
- Invariant = takes the same value in all coordinate systems
- Fundamental equations: Einstein equations, covariant conservation of stress-energy tensor:

$$\begin{aligned} G_{\mu\nu} &= 8\pi G T_{\mu\nu} \\ \nabla_\mu T^{\mu\nu} &= 0 \end{aligned}$$

- Preserve general covariance by keeping all degrees of freedom: 10 for each symmetric 4×4 tensor

1	2	3	4
	5	6	7
		8	9
			10

Metric Tensor

- Expand the metric tensor around the general FRW metric

$$g_{00} = -a^2, \qquad g_{ij} = a^2\gamma_{ij} \, .$$

where the "0" component is conformal time $\eta = dt/a$ and γ_{ij} is a spatial metric of constant curvature $K = H_0^2(\Omega_{\rm tot} - 1)$.

- Add in a general perturbation (Bardeen 1980)

$$\begin{aligned} g^{00} &= -a^{-2}(1 - 2A) \, , \\ g^{0i} &= -a^{-2}B^i \, , \\ g^{ij} &= a^{-2}(\gamma^{ij} - 2H_L\gamma^{ij} - 2H_T^{ij}) \, . \end{aligned}$$

- (1) $A \equiv$ a scalar potential; (3) B^i a vector shift, (1) H_L a perturbation to the spatial curvature; (5) H_T^{ij} a trace-free distortion to spatial metric = (10)

Matter Tensor

- Likewise expand the matter stress energy tensor around a homogeneous density ρ and pressure p:

$$
\begin{aligned}
T^0{}_0 &= -\rho - \delta\rho \,, \\
T^0{}_i &= (\rho + p)(v_i - B_i) \,, \\
T_0{}^i &= -(\rho + p)v^i \,, \\
T^i{}_j &= (p + \delta p)\delta^i{}_j + p\Pi^i{}_j \,,
\end{aligned}
$$

- (1) $\delta\rho$ a density perturbation; (3) v_i a vector velocity, (1) δp a pressure perturbation; (5) Π_{ij} an anisotropic stress perturbation
- So far this is fully general and applies to any type of matter or coordinate choice including non-linearities in the matter, e.g. cosmological defects.

Counting DOF's

20	Variables (10 metric; 10 matter)
-10	Einstein equations
-4	Conservation equations
$+4$	Bianchi identities
-4	Gauge (coordinate choice 1 time, 3 space)
———	
6	Degrees of freedom

- Without loss of generality these can be taken to be the 6 components of the matter stress tensor
- For the background, specify $p(a)$ or equivalently $w(a) \equiv p(a)/\rho(a)$ the equation of state parameter.

Scalar, Vector, Tensor

- In linear perturbation theory, perturbations may be separated by their transformation properties under rotation and translation.
- The eigenfunctions of the Laplacian operator form a complete set

$$
\begin{aligned}
\nabla^2 Q^{(0)} &= -k^2 Q^{(0)} && \mathrm{S}\,, \\
\nabla^2 Q_i^{(\pm 1)} &= -k^2 Q_i^{(\pm 1)} && \mathrm{V}\,, \\
\nabla^2 Q_{ij}^{(\pm 2)} &= -k^2 Q_{ij}^{(\pm 2)} && \mathrm{T}\,,
\end{aligned}
$$

- Vector and tensor modes satisfy divergence-free and transverse-traceless conditions

$$
\nabla^i Q_i^{(\pm 1)} = 0
$$

$$
\nabla^i Q_{ij}^{(\pm 2)} = 0
$$

$$
\gamma^{ij} Q_{ij}^{(\pm 2)} = 0
$$

Vector and Tensor Modes vs. Vector and Tensor Quantities

- A scalar mode carries with it associated vector (curl-free) and tensor (longitudinal) quantities
- A vector mode carries and associated tensor (neither longitudinal or transverse) quantities
- These are built from the mode basis out of covariant derivatives and the metric

$$
\begin{aligned}
Q_i^{(0)} &= -k^{-1}\nabla_i Q^{(0)} \,, \\
Q_{ij}^{(0)} &= (k^{-2}\nabla_i\nabla_j + \frac{1}{3}\gamma_{ij})Q^{(0)} \,, \\
Q_{ij}^{(\pm1)} &= -\frac{1}{2k}[\nabla_i Q_j^{(\pm1)} + \nabla_j Q_i^{(\pm1)}] \,,
\end{aligned}
$$

Spatially Flat Case

- For a spatially flat background metric, harmonics are related to plane waves:

$$
\begin{aligned}
Q^{(0)} &= \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k}\cdot\mathbf{x})
\end{aligned}
$$

  where $\hat{\mathbf{e}}_3 \parallel \mathbf{k}$. Chosen as spin states, c.f. polarization.
- For vectors, the harmonic points in a direction orthogonal to $\mathbf{k}$ suitable for the vortical component of a vector
- For tensors, the harmonic is transverse and traceless as appropriate for the decompositon of gravitational waves

Perturbation k-Modes

- For the kth eigenmode, the scalar components become

$$\begin{aligned} A(\mathbf{x}) &= A(k)\,Q^{(0)}\,, & H_L(\mathbf{x}) &= H_L(k)\,Q^{(0)}\,, \\ \delta\rho(\mathbf{x}) &= \delta\rho(k)\,Q^{(0)}\,, & \delta p(\mathbf{x}) &= \delta p(k)\,Q^{(0)}\,, \end{aligned}$$

the vectors components become

$$B_i(\mathbf{x}) = \sum_{m=-1}^{1} B^{(m)}(k)\,Q_i^{(m)}\,, \qquad v_i(\mathbf{x}) = \sum_{m=-1}^{1} v^{(m)}(k)\,Q_i^{(m)}\,,$$

and the tensors components

$$H_{Tij}(\mathbf{x}) = \sum_{m=-2}^{2} H_T^{(m)}(k)\,Q_{ij}^{(m)}\,, \quad \Pi_{ij}(\mathbf{x}) = \sum_{m=-2}^{2} \Pi^{(m)}(k)\,Q_{ij}^{(m)}\,,$$

Homogeneous Einstein Equations

- Einstein (Friedmann) equations:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho$$
$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p)$$

so that $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}$$

Homogeneous Einstein Equations

- Counting exercise:

20	Variables (10 metric; 10 matter)
-17	Homogeneity and Isotropy
-2	Einstein equations
-1	Conservation equations
$+1$	Bianchi identities
1	Degree of freedom

- without loss of generality choose ratio of homogeneous & isotropic component of the stress tensor to the density $w(a) = p(a)/\rho(a)$.

Acceleration Implies Negative Pressure

- Role of stresses in the background cosmology
- Homogeneous Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ imply the two Friedman equations (flat universe, or associating curvature $\rho_K = -3K/8\pi G a^2$)

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho$$
$$\frac{1}{a}\frac{d^2 a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p)$$

so that the total equation of state $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}$$

- so that ρ must scale more slowly than a^{-2}

Questions regarding Dark Energy

- Coincidence: given the very different scalings of matter and dark energy with a, why are they comparable now?
- Stability: why doesn't negative pressure imply accelerated collapse? or why doesn't the vacuum suck?
- Answer: stability is associated with stress (pressure) gradients not stress (pressure) itself.
- Example: the cosmological constant $w_\Lambda = -1$, a constant in time and space – no gradients.
- Example: a scalar field where $w = p/\rho \neq \delta p/\delta\rho$ = sound speed.

Covariant Scalar Equations

- Einstein equations (suppressing 0) superscripts (Hu & Eisenstein 1999):

$$
\begin{aligned}
(k^2 - 3K)[H_L + \frac{1}{3}H_T + \frac{\dot a}{a}\frac{1}{k^2}(kB - \dot H_T)]& \\
= 4\pi G a^2 \left[\delta\rho + 3\frac{\dot a}{a}(\rho + p)(v - B)/k\right]&, \qquad \text{Poisson Equation}\\
k^2(A + H_L + \frac{1}{3}H_T) + \left(\frac{d}{d\eta} + 2\frac{\dot a}{a}\right)(kB - \dot H_T)& \\
= 8\pi G a^2 p\Pi&, \\
\frac{\dot a}{a}A - \dot H_L - \frac{1}{3}\dot H_T - \frac{K}{k^2}(kB - \dot H_T)& \\
= 4\pi G a^2(\rho + p)(v - B)/k&, \\
\left[2\frac{\ddot a}{a} - 2\left(\frac{\dot a}{a}\right)^2 + \frac{\dot a}{a}\frac{d}{d\eta} - \frac{k^2}{3}\right]A - \left[\frac{d}{d\eta} + \frac{\dot a}{a}\right](\dot H_L + \frac{1}{3}kB)& \\
= 4\pi G a^2(\delta p + \frac{1}{3}\delta\rho)&.
\end{aligned}
$$

Covariant Scalar Equations

- Conservation equations: continuity and Navier Stokes

$$\begin{aligned}
\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho + 3\frac{\dot a}{a}\delta p &= -(\rho + p)(kv + 3\dot H_L)\,, \\
\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right]\left[(\rho + p)\frac{(v - B)}{k}\right] &= \delta p - \frac{2}{3}(1 - 3\frac{K}{k^2})p\Pi + (\rho + p)A\,,
\end{aligned}$$

- Equations are not independent since $\nabla_\mu G^{\mu\nu} = 0$ via the Bianchi identities.
- Related to the ability to choose a coordinate system or "gauge" to represent the perturbations.

Covariant Scalar Equations

- DOF counting exercise

8	Variables (4 metric; 4 matter)
-4	Einstein equations
-2	Conservation equations
$+2$	Bianchi identities
-2	Gauge (coordinate choice 1 time, 1 space)
——	
2	Degrees of freedom

- without loss of generality choose scalar components of the stress tensor δp, Π .

Covariant Vector Equations

- Einstein equations

$$
\begin{aligned}
(1-2K/k^2)(kB^{(\pm1)} - \dot{H}_T^{(\pm1)}) \\
= 16\pi G a^2(\rho+p)(v^{(\pm1)} - B^{(\pm1)})/k\,, \\
\left[\frac{d}{d\eta} + 2\frac{\dot a}{a}\right](kB^{(\pm1)} - \dot{H}_T^{(\pm1)}) \\
= -8\pi G a^2 p\Pi^{(\pm1)}\,.
\end{aligned}
$$

- Conservation Equations

$$
\begin{aligned}
\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right][(\rho+p)(v^{(\pm1)} - B^{(\pm1)})/k] \\
= -\frac{1}{2}(1-2K/k^2)p\Pi^{(\pm1)}\,,
\end{aligned}
$$

- Gravity provides no source to vorticity $\longrightarrow$ decay

Covariant Vector Equations

- DOF counting exercise

8	Variables (4 metric; 4 matter)
-4	Einstein equations
-2	Conservation equations
$+2$	Bianchi identities
-2	Gauge (coordinate choice 1 time, 1 space)
2	Degrees of freedom

- without loss of generality choose vector components of the stress tensor $\Pi^{(\pm 1)}$.

Covariant Tensor Equation

- Einstein equation

$$\left[\frac{d^2}{d\eta^2} + 2\frac{\dot{a}}{a}\frac{d}{d\eta} + (k^2 + 2K) \right] H_T^{(\pm 2)} = 8\pi G a^2 p \Pi^{(\pm 2)} \, .$$

- DOF counting exercise

4	Variables (2 metric; 2 matter)
-2	Einstein equations
-0	Conservation equations
$+0$	Bianchi identities
-0	Gauge (coordinate choice 1 time, 1 space)
2	Degrees of freedom

- wlog choose tensor components of the stress tensor $\Pi^{(\pm 2)}$.

Arbitrary Dark Components

- Total stress energy tensor can be broken up into individual pieces
- Dark components interact only through gravity and so satisfy separate conservation equations
- Einstein equation source remains the sum of components.
- To specify an arbitrary dark component, give the behavior of the stress tensor: 6 components: δp, $\Pi^{(i)}$, where $i = -2, ..., 2$.
- Many types of dark components (dark matter, scalar fields, massive neutrinos,..) have simple forms for their stress tensor in terms of the energy density, i.e. described by equations of state.
- An equation of state for the background $w = p/\rho$ is *not* sufficient to determine the behavior of the perturbations.

Gauge

- Metric and matter fluctuations take on different values in different coordinate system
- No such thing as a "gauge invariant" density perturbation!
- General coordinate transformation:

$$\begin{aligned} \tilde{\eta} &= \eta + T \\ \tilde{x}^i &= x^i + L^i \end{aligned}$$

  free to choose (T, L^i) to simplify equations or physics. Decompose these into scalar and vector harmonics.
- $G_{\mu\nu}$ and $T_{\mu\nu}$ transform as tensors, so components in different frames can be related

Gauge Transformation

- Scalar Metric:

$$
\begin{aligned}
\tilde{A} &= A - \dot{T} - \frac{\dot{a}}{a} T \,, \\
\tilde{B} &= B + \dot{L} + kT \,, \\
\tilde{H}_L &= H_L - \frac{k}{3} L - \frac{\dot{a}}{a} T \,, \\
\tilde{H}_T &= H_T + kL \,,
\end{aligned}
$$

- Scalar Matter (Jth component):

$$
\begin{aligned}
\delta\tilde{\rho}_J &= \delta\rho_J - \dot{\rho}_J T, \\
\delta\tilde{p}_J &= \delta p_J - \dot{p}_J T, \\
\tilde{v}_J &= v_J + \dot{L},
\end{aligned}
$$

- Vector:

$$
\tilde{B}^{(\pm1)} = B^{(\pm1)} + \dot{L}^{(\pm1)}, \; \tilde{H}_T^{(\pm1)} = H_T^{(\pm1)} + kL^{(\pm1)}, \; \tilde{v}_J^{(\pm1)} = v_J^{(\pm1)} + \dot{L}^{(\pm1)},
$$

Common Scalar Gauge Choices

- A coordinate system is fully specified if there is an explicit prescription for (T, L^i) or for scalars (T, L)
- Newtonian:

$$
\begin{aligned}
\tilde{B} &= \tilde{H}_T = 0 \\
\Psi &\equiv \tilde{A} \quad \text{(Newtonian potential)} \\
\Phi &\equiv \tilde{H}_L \quad \text{(Newtonian curvature)} \\
L &= -H_T/k \\
T &= -B/k + \dot{H}_T/k^2
\end{aligned}
$$

Good: intuitive Newtonian like gravity; matter and metric algebraically related; commonly chosen for analytic CMB and lensing work

Bad: numerically unstable

Example: Newtonian Reduction

- In the general equations, set $B = H_T = 0$:

$$
\begin{aligned}
(k^2 - 3K)\Phi &= 4\pi G a^2 \left[\delta\rho + 3\frac{\dot a}{a}(\rho + p)v/k \right] \\
k^2(\Psi + \Phi) &= 8\pi G a^2 p \Pi
\end{aligned}
$$

so $\Psi = -\Phi$ if anisotropic stress $\Pi = 0$ and

$$
\begin{aligned}
\left[\frac{d}{d\eta} + 3\frac{\dot a}{a} \right] \delta\rho + 3\frac{\dot a}{a}\delta p &= -(\rho + p)(kv + 3\dot\Phi)\,, \\
\left[\frac{d}{d\eta} + 4\frac{\dot a}{a} \right] (\rho + p)v &= k\delta p - \frac{2}{3}(1 - 3\frac{K}{k^2})p\,k\Pi + (\rho + p)\,k\Psi\,,
\end{aligned}
$$

- Competition between stress (pressure and viscosity) and potential gradients

Common Scalar Gauge Choices

- Comoving:

$$
\begin{aligned}
\tilde{B} &= \tilde{v} \quad (T^0_i = 0) \\
H_T &= 0 \\
\xi &= \tilde{A} \\
\zeta &= \tilde{H}_L \quad \text{(Bardeen curvature)} \\
\Delta &= \tilde{\delta} \quad \text{(comoving density pert)} \\
T &= (v - B)/k \\
L &= -H_T/k
\end{aligned}
$$

Good: Algebraic relations between matter and metric

- Euler equation becomes an algebraic relation between stress and potential

$$
(\rho + p)\xi = -\delta p + \frac{2}{3}\left(1 - \frac{3K}{k}\right) p\Pi
$$

Common Scalar Gauge Choices

- Einstein equation lacks momentum density source

$$\frac{\dot a}{a}\xi - \dot\zeta - \frac{K}{k^2}kv = 0$$

- Combine: ζ is conserved if stress fluctuations negligible, e.g. above the horizon if $|K| \ll H^2$

$$\dot\zeta + Kv/k = \frac{\dot a}{a}\left[-\frac{\delta p}{\rho + p} + \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\frac{p}{\rho + p}\Pi\right] \to 0$$

Bad: explicitly relativistic choice

Common Scalar Gauge Choices

- Synchronous:

$$
\begin{aligned}
\tilde{A} &= \tilde{B} = 0 \\
\eta_L &\equiv -\tilde{H}_L - \frac{1}{3}\tilde{H}_T \\
h_T &= \tilde{H}_T \quad \text{or} \quad h = 6H_L \\
T &= a^{-1}\int d\eta a A + c_1 a^{-1} \\
L &= -\int d\eta (B + kT) + c_2
\end{aligned}
$$

Good: stable, the choice of numerical codes

Bad: residual gauge freedom in constants c_1, c_2 must be specified as an initial condition, intrinsically relativistic.

Common Scalar Gauge Choices

- Spatially Unperturbed:

$$\begin{aligned}
\tilde{H}_L &= \tilde{H}_T = 0 \\
L &= -H_T/k \\
\tilde{A}\,, \tilde{B} &= \quad \text{metric perturbations} \\
T &= \left(\frac{\dot{a}}{a}\right)^{-1} \left(H_L + \frac{1}{3} H_T\right)
\end{aligned}$$

  Good: eliminates spatial metric in evolution equations; useful in inflationary calculations (Mukhanov et al)

  Bad: intrinsically relativistic.

- Caution: perturbation evolution is governed by the behavior of stress fluctuations and an isotropic stress fluctuation δp is gauge dependent.

Hybrid “Gauge Invariant” Approach

- With the gauge transformation relations, express variables of one gauge in terms of those in another – allows a mixture in the equations of motion
- Example: Newtonian curvature and comoving density

$$(k^2 - 3K)\Phi = 4\pi G a^2 \rho \Delta$$

  ordinary Poisson equation then implies Φ approximately constant if stresses negligible.
- Example: Exact Newtonian curvature above the horizon derived through Bardeen curvature conservation

  Gauge transformation

$$\Phi = \zeta + \frac{\dot{a}}{a}\frac{v}{k}$$

Hybrid "Gauge Invariant" Approach

Einstein equation to eliminate velocity

$$\frac{\dot a}{a}\Psi - \dot\Phi = 4\pi G a^2(\rho + p)v/k$$

Friedman equation with no spatial curvature

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}a^2\rho$$

With $\dot\Phi = 0$ and $\Psi \approx -\Phi$

$$\frac{\dot a}{a}\frac{v}{k} = -\frac{2}{3(1+w)}\Phi$$

Hybrid "Gauge Invariant" Approach

Combining gauge transformation with velocity relation

$$\Phi = \frac{3 + 3w}{5 + 3w}\zeta$$

Usage: calculate ζ from inflation determines Φ for any choice of matter content or causal evolution.

- Example: Scalar field ("quintessence" dark energy) equations in comoving gauge imply a sound speed $\delta p/\delta\rho = 1$ independent of potential $V(\phi)$. Solve in synchronous gauge (Hu 1998).

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Inflationary Perturbations

Scalar Fields

- Stress-energy tensor of a scalar field

$$T^{\mu}{}_{\nu} = \nabla^{\mu}\varphi\,\nabla_{\nu}\varphi - \frac{1}{2}(\nabla^{\alpha}\varphi\,\nabla_{\alpha}\varphi + 2V)\delta^{\mu}{}_{\nu}\,.$$

- For the background $\langle\phi\rangle \equiv \phi_0$

$$\rho_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + V \quad p_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + -V$$

- So for kinetic dominated $w_\phi = p_\phi/\rho_\phi \to 1$
- And potential dominated $w_\phi = p_\phi/\rho_\phi \to -1$
- A slowly rolling (potential dominated) scalar field can accelerate the expansion and so solve the horizon problem or act as a dark energy candidate

Equation of Motion

- Can use general equations of motion of dictated by stress energy conservation

$$\dot{\rho}_\phi = -3(\rho_\phi + p_\phi)\frac{\dot{a}}{a}\,,$$

to obtain the equation of motion of the background field ϕ

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2 V' = 0\,,$$

- Likewise for the perturbations $\phi = \phi_0 + \phi_1$

$$\begin{aligned}
\delta\rho_\phi &= a^{-2}(\dot{\phi}_0\dot{\phi}_1 - \dot{\phi}_0^2 A) + V'\phi_1\,,\\
\delta p_\phi &= a^{-2}(\dot{\phi}_0\dot{\phi}_1 - \dot{\phi}_0^2 A) - V'\phi_1\,,\\
(\rho_\phi + p_\phi)(v_\phi - B) &= a^{-2}k\dot{\phi}_0\phi_1\,,\\
p_\phi\pi_\phi &= 0\,,
\end{aligned}$$

Equation of Motion

- The stress of the perturbations is defined through

$$\delta p_\phi = \delta\rho_\phi + 3(\rho_\phi + p_\phi)\frac{v_\phi - B}{k}\frac{\dot{a}}{a}(1 - c_\phi^2)$$

where $c_\phi^2 \equiv \dot{p}_\phi/\dot{\rho}_\phi$ is the "adiabatic" sound speed

- So for the comoving gauge where $v_\phi = B$, $\delta p_\phi = \delta\rho_\phi$ so the sound speed relevant for stability is $\delta p_\phi/\delta\rho_\phi = 1$. Very useful for solving system since in this gauge everything is specified by $w(a)$
- Scalar field fluctuations are stable inside the horizon and are a good candidate for the smooth dark energy
- More generally, continuity and Euler equations imply

$$\ddot{\phi}_1 = -2\frac{\dot{a}}{a}\dot{\phi}_1 - (k^2 + a^2 V'')\phi_1 + (\dot{A} - 3\dot{H}_L - kB)\dot{\phi}_0 - 2Aa^2 V' \,.$$

Inflationary Perturbations

- Classical equations of motion for scalar field inflaton determine the evolution of scalar field fluctuations generated by quantum fluctuations
- Since the Bardeen or comoving curvature ζ is conserved in the absence of stress fluctuations (i.e. outside the apparent horizon, calculate this and we're done no matter what happens in between inflation and the late universe (reheating etc.)
- But in the comoving gauge $\phi_1 = 0$! no scalar-field perturbation
- Solution: solve the scalar field equation in the dual gauge where the curvature $H_L = 0$ (and $H_T = 0$ to fix the gauge completely, as the "spatially unperturbed" or "spatially flat" gauge) and transform the result to the comoving gauge

Transformation to Comoving Gauge

- Scalar field transforms as scalar field

$$\tilde{\phi}_1 = \phi_1 - \dot{\phi}_0 T$$

- To get to comoving frame $\tilde{\phi}_1 = 0$, $T = \phi_1/\dot{\phi}_0$, and $\tilde{H}_T = H_T + kL$ so

$$\begin{aligned}\zeta &= H_L - \frac{k}{3}L - \frac{\dot{a}}{a}T\,,\\ &= H_L + \frac{H_T}{3} - \frac{\dot{a}}{a}\frac{\phi_1}{\dot{\phi}_0}\end{aligned}$$

- Transformation particularly simple from a gauge with $H_T = H_L = 0$, i.e. spatially unperturbed metric

$$\zeta = -\frac{\dot{a}}{a}\frac{\phi_1}{\dot{\phi}_0}$$

Scalar Field Eqn of Motion

- Scalar field perturbation in spatially unperturbed gauge is simply proportional to resulting Bardeen curvature with the proportionality constant as the expansion rate over roll rate – enhanced
- Scalar field fluctuation satisfies classical equation of motion

$$\ddot{\phi}_1 = -2\frac{\dot{a}}{a}\dot{\phi}_1 - (k^2 + a^2 V'')\phi_1 + (\dot{A} - kB)\dot{\phi}_0 - 2Aa^2 V' \,.$$

- Metric terms may be eliminated through Einstein equations

$$\begin{aligned} A &= 4\pi G a^2 \left(\frac{\dot{a}}{a}\right)^{-1} (\rho_\phi + p_\phi)(v_\phi - B)/k \\ &= 4\pi G \left(\frac{\dot{a}}{a}\right)^{-1} \dot{\phi}_0 \phi_1 \end{aligned}$$

Scalar Field Eqn of Motion

- And

$$kB = 4\pi G a^2 [\left(\frac{\dot a}{a}\right)^{-1} \delta\rho_\phi + 3\frac{\dot a}{a}(\rho_\phi + p_\phi)(v_\phi - B)/k]$$

$$= 4\pi G [\left(\frac{\dot a}{a}\right)^{-1} (\dot\phi_0 \dot\phi_1 + a^2 V' \phi_1) - \left(\frac{\dot a}{a}\right)^{-2} (4\pi G \dot\phi_0)^2 \dot\phi_0 \phi_1 + 3\dot\phi_0 \phi_1]$$

- So $\dot A - kB \propto \phi_1$ with proportionality that depends only on the background evolution – Einstein & scalar field equations reduce to a single second order diff eq!
- Equation resembles a damped oscillator equation with a particular dispersion relation

$$\ddot\phi_1 + 2\frac{\dot a}{a}\dot\phi_1 + [k^2 + f(\eta)]\phi_1$$

Exact Equation

- Rewrite equations of motion in terms of slow roll parameters but do not require them to be small or constant.
- Deviation from de Sitter expansion

$$\epsilon \equiv \frac{3}{2}(1 + w_\phi)$$
$$= \frac{\frac{3}{2}\dot{\phi}_0^2/a^2 V}{1 + \frac{1}{2}\dot{\phi}_0^2/a^2 V}$$

- Deviation from overdamped limit of $d^2\phi_0/dt^2 = 0$

$$\delta \equiv \frac{\ddot{\phi}_0}{\dot{\phi}} \left(\frac{\dot{a}}{a}\right)^{-1} - 1$$

Exact Equation

- Friedman equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = 4\pi G\dot{\phi}_0^2\epsilon^{-1}$$

$$\frac{d}{d\eta}\left(\frac{\dot{a}}{a}\right) = \left(\frac{\dot{a}}{a}\right)^2(1-\epsilon)$$

- Homogenous scalar field equation

$$\dot{\phi}_0\frac{\dot{a}}{a}(3+\delta) = -a^2V'$$

- Combination

$$\dot{\epsilon} = 2\epsilon(\delta+\epsilon)\frac{\dot{a}}{a}$$

Exact equation

- Rewrite in $u \equiv a\phi$ to remove expansion damping

$$\ddot{u} + [k^2 + g(\eta)]u = 0$$

where Mukhanov

$$\begin{aligned} g(\eta) &\equiv f(\eta) + \epsilon - 2 \\ &= -\left(\frac{\dot{a}}{a}\right)^2 [2 + 3\delta + 2\epsilon + (\delta + \epsilon)(\delta + 2\epsilon)] - \frac{\dot{a}}{a}\dot{\delta} \\ &= -\frac{\ddot{z}}{z} \end{aligned}$$

and

$$z \equiv a \left(\frac{\dot{a}}{a}\right)^{-1} \dot{\phi}_0$$

Slow Roll Limit

- Slow roll $\epsilon \ll 1$, $\delta \ll 1$, $\dot{\delta} \ll \frac{\dot{a}}{a}$

$$\ddot{u} + [k^2 - 2\left(\frac{\dot{a}}{a}\right)^2]u = 0$$

- or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\rm end}$$

$$\tilde{\eta} = \int_{a_{\rm end}}^{a} \frac{da}{Ha^2} \approx -\frac{1}{aH}$$

- Compact, slow-roll equation:

$$\ddot{u} + [k^2 - \frac{2}{\tilde{\eta}^2}]u = 0$$

Slow Roll Limit

- Slow roll equation has the exact solution:

$$u = A(k \pm \frac{i}{\tilde{\eta}})e^{\mp ik\tilde{\eta}}$$

- For $|k\tilde{\eta}| \gg 1$ (early times, inside Hubble length) behaves as free oscillator

$$\lim_{|k\tilde{\eta}|\to\infty} u = Ake^{\mp ik\tilde{\eta}}$$

- Normalization A will be set by origin in quantum fluctuations of free field

Slow Roll Limit

- For $|k\tilde{\eta}| \ll 1$ (late times, $\gg$ Hubble length) fluctuation freezes in

$$\lim_{|k\tilde{\eta}|\to 0} u = \pm\frac{i}{\tilde{\eta}}A = \pm iHaA$$

$$\phi_1 = \pm iHA$$

$$\zeta = \mp iHA\left(\frac{\dot{a}}{a}\right)\frac{1}{\dot{\phi}_0}$$

- Slow roll replacement

$$\left(\frac{\dot{a}}{a}\right)^2\frac{1}{\dot{\phi}_0^2} = \frac{8\pi G a^2 V}{3}\frac{3}{2a^2 V\epsilon} = 4\pi G = \frac{4\pi}{m_{\rm pl}^2}$$

- Bardeen curvature power spectrum

$$\Delta_\zeta^2 \equiv \frac{k^3|\zeta|^2}{2\pi^2} = \frac{2k^3}{\pi}\frac{H^2}{\epsilon m_{\rm pl}^2}A^2$$

Quantum Fluctuations

- Simple harmonic oscillator $\ll$ Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k,\eta)\hat{a} + u^*(k,\eta)\hat{a}^\dagger$$

where $u(k,\eta)$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \qquad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\eta] = i$

$$u(k,\eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

Quantum Fluctuations

- Zero point fluctuations of ground state

$$
\begin{aligned}
\langle u^2 \rangle &= \langle 0 | u^\dagger u | 0 \rangle \\
&= \langle 0 | (u^* \hat{a}^\dagger + u \hat{a})(u \hat{a} + u^* \hat{a}^\dagger) | 0 \rangle \\
&= \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle |u(k, \tilde{\eta})|^2 \\
&= \langle 0 | [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} | 0 \rangle |u(k, \tilde{\eta})|^2 \\
&= |u(k, \tilde{\eta})|^2 = \frac{1}{2k}
\end{aligned}
$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in. Slow roll equation
- So $A = (2k^3)^{1/2}$ and curvature power spectrum

$$
\Delta_\zeta^2 \equiv \frac{1}{\pi} \frac{H^2}{\epsilon m_{\rm pl}^2}
$$

Tilt

- Curvature power spectrum is scale invariant to the extent that H is constant
- Scalar spectral index

$$
\begin{aligned}
\frac{d \ln \Delta_\zeta^2}{d \ln k} &\equiv n_S - 1 \\
&= 2\frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon}{d \ln k}
\end{aligned}
$$

- Evaluate at horizon crossing where fluctuation freezes

$$
\begin{aligned}
\frac{d \ln H}{d \ln k}\Big|_{-k\tilde\eta=1} &= \frac{k}{H}\frac{dH}{d\tilde\eta}\Big|_{-k\tilde\eta=1}\frac{d\tilde\eta}{dk}\Big|_{-k\tilde\eta=1} \\
&= \frac{k}{H}(-aH^2\epsilon)\big|_{-k\tilde\eta=1}\frac{1}{k^2} = -\epsilon
\end{aligned}
$$

where $aH = -1/\tilde\eta = k$

Tilt

- Evolution of ϵ

$$\frac{d\ln\epsilon}{d\ln k} = -\frac{d\ln\epsilon}{d\ln\tilde{\eta}} = -2(\delta+\epsilon)\frac{\dot{a}}{a}\tilde{\eta} = 2(\delta+\epsilon)$$

- Tilt in the slow-roll approximation

$$n_S = 1 - 4\epsilon - 2\delta$$

Relationship to Potential

- To leading order in slow roll parameters

$$
\begin{aligned}
\epsilon &= \frac{\frac{3}{2}\dot\phi_0^2/a^2V}{1+\frac{1}{2}\dot\phi_0^2/a^2V} \\
&\approx \frac{3}{2}\dot\phi_0^2/a^2V \\
&\approx \frac{3}{a^2V}\frac{a^4V'^2}{9(\dot a/a)^2}, \qquad\qquad (3\dot\phi_0\frac{\dot a}{a} = -a^2V') \\
&\approx \frac{1}{6}\frac{3}{8\pi G}\left(\frac{V'}{V}\right)^2, \qquad\qquad \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}a^2V \\
&\approx \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2
\end{aligned}
$$

so $\epsilon \ll 1$ is related to the first derivative of potential being small

Relationship to Potential

- And

$$\delta = \frac{\ddot{\phi}_0}{\dot{\phi}_0} \left(\frac{\dot{a}}{a} \right)^{-1} - 1$$

$$(\dot{\phi}_0 \approx -a^2 \left(\frac{\dot{a}}{a} \right)^{-1} \frac{V'}{3})$$

$$(\ddot{\phi}_0 \approx -\frac{a^2 V'}{3}(1+\epsilon) + a^4 \left(\frac{\dot{a}}{a} \right)^{-2} \frac{V' V''}{9})$$

$$\approx -\frac{1}{a^2 V'/3} \left(-\frac{a^2 V'}{3}(1+\epsilon) + \frac{a^2}{9} \frac{3}{8\pi G} \frac{V' V''}{V} \right) - 1 \quad \approx \epsilon - \frac{1}{8\pi G} \frac{V''}{V}$$

so δ is related to second derivative of potential being small. Very flat potential.

Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation $(K = 0)$, same as scalar field

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0 \, .$$

- Acquires quantum fluctuations in same manner as ϕ. Lagrangian sets the normalization

$$\phi_1 \to H_T^{(\pm 2)}\sqrt{\frac{3}{16\pi G}}$$

- Scale-invariant gravitational wave amplitude (each component: NB more traditional notation $H_T^{(\pm 2)} = (h_+ \pm i h_\times)/\sqrt{6}$)

$$\Delta_H^2 = \frac{16\pi G}{3 \cdot 2\pi^2}\frac{H^2}{2} = \frac{4}{3\pi}\frac{H^2}{m_{\rm pl}^2}$$

Gravitational Waves

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation
- Tensor tilt:

$$\frac{d\ln\Delta_H^2}{d\ln k} \equiv n_T = 2\frac{d\ln H}{d\ln k} = -2\epsilon$$

- Consistency relation between tensor-scalar ratio and tensor tilt

$$\frac{\Delta_H^2}{\Delta_\zeta^2} = \frac{4}{3}\epsilon = -\frac{2}{3}\epsilon$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparision of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself

Gravitational Wave Phenomenology

- Equation of motion

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0\,.$$

- has solutions

$$H_T^{(\pm 2)} = C_1 H_1(k\eta) + C_2 H_2(k\eta)$$
$$H_1 \propto x^{-m} j_m(x)$$
$$H_2 \propto x^{-m} n_m(x)$$

where $m = (1-3w)/(1+3w)$

- If $w > -1/3$ then gravity wave is constant above horizon $x \ll 1$ and then oscillates and damps
- If $w < -1/3$ then gravity wave oscillates and freezes into some value, just like scalar field

Gravitational Wave Phenomenology

- A gravitational wave makes a quadrupolar (transverse-traceless) distortion to metric
- Just like the scale factor or spatial curvature, a temporal variation in its amplitude leaves a residual temperature variation in CMB photons – here anisotropic
- Before recombination, anisotropic variation is eliminated by scattering
- Gravitational wave temperature effect drops sharply at the horizon scale at recombination
- Source to polarization goes as $\dot{\tau}/\dot{H}_T$ and peaks at the horizon not damping scale
- B modes formed as photons propagate – the spatial variation in the plane waves modulate the signal: described by Boltzmann eqn.

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Boltzmann Formalism

Boltzmann Equation

- CMB radiation is generally described by the phase space distribution function for each polarization state $f_a(\mathbf{x}, \mathbf{q}, \eta)$, where $\mathbf{x}$ is the comoving position and $\mathbf{q}$ is the photon momentum
- Boltzmann equation describes the evolution of the distribution function under gravity and collisions
- Low order moments of the Boltzmann equation are simply the covariant conservation equations
- Higher moments provide the closure condition to the conservation law (specification of stress tensor) and the CMB observable – fine scale anisotropy
- Higher moments mainly describe the simple geometry of source projection

Liouville Equation

- In absence of scattering, the phase space distribution of photons is conserved along the propagation path
- Rewrite variables in terms of the photon propagation direction $\mathbf{q} = q\hat{\mathbf{n}}$, so $f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta)$ and

$$\begin{aligned}\frac{d}{d\eta} f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta) &= 0 \\ &= \left(\frac{\partial}{\partial\eta} + \frac{d\mathbf{x}}{d\eta} \cdot \frac{\partial}{\partial\mathbf{x}} + \frac{d\hat{\mathbf{n}}}{d\eta} \cdot \frac{\partial}{\partial\hat{\mathbf{n}}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a\end{aligned}$$

- For simplicity, assume spatially flat universe $K = 0$ then $d\hat{\mathbf{n}}/d\eta = 0$ and $d\mathbf{x} = \hat{\mathbf{n}} d\eta$

$$\dot{f}_a + \hat{\mathbf{n}} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0$$

Correspondence to Einstein Eqn.

- Geodesic equation gives the redshifting term

$$\frac{\dot{q}}{q} = -\frac{\dot{a}}{a} - \frac{1}{2} n^i n^j \dot{H}_{Tij} - \dot{H}_L + n^i \dot{B}_i - \hat{\mathbf{n}} \cdot \nabla A$$

- which is incorporated in the conservation and gauge transformation equations
- Stress energy tensor involves integrals over the distribution function the two polarization states

$$T^{\mu\nu} = \int \frac{d^3 q}{(2\pi)^3} \frac{q^\mu q^\nu}{E} (f_a + f_b)$$

- Components are simply the low order angular moments of the distribution function

Angular Moments

- Define the angularly dependent temperature perturbation

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \frac{1}{4\rho_\gamma} \int \frac{q^3 dq}{2\pi^2} (f_a + f_b) - 1$$

  and likewise for the linear polarization states Q and U

- Decompose into normal modes: plane waves for spatial part and spherical harmonics for angular part

$$G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k}\cdot\mathbf{x})$$

$${}_{\pm 2}G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} {}_{\pm 2}Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k}\cdot\mathbf{x})$$

- In a spatially curved universe generalize the plane wave part

Normal Modes

- Temperature and polarization fields

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} \Theta_\ell^{(m)} G_\ell^m$$

$$[Q \pm iU](\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} [E_\ell^{(m)} \pm iB_\ell^{(m)}]_{\pm 2} G_\ell^m$$

- For each **k** mode, work in coordinates where $\mathbf{k} \parallel \mathbf{z}$ and so $m = 0$ represents scalar modes, $m = \pm 1$ vector modes, $m = \pm 2$ tensor modes, $|m| > 2$ vanishes. Since modes add incoherently and $Q \pm iU$ is invariant up to a phase, rotation back to a fixed coordinate system is trivial.

Scalar, Vector, Tensor

- Normalization of modes is chosen so that the lowest angular mode for scalars, vectors and tensors are normalized in the same way as the mode function

$$G_0^0 = Q^{(0)} \quad G_1^0 = n^i Q_i^{(0)} \quad G_2^0 \propto n^i n^j Q_{ij}^{(0)}$$
$$G_1^{\pm 1} = n^i Q_i^{(\pm 1)} \quad G_2^{\pm 1} \propto n^i n^j Q_{ij}^{(\pm 1)}$$
$$G_2^{\pm 2} = n^i n^j Q_{ij}^{(\pm 2)}$$

where recall

$$\begin{aligned} Q^{(0)} &= \exp(i\mathbf{k}\cdot\mathbf{x}) \\ Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k}\cdot\mathbf{x}) \\ Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k}\cdot\mathbf{x}) \end{aligned}$$

Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies
- Spatial gradient term hits plane wave:

$$\hat{\mathbf{n}} \cdot \nabla e^{i\mathbf{k}\cdot\mathbf{x}} = i\hat{\mathbf{n}} \cdot \mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} = i\sqrt{\frac{4\pi}{3}} k Y_1^0(\hat{\mathbf{n}}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

- Dipole term adds to angular dependence through the addition of angular momentum

$$\sqrt{\frac{4\pi}{3}} Y_1^0 Y_\ell^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell+1)(2\ell-1)}} Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell+1)(2\ell+3)}} Y_{\ell+1}^m$$

where $\kappa_\ell^m = \sqrt{\ell^2 - m^2}$ is given by Clebsch-Gordon coefficients.

Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

$$\dot{\Theta}_\ell^{(m)} = k\left[\frac{\kappa_\ell^m}{2\ell+1}\Theta_{\ell-1}^{(m)} - \frac{\kappa_{\ell+1}^m}{2\ell+3}\Theta_{\ell+1}^{(m)}\right] - \dot{\tau}\Theta_\ell^{(m)} + S_\ell^{(m)}$$

  where $S_\ell^{(m)}$ are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic $\ell = 0$ temperature perturbation will eventually become a high order anisotropy by "free streaming" or simple projection
- Original CMB codes solved the full hierarchy equations out to the ℓ of interest.

Integral Solution

- Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface
- In general, the solution describes the decomposition of the source $S_\ell^{(m)}$ with its local angular dependence as seen at a distance $\mathbf{x} = D\hat{\mathbf{n}}$.
- Proceed by decomposing the angular dependence of the plane wave

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} j_\ell(kD) Y_\ell^0(\hat{\mathbf{n}})$$

- Recouple to the local angular dependence of G_ℓ^m

$$G_{\ell_s}^m = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} \alpha_{\ell_s \ell}^{(m)}(kD) Y_\ell^m(\hat{\mathbf{n}})$$

Integral Solution

- Projection kernels:

$$\ell_s = 0, \quad m = 0 \qquad\qquad \alpha_{0\ell}^{(0)} \equiv j_\ell$$

$$\ell_s = 1, \quad m = 0 \qquad\qquad \alpha_{1\ell}^{(0)} \equiv j'_\ell$$

- Integral solution:

$$\frac{\Theta_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S_{\ell_s}^{(m)} \alpha_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

- Power spectrum:

$$C_\ell = \frac{2}{\pi} \int \frac{dk}{k} \sum_m \frac{k^3 \langle \Theta_\ell^{(m)*} \Theta_\ell^{(m)} \rangle}{(2\ell+1)^2}$$

- Solving for C_ℓ reduces to solving for the behavior of a handful of sources

Polarization Hiearchy

- In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hiearchy:

$$\dot{E}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell+1)}B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}\right] - \dot{\tau}E_\ell^{(m)} + \mathcal{E}_\ell^{(m)}$$

$$\dot{B}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell+1)}B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}\right] - \dot{\tau}E_\ell^{(m)} + \mathcal{B}_\ell^{(m)}$$

  where ${}_2\kappa_\ell^m = \sqrt{(\ell^2-m^2)(\ell^2-4)/\ell^2}$ is given by the Clebsch-Gordon coefficients and $\mathcal{E}$, $\mathcal{B}$ are the sources (scattering only).

- Note that for vectors and tensors $|m| > 0$ and B modes may be generated from E modes by projection. Cosmologically $\mathcal{B}_\ell^{(m)} = 0$

Polarization Integral Solution

- Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

$$\frac{E_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \epsilon_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

$$\frac{B_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \beta_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

- The only source to the polarization is from the quadrupole anisotropy so we only need $\ell_s = 2$, e.g. for scalars

$$\epsilon_{2\ell}^{(0)}(x) = \sqrt{\frac{3}{8}\frac{(\ell+2)!}{(\ell-2)!}}\frac{j_\ell(x)}{x^2} \qquad \beta_{2\ell}^{(0)} = 0$$

Truncated Hierarchy

- CMBFast uses the integral solution and relies on a fast j_ℓ generator
- However sources are not external to system and are defined through the Boltzmann hierarchy itself
- Solution: recall that we used this technique in the tight coupling regime by applying a closure condition from tight coupling
- CMBFast extends this idea by solving a truncated hierarchy of equations, e.g. out to $\ell = 25$ with non-reflecting boundary conditions

Thomson Collision Term

- Full Boltzmann equation

$$\frac{d}{d\eta} f_{a,b} = C[f_a, f_b]$$

- Collision term describes the scattering out of and into a phase space element
- Thomson collision based on differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \,,$$

where $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

Scattering Calculation

- Start in the electron rest frame and in a coordinate system fixed by the scattering plane, spanned by incoming and outgoing directional vectors $-\hat{\mathbf{n}}' \cdot \hat{\mathbf{n}} = \cos\beta$, where β is the scattering angle
- $\Theta_{\|}$: in-plane polarization state; $\Theta_{\perp}$: $\perp$-plane polarization state
- Transfer probability (constant set by $\dot{\tau}$)

$$\Theta_{\|} \propto \cos^2\beta\, \Theta'_{\|}, \qquad \Theta_{\perp} \propto \Theta'_{\perp}$$

- and with the $45°$ axes as

$$\hat{\mathbf{E}}_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\|} + \hat{\mathbf{E}}_{\perp}), \qquad \hat{\mathbf{E}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\|} - \hat{\mathbf{E}}_{\perp})$$

Stokes Parameters

- Define the temperature in this basis

$$
\begin{aligned}
\Theta_1 &\propto |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_1|^2 \Theta_1' + |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2|^2 \Theta_2' \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta_1' + \frac{1}{4}(\cos\beta - 1)^2 \Theta_2' \\
\Theta_2 &\propto |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_2|^2 \Theta_2' + |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_1|^2 \Theta_1' \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta_2' + \frac{1}{4}(\cos\beta - 1)^2 \Theta_1'
\end{aligned}
$$

or $\Theta_1 - \Theta_2 \propto \cos\beta(\Theta_1' - \Theta_2')$

- Define Θ, Q, U in the scattering coordinates

$$
\Theta \equiv \frac{1}{2}(\Theta_\parallel + \Theta_\perp), \quad Q \equiv \frac{1}{2}(\Theta_\parallel - \Theta_\perp), \quad U \equiv \frac{1}{2}(\Theta_1 - \Theta_2)
$$

Scattering Matrix

- Transfer of Stokes states, e.g.

$$\Theta = \frac{1}{2}(\Theta_{\parallel} + \Theta_{\perp}) \propto \frac{1}{4}(\cos^2\beta + 1)\Theta' + \frac{1}{4}(\cos^2\beta - 1)Q'$$

- Transfer matrix of Stokes state $\mathbf{T} \equiv (\Theta, Q + iU, Q - iU)$

$$\mathbf{T} \propto \mathbf{S}(\beta)\mathbf{T}'$$

$$\mathbf{S}(\beta) = \frac{3}{4}\begin{pmatrix} \cos^2\beta + 1 & -\frac{1}{2}\sin^2\beta & -\frac{1}{2}\sin^2\beta \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta + 1)^2 & \frac{1}{2}(\cos\beta - 1)^2 \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta - 1)^2 & \frac{1}{2}(\cos\beta + 1)^2 \end{pmatrix}$$

normalization factor of 3 is set by photon conservation in scattering

Scattering Matrix

- Transform to a fixed basis, by a rotation of the incoming and outgoing states $\mathbf{T} = \mathbf{R}(\psi)\mathbf{T}$ where

$$\mathbf{R}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & e^{2i\psi} \end{pmatrix}$$

giving the scattering matrix

$$\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha) =$$

$$\frac{1}{2}\sqrt{\frac{4\pi}{5}}\begin{pmatrix} Y_2^0(\beta,\alpha) + 2\sqrt{5}Y_0^0(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^{-2}(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^{2}(\beta,\alpha) \\ -\sqrt{6}\,{}_2Y_2^0(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^{-2}(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^{2}(\beta,\alpha)e^{2i\gamma} \\ -\sqrt{6}\,{}_{-2}Y_2^0(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^{-2}(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^{2}(\beta,\alpha)e^{-2i\gamma} \end{pmatrix}$$

Addition Theorem for Spin Harmonics

- Spin harmonics are related to rotation matrices as

$$ {}_sY_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\mathcal{D}^\ell_{-ms}(\phi,\theta,0) $$

Note: for explicit evaluation sign convention differs from usual (e.g. Jackson) by $(-1)^m$

- Multiplication of rotations

$$ \sum_{m''}\mathcal{D}^\ell_{mm''}(\alpha_2,\beta_2,\gamma_2)\mathcal{D}^\ell_{m''m}(\alpha_1,\beta_1,\gamma_1) = \mathcal{D}^\ell_{mm'}(\alpha,\beta,\gamma) $$

- Implies

$$ \sum_m {}_{s_1}Y_\ell^{m*}(\theta',\phi')\ {}_{s_2}Y_\ell^m(\theta,\phi) = (-1)^{s_1-s_2}\sqrt{\frac{2\ell+1}{4\pi}}\ {}_{s_2}Y_\ell^{-s_1}(\beta,\alpha)e^{is_2\gamma} $$

Sky Basis

- Scattering into the state (rest frame)

$$C_{\rm in}[\mathbf{T}] = \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi} \mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha)\mathbf{T}(\hat{\mathbf{n}}')\,,$$

$$= \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi}(\Theta', 0, 0) + \frac{1}{10}\dot\tau \int d\hat{\mathbf{n}}' \sum_{m=-2}^{2} \mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}')\mathbf{T}(\hat{\mathbf{n}}')\,.$$

where the quadrupole coupling term is $\mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') =$

$$\left(\begin{array}{ccc} Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\, {}_2Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\, {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}')\, {}_2Y_2^m(\hat{\mathbf{n}}) & 3\, {}_2Y_2^{m*}(\hat{\mathbf{n}}')\, {}_2Y_2^m(\hat{\mathbf{n}}) & 3\, {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\, {}_2Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}')\, {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\, {}_2Y_2^{m*}(\hat{\mathbf{n}}')\, {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\, {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\, {}_{-2}Y_2^m(\hat{\mathbf{n}}) \end{array} \right),$$

expression uses angle addition relation above. We call this term C_Q.

Scattering Matrix

- Full scattering matrix involves difference of scattering into and out of state

$$C[\mathbf{T}] = C_{\rm in}[\mathbf{T}] - C_{\rm out}[\mathbf{T}]$$

- In the electron rest frame

$$C[\mathbf{T}] = \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi} (\Theta', 0, 0) - \dot\tau \mathbf{T} + C_Q[\mathbf{T}]$$

which describes isotropization in the rest frame. All moments have $e^{-\tau}$ suppression except for isotropic temperature Θ_0.
Transformation into the background frame simply induces a dipole term

$$C[\mathbf{T}] = \dot\tau \left(\hat{\mathbf{n}} \cdot \mathbf{v}_b + \int \frac{d\hat{\mathbf{n}}'}{4\pi} \Theta', 0, 0 \right) - \dot\tau \mathbf{T} + C_Q[\mathbf{T}]$$

Source Terms

- Temperature source terms $S_l^{(m)}$ (rows $\pm|m|$; flat assumption

$$
\begin{pmatrix}
\dot\tau\Theta_0^{(0)} - \dot H_L^{(0)} & \dot\tau v_b^{(0)} + \dot B^{(0)} & \dot\tau P^{(0)} - \frac{2}{3}\dot H_T^{(0)} \\
0 & \dot\tau v_b^{(\pm1)} + \dot B^{(\pm1)} & \dot\tau P^{(\pm1)} - \frac{\sqrt{3}}{3}\dot H_T^{(\pm1)} \\
0 & 0 & \dot\tau P^{(\pm2)} - \dot H_T^{(\pm2)}
\end{pmatrix}
$$

where

$$
P^{(m)} \equiv \frac{1}{10}(\Theta_2^{(m)} - \sqrt{6}E_2^{(m)})
$$

- Polarization source term

$$
\mathcal{E}_\ell^{(m)} = -\dot\tau\sqrt{6}P^{(m)}\delta_{\ell,2}
$$

$$
\mathcal{B}_\ell^{(m)} = 0
$$

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Secondary Anisotropy

Secondary Anisotropy

- CMB photons traverse the large-scale structure of the universe from $z = 1000$ to the present.
- With the nearly scale-invariant adiabatic fluctuations observed in the CMB, structures form from the bottom up, i.e. small scales first, a.k.a. hierarchical structure formation.
- First objects reionize the universe between $z \sim 7 - 30$
- Main sources of secondary anisotropy
- Gravitational: Integrated Sachs-Wolfe effect (gravitational redshift) and gravitational lensing
- Scattering: peak suppression, large-angle polarization, Doppler effect(s), inverse Compton scattering

Transfer Function

- Transfer function transfers the initial Newtonian curvature to its value today (linear response theory)

$$T(k) = \frac{\Phi(k, a=1)}{\Phi(k, a_{\rm init})} \frac{\Phi(k_{\rm norm}, a_{\rm init})}{\Phi(k_{\rm norm}, a=1})$$

- Conservation of Bardeen curvature: Newtonian curvature is a constant when stress perturbations are negligible: above the horizon during radiation and dark energy domination, on all scales during matter domination
- When stress fluctuations dominate, perturbations are stabilized by the Jeans mechanism
- Hybrid Poisson equation: Newtonian curvature, comoving density perturbation $\Delta \equiv (\delta\rho/\rho)_{\rm com}$ implies Φ decays

$$(k^2 - 3K)\Phi = 4\pi G\rho\Delta \sim \eta^{-2}\Delta$$

Transfer Function

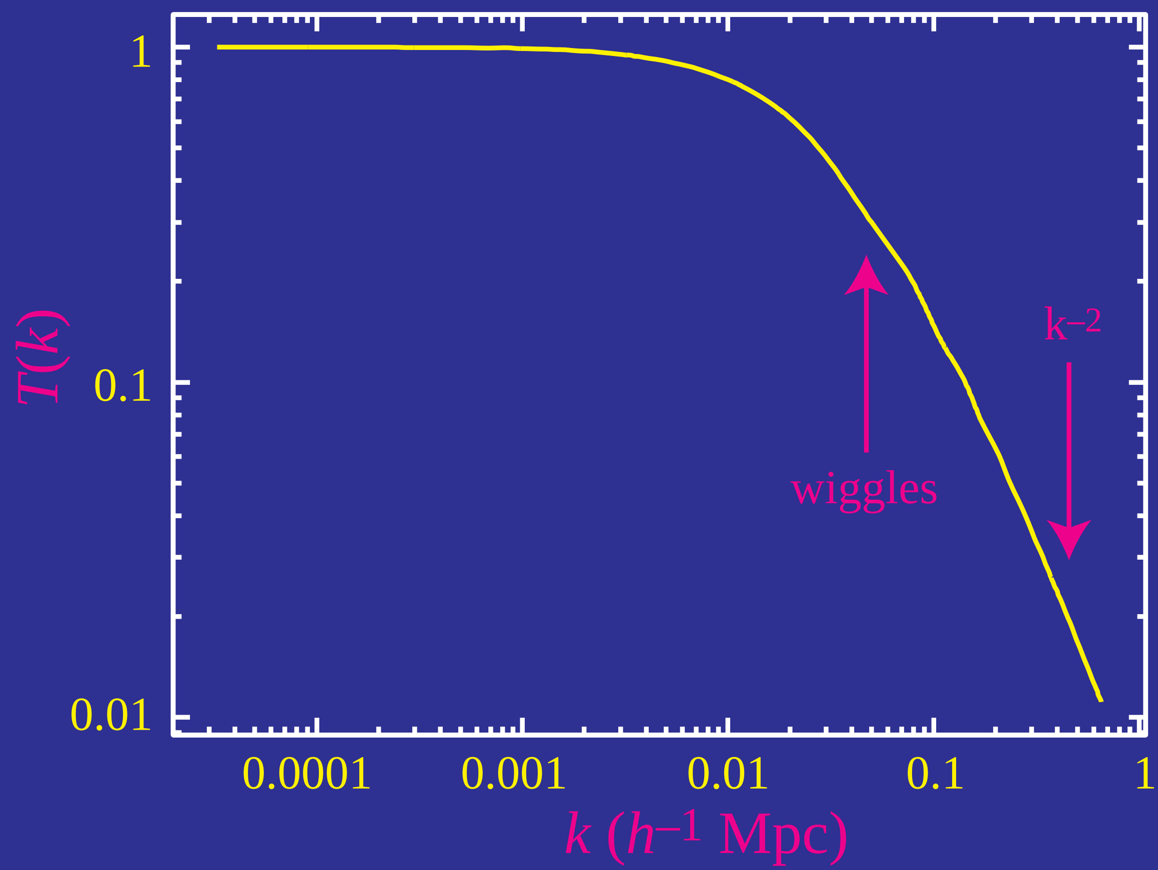
- Matter-radiation example: Jeans scale is horizon scale and Δ freezes into its value at horizon crossing $\Delta_H \approx \Phi_{\rm init}$
- Freezing of Δ stops at $\eta_{\rm eq}$

$$\Phi \sim (k\eta_{\rm eq})^{-2}\Delta_H \sim (k\eta_{\rm eq})^{-2}\Phi_{\rm init}$$

- Conventionally $k_{\rm norm}$ is chosen as a scale between the horizon at matter radiation equality and dark energy domination.
- Small correction since growth with a smooth radiation component is logarithmic not frozen
- Run CMBfast to get transfer function or use fits

Transfer Function

- Transfer function has a k^{-2} fall-off beyond $k_{\rm eq} \sim \eta_{\rm eq}^{-1}$



- Additional baryon wiggles are due to acoustic oscillations at recombination – an interesting means of measuring distances

Growth Function

- Same physics applies to the dark energy dominated universe
- Under the dark energy sound horizon, dark energy density frozen. Potential decays at the same rate for all scales

$$g(a) = \frac{\Phi(k_{\rm norm}, a)}{\Phi(k_{\rm norm}, a_{\rm init})}$$

- Pressure growth suppression: $\delta \equiv \delta\rho_m/\rho_m \propto a\phi$

$$\frac{d^2 g}{d\ln a^2} + \left[\frac{5}{2} - \frac{3}{2} w(z)\Omega_{DE}(z)\right]\frac{dg}{d\ln a} + \frac{3}{2}[1 - w(z)]\Omega_{DE}(z) g = 0\,,$$

where $w \equiv p_{DE}/\rho_{DE}$ and $\Omega_{DE} \equiv \rho_{DE}/(\rho_m + \rho_{DE})$ with initial conditions $g = 1$, $dg/d\ln a = 0$

- As $\Omega_{DE} \to 0$ $g =$const. is a solution. The other solution is the decaying mode, elimated by initial conditions

ISW effect

- Potential decay leads to gravitational redshifts through the integrated Sachs-Wolfe effect
- Intrinsically a large effect since $2\Delta\Phi = 6\Psi_{\rm init}/3$
- But net redshift is integral along along line of sight

$$\frac{\Theta_\ell(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau}[2\dot{\Phi}(k,\eta)] j_\ell(k(\eta_0-\eta))$$
$$= 2\Phi(k,\eta_{MD}) \int_0^{\eta_0} d\eta e^{-\tau} \dot{g}(D) j_\ell(kD)$$

- On small scales where $k \gg \dot{g}/g$, can pull source out of the integral

$$\int_0^{\eta_0} d\eta \dot{g}(D) j_\ell(kD) \approx \dot{g}(D=\ell/k)\frac{1}{k}\sqrt{\frac{\pi}{2\ell}}$$

evaluated at peak, where we have used $\int dx j_\ell(x) = \sqrt{\pi/2\ell}$

ISW effect

- Power spectrum

$$C_\ell = \frac{2}{\pi}\int \frac{dk}{k}\frac{k^3\langle\Theta_\ell^*(k,\eta_0)\Theta_\ell(k,\eta_0)\rangle}{(2\ell+1)^2}$$
$$= \frac{2\pi^2}{l^3}\int d\eta D\dot{g}^2(\eta)\Delta_\Phi^2(\ell/D,\eta_{MD})$$

- Or $l^2C_l/2\pi \propto 1/\ell$ for scale invariant potential. This is the Limber equation in spherical coordinates. Projection of $3D$ power retains only the transverse piece. For a general dark energy model, add in the scale dependence of growth rate on large scales.
- Cancellation of redshifts and blueshifts as the photon traverses many crests and troughs of a small scale fluctuation during decay. Enhancement of the $\ell < 10$ multipoles. Difficult to extract from cosmic variance and galaxy. Current ideas: cross correlation with other tracers of structure

Gravitational Lensing

- Lensing is a surface brightness conserving remapping of source to image planes by the gradient of the projected potential

$$\phi(\hat{\mathbf{n}}) = 2 \int_{\eta_*}^{\eta_0} d\eta \, \frac{(D_* - D)}{D \, D_*} \Phi(D\hat{\mathbf{n}}, \eta) \, .$$

such that the fields are remapped as

$$x(\hat{\mathbf{n}}) \rightarrow x(\hat{\mathbf{n}} + \nabla\phi) \, ,$$

where $x \in \{\Theta, Q, U\}$ temperature and polarization.

- Taylor expansion leads to product of fields and Fourier mode-coupling

Flat-sky Treatment

- Talyor expand

$$\Theta(\hat{\mathbf{n}}) = \tilde{\Theta}(\hat{\mathbf{n}} + \nabla\phi)$$

$$= \tilde{\Theta}(\hat{\mathbf{n}}) + \nabla_i\phi(\hat{\mathbf{n}})\nabla^i\tilde{\Theta}(\hat{\mathbf{n}}) + \frac{1}{2}\nabla_i\phi(\hat{\mathbf{n}})\nabla_j\phi(\hat{\mathbf{n}})\nabla^i\nabla^j\tilde{\Theta}(\hat{\mathbf{n}}) + \ldots$$

- Fourier decomposition

$$\phi(\hat{\mathbf{n}}) = \int \frac{d^2l}{(2\pi)^2}\phi(\mathbf{l})e^{i\mathbf{l}\cdot\hat{\mathbf{n}}}$$

$$\tilde{\Theta}(\hat{\mathbf{n}}) = \int \frac{d^2l}{(2\pi)^2}\tilde{\Theta}(\mathbf{l})e^{i\mathbf{l}\cdot\hat{\mathbf{n}}}$$

Flat-sky Treatment

- Mode coupling of harmonics

$$
\begin{aligned}
\Theta(\mathbf{l}) &= \int d\hat{\mathbf{n}}\, \Theta(\hat{\mathbf{n}}) e^{-il\cdot\hat{\mathbf{n}}} \\
&= \tilde{\Theta}(\mathbf{l}) - \int \frac{d^2\mathbf{l}_1}{(2\pi)^2} \tilde{\Theta}(\mathbf{l}_1) L(\mathbf{l}, \mathbf{l}_1)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
L(\mathbf{l}, \mathbf{l}_1) &= \phi(\mathbf{l} - \mathbf{l}_1)\,(\mathbf{l} - \mathbf{l}_1)\cdot \mathbf{l}_1 \\
&+ \frac{1}{2}\int \frac{d^2\mathbf{l}_2}{(2\pi)^2} \phi(\mathbf{l}_2)\phi^*(\mathbf{l}_2 + \mathbf{l}_1 - \mathbf{l})\,(\mathbf{l}_2 \cdot \mathbf{l}_1)(\mathbf{l}_2 + \mathbf{l}_1 - \mathbf{l})\cdot \mathbf{l}_1\,.
\end{aligned}
$$

- Represents a coupling of harmonics separated by $L \approx 60$ peak of deflection power

Power Spectrum

- Power spectra

$$\langle \Theta^*(\mathbf{l})\Theta(\mathbf{l}')\rangle = (2\pi)^2\delta(\mathbf{l}-\mathbf{l}')\, C_l^{\Theta\Theta}\,,$$

$$\langle \phi^*(\mathbf{l})\phi(\mathbf{l}')\rangle = (2\pi)^2\delta(\mathbf{l}-\mathbf{l}')\, C_l^{\phi\phi}\,,$$

becomes

$$C_l^{\Theta\Theta} = \left(1 - l^2 R\right)\tilde{C}_l^{\Theta\Theta} + \int \frac{d^2\mathbf{l}_1}{(2\pi)^2}\tilde{C}^{\Theta\Theta}_{|\mathbf{l}-\mathbf{l}_1|} C^{\phi\phi}_{l_1}[(\mathbf{l}-\mathbf{l}_1)\cdot\mathbf{l}_1]^2\,,$$

where

$$R = \frac{1}{4\pi}\int \frac{dl}{l}\, l^4 C_l^{\phi\phi}\,. \tag{3}$$

Smoothing Power Spectrum

- If $\tilde{C}_l^{\Theta\Theta}$ slowly varying then two term cancel

$$\tilde{C}_l^{\Theta\Theta} \int \frac{d^2\mathbf{l}_1}{(2\pi)^2} C_l^{\phi\phi} (\mathbf{l} \cdot \mathbf{l}_1)^2 \approx l^2 R \tilde{C}_l^{\Theta\Theta} \, .$$

- So lensing acts to smooth features in the power spectrum. Smoothing kernel is $L \sim 60$ the peak of deflection power spectrum
- Because acoustic feature appear on a scale $l_A \sim 300$, smoothing is a subtle effect in the power spectrum.
- Lensing generates power below the damping scale which directly reflect power in deflections on the same scale

Polarization Lensing

- Polarization field harmonics lensed similarly

$$[Q \pm iU](\hat{\mathbf{n}}) = -\int \frac{d^2l}{(2\pi)^2}[E \pm iB](\mathbf{l})e^{\pm 2i\phi_{\mathbf{l}}}e^{\mathbf{l}\cdot\hat{\mathbf{n}}}$$

so that

$$\begin{aligned}[Q \pm iU](\hat{\mathbf{n}}) &= [\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}} + \nabla\phi) \\ &\approx [\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}}) + \nabla_i\phi(\hat{\mathbf{n}})\nabla^i[\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}}) \\ &\quad + \frac{1}{2}\nabla_i\phi(\hat{\mathbf{n}})\nabla_j\phi(\hat{\mathbf{n}})\nabla^i\nabla^j[\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}})\end{aligned}$$

Polarization Power Spectra

- Carrying through the algebra

$$
\begin{aligned}
C_l^{EE} &= \left(1 - l^2 R\right) \tilde{C}_l^{EE} + \frac{1}{2} \int \frac{d^2 \mathbf{l}_1}{(2\pi)^2} [(\mathbf{l} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2 C^{\phi\phi}_{|\mathbf{l}-\mathbf{l}_1|} \\
&\quad \times \left[(\tilde{C}_{l_1}^{EE} + \tilde{C}_{l_1}^{BB}) + \cos(4\varphi_{l_1})(\tilde{C}_{l_1}^{EE} - \tilde{C}_{l_1}^{BB}) \right], \\
C_l^{BB} &= \left(1 - l^2 R\right) \tilde{C}_l^{BB} + \frac{1}{2} \int \frac{d^2 \mathbf{l}_1}{(2\pi)^2} [(\mathbf{l} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2 C^{\phi\phi}_{|\mathbf{l}-\mathbf{l}_1|} \\
&\quad \times \left[(\tilde{C}_{l_1}^{EE} + \tilde{C}_{l_1}^{BB}) - \cos(4\varphi_{l_1})(\tilde{C}_{l_1}^{EE} - \tilde{C}_{l_1}^{BB}) \right], \\
C_l^{\Theta E} &= \left(1 - l^2 R\right) \tilde{C}_l^{\Theta E} + \int \frac{d^2 \mathbf{l}_1}{(2\pi)^2} [(\mathbf{l} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2 C^{\phi\phi}_{|\mathbf{l}-\mathbf{l}_1|} \\
&\quad \times \tilde{C}_{l_1}^{\Theta E} \cos(2\varphi_{l_1}),
\end{aligned}
$$

- Lensing generates B-modes out of the acoustic polaraization E-modes contaminates gravitational wave signature if $E_i < 10^{16}$GeV.

Reconstruction from the CMB

- Correlation between Fourier moments reflect lensing potential

$$\langle x(\mathbf{l})x'(\mathbf{l}')\rangle_{\rm CMB} = f_\alpha(\mathbf{l},\mathbf{l}')\phi(\mathbf{l}+\mathbf{l}')\,,$$

  where $x \in$ temperature, polarization fields and f_α is a fixed weight that reflects geometry

- Each pair forms a noisy estimate of the potential or projected mass - just like a pair of galaxy shears
- Minimum variance weight all pairs to form an estimator of the lensing mass

Scattering Secondaries

- Optical depth during reionization

$$\tau \approx 0.066 \left(\frac{\Omega_b h^2}{0.02}\right) \left(\frac{\Omega_m h^2}{0.15}\right)^{-1/2} \left(\frac{1+z}{10}\right)^{3/2}$$

- Anisotropy suppressed as $e^{-\tau}$. Integral solution

$$\frac{\Theta_\ell(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} S_0^{(0)} j_\ell(k(\eta_0-\eta)) + \ldots$$

- Isotropic (lare scale) fluctuations not supressed since suppression represents isotropization by scattering
- Quadrupole from the Sachs-Wolfe effect scatters into a large angle polarization bump

Doppler Effects

- Velocity fields of 10^{-3} and optical depths of 10^{-2} would imply large Doppler effect due to reionization
- Limber approximation says only fluctuations transverse to line of sight survive
- In linear theory, transverse fluctuations have no line of sight velocity and so Doppler effect is highly suppressed.
- Beyond linear theory: modulate the optical depth in the transverse direction using density fluctuations or ionization fraction fluctuations. Generate a modulated Doppler effect
- Linear fluctuations: Vishniac effect; Clusters: kinetic SZ effect; ionization patches: inhomogeneous reionization effect

Thermal SZ Effect

- Thermal velocities also lead to Doppler effect but first order contribution cancels because of random directions
- Residual effect is of order $v^2\tau \approx T_e/m_e\,\tau$ and can reach a sizeable level for clusters with $T_e \approx 10$keV.
- Raleigh-Jeans decrement and Wien enhancement described by second order collision term in Boltzmann equation: Kompaneets equation
- Clusters are rare objects so contribution to power spectrum suppressed, but may have been detected by CBI/BIMA: extremely sensitive to power spectrum normalization σ_8
- White noise on large-scales $(l < 2000)$, turnover as cluster profile is resolved

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Data Pipeline

Gaussian Statistics

- Statistical isotropy says two-point correlation depends only on the power spectrum

$$\Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m}(\hat{\mathbf{n}})$$

$$\langle \Theta^*_{\ell m} \Theta_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{\Theta\Theta}$$

- Reality of field says $\Theta_{\ell m} = (-1)^m \Theta_{\ell(-m)}$
- For a Gaussian random field, power spectrum defines all higher order statistics, e.g.

$$\langle \Theta_{\ell_1 m_1} \Theta_{\ell_2 m_2} \Theta_{\ell_3 m_3} \Theta_{\ell_4 m_4} \rangle$$
$$= (-1)^{m_1+m_2} \delta_{\ell_1 \ell_3} \delta_{m_1(-m_3)} \delta_{\ell_2 \ell_4} \delta_{m_2(-m_4)} C_{\ell_1}^{\Theta\Theta} C_{\ell_2}^{\Theta\Theta} + \text{all pairs}$$

Idealized Statistical Errors

- Take a noisy estimator of the multipoles in the map

$$\hat{\Theta}_{\ell m} = \Theta_{\ell m} + N_{\ell m}$$

and take the noise to be statistically isotropic

$$\langle N^*_{\ell m} N_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell^{NN}$$

- Construct an unbiased estimator of the power spectrum $\langle \hat{C}_\ell^{\Theta\Theta} \rangle = C_\ell^{\Theta\Theta}$

$$\hat{C}_\ell^{\Theta\Theta} = \frac{1}{2\ell+1} \sum_{m=-l}^{l} \hat{\Theta}^*_{\ell m} \hat{\Theta}_{\ell m} - C_\ell^{NN}$$

- Variance in estimator

$$\langle \hat{C}_\ell^{\Theta\Theta} \hat{C}_\ell^{\Theta\Theta} \rangle - \langle \hat{C}_\ell^{\Theta\Theta} \rangle^2 = \frac{2}{2\ell+1} (C_\ell^{\Theta\Theta} + C_\ell^{NN})^2$$

Cosmic and Noise Variance

- RMS in estimator is simply the total power spectrum reduced by $\sqrt{2/N_{\rm modes}}$ where $N_{\rm modes}$ is the number of m-mode measurements
- Even a perfect experiment where $C_\ell^{NN} = 0$ has statistical variance due to the Gaussian random realizations of the field. This cosmic variance is the result of having only one realization to measure.
- Noise variance is often approximated as white detector noise. Removing the beam to place the measurement on the sky

$$N_\ell^{\Theta\Theta} = \left(\frac{T}{d_T}\right)^2 e^{\ell(\ell+1)\sigma^2} = \left(\frac{T}{d_T}\right)^2 e^{\ell(\ell+1){\rm FWHM}^2/8\ln 2}$$

where d_T can be thought of as a noise level per steradian of the temperature measurement, σ is the Gaussian beam width, FWHM is the full width at half maximum of the beam

Idealized Parameter Forecasts

- A crude propagation of errors is often useful for estimation purposes.
- Suppose $C_{\alpha\beta}$ describes the covariance matrix of the estimators for a given parameter set π_α.
- Define $\mathbf{F} = \mathbf{C}^{-1}$ [formalized as the Fisher matrix later]. Making an infinitesimal transformation to a new set of parameters p_μ

$$F_{\mu\nu} = \sum_{\alpha\beta} \frac{\partial \pi_\alpha}{\partial p_\mu} F_{\alpha\beta} \frac{\partial \pi_\beta}{\partial p_\nu}$$

- In our case π_α are the C_ℓ the covariance is diagonal and p_μ are cosmological parameters

$$F_{\mu\nu} = \sum_\ell \frac{2\ell+1}{2(C_\ell^{\Theta\Theta} + C_\ell^{NN})^2} \frac{\partial C_\ell^{\Theta\Theta}}{\partial p_\mu} \frac{\partial C_\ell^{\Theta\Theta}}{\partial p_\nu}$$

Idealized Parameter Forecasts

- Polarization handled in same way (requires covariance)
- Fisher matrix represents a local approximation to the transformation of the covariance and hence is only accurate for well constrained directions in parameter space
- Derivatives evaluated by finite difference
- Fisher matrix identifies parameter degeneracies but only the local direction – i.e. all errors are ellipses not bananas

Beyond Idealizations: Time Ordered Data

- For the data analyst the starting point is a string of "time ordered" data coming out of the instrument (post removal of systematic errors!)
- Begin with a model of the time ordered data as

$$d_t = P_{ti}\Theta_i + n_t$$

  where i denotes pixelized positions indexed by i, d_t is the data in a time ordered stream indexed by t. Number of time ordered data will be of the order 10^{10} for a satellite! number of pixels $10^6 - 10^7$.
- The noise n_t is drawn from a distribution with a known power spectrum

$$\langle n_t n_{t'} \rangle = C_{d,tt'}$$

Pointing Matrix

- The pointing matrix $\mathbf{P}$ is the mapping between pixel space and the time ordered data
- Simplest incarnation: row with all zeros except one column which just says what point in the sky the telescope is pointing at that time

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 1 & \ldots & 0 \end{pmatrix}$$

- More generally encorporates differencing, beam, rotation (for polarization)

Maximum Likelihood Mapmaking

- What is the best estimator of the underlying map Θ_i
- Likelihood function: the probability of getting the data given the theory $\mathcal{L} \equiv P[\mathrm{data}|\mathrm{theory}]$. In this case, the *theory* is the set of parameters Θ_i.

$$\mathcal{L}_\Theta(d_t) = \frac{1}{(2\pi)^{N_t/2}\sqrt{\det \mathbf{C}_d}} \exp\left[-\frac{1}{2}\left(d_t - P_{ti}\Theta_i\right) C^{-1}_{d,tt'} \left(d_{t'} - P_{t'j}\Theta_j\right)\right] .$$

- Bayes theorem says that $P[\Theta_i|d_t]$, the probability that the temperatures are equal to Θ_i given the data, is proportional to the likelihood function times a *prior* $P(\Theta_i)$, taken to be uniform

$$P[\Theta_i|d_t] \propto P[d_t|\Theta_i] \equiv \mathcal{L}_\Theta(d_t)$$

Maximum Likelihood Mapmaking

- Maximizing the likelihood of Θ_i is simple since the log-likelihood is quadratic.
- Differentiating the argument of the exponential with respect to Θ_i and setting to zero leads immediately to the estimator

$$\hat{\Theta}_i = C_{N,ij} P_{jt} C^{-1}_{d,tt'} d_{t'} \ ,$$

where $\mathbf{C}_N \equiv (\mathbf{P}^{\mathrm{tr}} \mathbf{C}_d^{-1} \mathbf{P})^{-1}$ is the covariance of the estimator

- Given the large dimension of the time ordered data, direct matrix manipulation is unfeasible. A key simplifying assumption is the stationarity of the noise, that $C_{d,tt'}$ depends only on $t - t'$ (temporal statistical homogeneity)

Power Spectrum

- The next step in the chain of inference is the power spectrum extraction. Here the correlation between pixels is modelled through the power spectrum

$$C_{S,ij} \equiv \langle \Theta_i \Theta_j \rangle = \sum_{\ell} \Delta^2_{T,\ell} W_{\ell,ij}$$

- Here W_ℓ, the window function, is derived by writing down the expansion of $\Theta(\hat{\mathbf{n}})$ in harmonic space, including smoothing by the beam and pixelization
- For example in the simple case of a gaussian beam of width σ it is proportional to the Legendre polynomial $P_\ell(\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j)$ for the pixel separation multiplied by $b_\ell^2 \propto e^{-\ell(\ell+1)\sigma^2}$

Band Powers

- In principle the underlying theory to extract from maximum likelihood is the power spectrum at every ℓ
- However with a finite patch of sky, it is not possible to extract multipoles separated by $\Delta\ell < 2\pi/L$ where L is the dimension of the survey
- So consider instead a theory parameterization of $\Delta^2_{T,\ell}$ constant in bands of $\Delta\ell$ chosen to match the survey forming a set of band powers B_a
- The likelihood of the bandpowers given the pixelized data is

$$\mathcal{L}_B(\Theta_i) = \frac{1}{(2\pi)^{N_p/2}\sqrt{\det \mathbf{C}_\Theta}} \exp\left(-\frac{1}{2}\Theta_i C^{-1}_{\Theta,ij}\Theta_j\right)$$

where $\mathbf{C}_\Theta = \mathbf{C}_S + \mathbf{C}_N$ and N_p is the number of pixels in the map.

Band Power Esitmation

- As before, $\mathcal{L}_B$ is Gaussian in the anisotropies Θ_i, but in this case Θ_i are *not* the parameters to be determined; the theoretical parameters are the B_a, upon which the covariance matrix depends.
- The likelihood function is not Gaussian in the parameters, and there is no simple, analytic way to find the maximum likelihood bandpowers
- Iterative approach to maximizing the likelihood: take a trial point $B_a^{(0)}$ and improve estimate based a Newton-Rhapson approach to finding zeros

$$
\begin{aligned}
\hat{B}_a &= \hat{B}_a^{(0)} + F_{B,ab} \frac{\partial \ln \mathcal{L}_B}{\partial B_b} \\
&= \hat{B}_a^{(0)} + \frac{1}{2} F_{B,ab}^{-1} \left(\Theta_i C_{\Theta,ij}^{-1} \frac{\partial C_{\Theta,jk}}{\partial B_b} C_{\Theta,kl}^{-1} \Theta_l - C_{\Theta,ij}^{-1} \frac{\partial C_{\Theta,ji}}{\partial B_b} \right) ,
\end{aligned}
$$

Fisher Matrix

- The expectation value of the local curvature is the Fisher matrix

$$F_{B,ab} \equiv \left\langle -\frac{\partial^2 \ln \mathcal{L}_\Theta}{\partial B_a \partial B_b} \right\rangle$$
$$= \frac{1}{2} C^{-1}_{\Theta,ij} \frac{\partial C_{\Theta,jk}}{\partial B_a} C^{-1}_{\Theta,kl} \frac{\partial C_{\Theta,li}}{\partial B_b} \, .$$

- This is a general statement: for a gaussian distribution the Fisher matrix

$$F_{ab} = \frac{1}{2} \mathrm{Tr}[\mathbf{C}^{-1}\mathbf{C}_{,a}\mathbf{C}^{-1}\mathbf{C}_{,b}]$$

- Kramer-Rao identity says that the best possible covariance matrix on a set of parameters is $\mathbf{C} = \mathbf{F}^{-1}$
- Thus, the iteration returns an estimate of the covariance matrix of the estimators $\mathbf{C}_B$

Cosmological Parameters

- The probability distribution of the bandpowers given the cosmological parameters c_i is not Gaussian but it is often an adequate approximation

$$\mathcal{L}_c(\hat{B}_a) \approx \frac{1}{(2\pi)^{N_c/2}\sqrt{\det \mathbf{C}_B}} \exp\left[-\frac{1}{2}(\hat{B}_a - B_a) C^{-1}_{B,ab} (\hat{B}_b - B_b)\right]$$

- Grid based approaches evaluate the likelihood in cosmological parameter space and maximize
- Faster approaches monte carlo the exploration of the likelihood space intelligently ("Monte Carlo Markov Chains")
- Since the number of cosmological parameters in the working model is $N_c \sim 10$ this represents a final radical compression of information in the original timestream which recall has up to $N_t \sim 10^{10}$ data points.

Parameter Forecasts

- The Fisher matrix of the cosmological parameters becomes

$$F_{c,ij} = \frac{\partial B_a}{\partial c_i} C^{-1}_{B,ab} \frac{\partial B_b}{\partial c_j} \, .$$

which is the error propagation formula discussed above

- The Fisher matrix can be more accurately defined for an experiment by taking the pixel covariance and using the general formula for the Fisher matrix of gaussian data
- Corrects for edge effects with the approximate effect of

$$F_{\mu\nu} = \sum_\ell \frac{(2\ell+1) f_{\rm sky}}{2(C_\ell^{\Theta\Theta} + C_\ell^{NN})^2} \frac{\partial C_\ell^{\Theta\Theta}}{\partial p_\mu} \frac{\partial C_\ell^{\Theta\Theta}}{\partial p_\nu}$$

where the sky fraction $f_{\rm sky}$ quantifies the loss of independent modes due to the sky cut