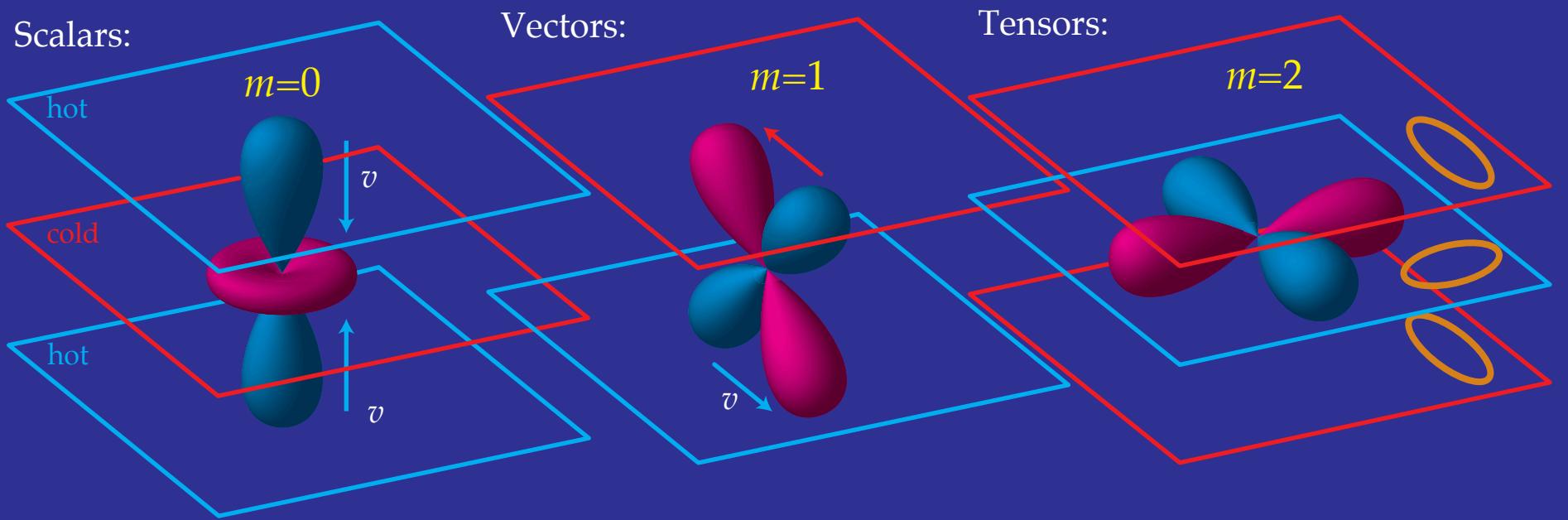


# Lecture IV



## Formalism & Codes

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Tenerife, November 2007

# Linear Perturbation Theory

# Covariant Perturbation Theory

- Covariant = takes same form in all coordinate systems
- Invariant = takes the same value in all coordinate systems
- Fundamental equations: Einstein equations, covariant conservation of stress-energy tensor:

$$\begin{aligned} G_{\mu\nu} &= 8\pi G T_{\mu\nu} \\ \nabla_\mu T^{\mu\nu} &= 0 \end{aligned}$$

- Preserve general covariance by keeping all degrees of freedom: 10 for each symmetric  $4 \times 4$  tensor

1	2	3	4
	5	6	7
		8	9
			10

# Metric Tensor

- Expand the metric tensor around the general FRW metric

$$g_{00} = -a^2, \quad g_{ij} = a^2 \gamma_{ij}.$$

where the “0” component is conformal time  $\eta = dt/a$  and  $\gamma_{ij}$  is a spatial metric of constant curvature  $K = H_0^2(\Omega_{\text{tot}} - 1)$ .

- Add in a general perturbation (Bardeen 1980)

$$\begin{aligned} g^{00} &= -a^{-2}(1 - 2A), \\ g^{0i} &= -a^{-2}B^i, \\ g^{ij} &= a^{-2}(\gamma^{ij} - 2H_L\gamma^{ij} - 2H_T^{ij}). \end{aligned}$$

- (1)  $A$   $\equiv$  a scalar potential; (3)  $B^i$  a vector shift, (1)  $H_L$  a perturbation to the spatial curvature; (5)  $H_T^{ij}$  a trace-free distortion to spatial metric = (10)

# Matter Tensor

- Likewise expand the matter stress energy tensor around a homogeneous density  $\rho$  and pressure  $p$ :

$$\begin{aligned} T^0_0 &= -\rho - \delta\rho, \\ T^0_i &= (\rho + p)(v_i - B_i), \\ T_0^i &= -(\rho + p)v^i, \\ T^i_j &= (p + \delta p)\delta^i_j + p\Pi^i_j, \end{aligned}$$

- (1)  $\delta\rho$  a density perturbation; (3)  $v_i$  a vector velocity, (1)  $\delta p$  a pressure perturbation; (5)  $\Pi_{ij}$  an anisotropic stress perturbation
- So far this is fully general and applies to any type of matter or coordinate choice including non-linearities in the matter, e.g. cosmological defects.

# Counting DOF's

20	Variables (10 metric; 10 matter)
-10	Einstein equations
-4	Conservation equations
+4	Bianchi identities
-4	Gauge (coordinate choice 1 time, 3 space)
<hr/>	
6	Degrees of freedom

- Without loss of generality these can be taken to be the 6 components of the matter stress tensor
- For the background, specify  $p(a)$  or equivalently  $w(a) \equiv p(a)/\rho(a)$  the equation of state parameter.

# Scalar, Vector, Tensor

- In linear perturbation theory, perturbations may be separated by their transformation properties under rotation and translation.
- The eigenfunctions of the Laplacian operator form a complete set

$$\begin{aligned}\nabla^2 Q^{(0)} &= -k^2 Q^{(0)} & \text{S ,} \\ \nabla^2 Q_i^{(\pm 1)} &= -k^2 Q_i^{(\pm 1)} & \text{V ,} \\ \nabla^2 Q_{ij}^{(\pm 2)} &= -k^2 Q_{ij}^{(\pm 2)} & \text{T ,}\end{aligned}$$

- Vector and tensor modes satisfy divergence-free and transverse-traceless conditions

$$\nabla^i Q_i^{(\pm 1)} = 0$$

$$\nabla^i Q_{ij}^{(\pm 2)} = 0$$

$$\gamma^{ij} Q_{ij}^{(\pm 2)} = 0$$

# Vector and Tensor Modes vs. Vector and Tensor Quantities

- A scalar mode carries with it associated vector (curl-free) and tensor (longitudinal) quantities
- A vector mode carries an associated tensor (neither longitudinal or transverse) quantities
- These are built from the mode basis out of covariant derivatives and the metric

$$Q_i^{(0)} = -k^{-1} \nabla_i Q^{(0)},$$

$$Q_{ij}^{(0)} = (k^{-2} \nabla_i \nabla_j + \frac{1}{3} \gamma_{ij}) Q^{(0)},$$

$$Q_{ij}^{(\pm 1)} = -\frac{1}{2k} [\nabla_i Q_j^{(\pm 1)} + \nabla_j Q_i^{(\pm 1)}],$$

# Spatially Flat Case

- For a spatially flat background metric, harmonics are related to plane waves:

$$Q^{(0)} = \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$Q_i^{(\pm 1)} = \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$Q_{ij}^{(\pm 2)} = -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k} \cdot \mathbf{x})$$

where  $\hat{\mathbf{e}}_3 \parallel \mathbf{k}$ . Chosen as spin states, c.f. polarization.

- For vectors, the harmonic points in a direction orthogonal to  $\mathbf{k}$  suitable for the vortical component of a vector
- For tensors, the harmonic is transverse and traceless as appropriate for the decompositon of gravitational waves

# Perturbation $k$ -Modes

- For the  $k$ th eigenmode, the scalar components become

$$\begin{aligned} A(\mathbf{x}) &= A(k) Q^{(0)}, & H_L(\mathbf{x}) &= H_L(k) Q^{(0)}, \\ \delta\rho(\mathbf{x}) &= \delta\rho(k) Q^{(0)}, & \delta p(\mathbf{x}) &= \delta p(k) Q^{(0)}, \end{aligned}$$

the vectors components become

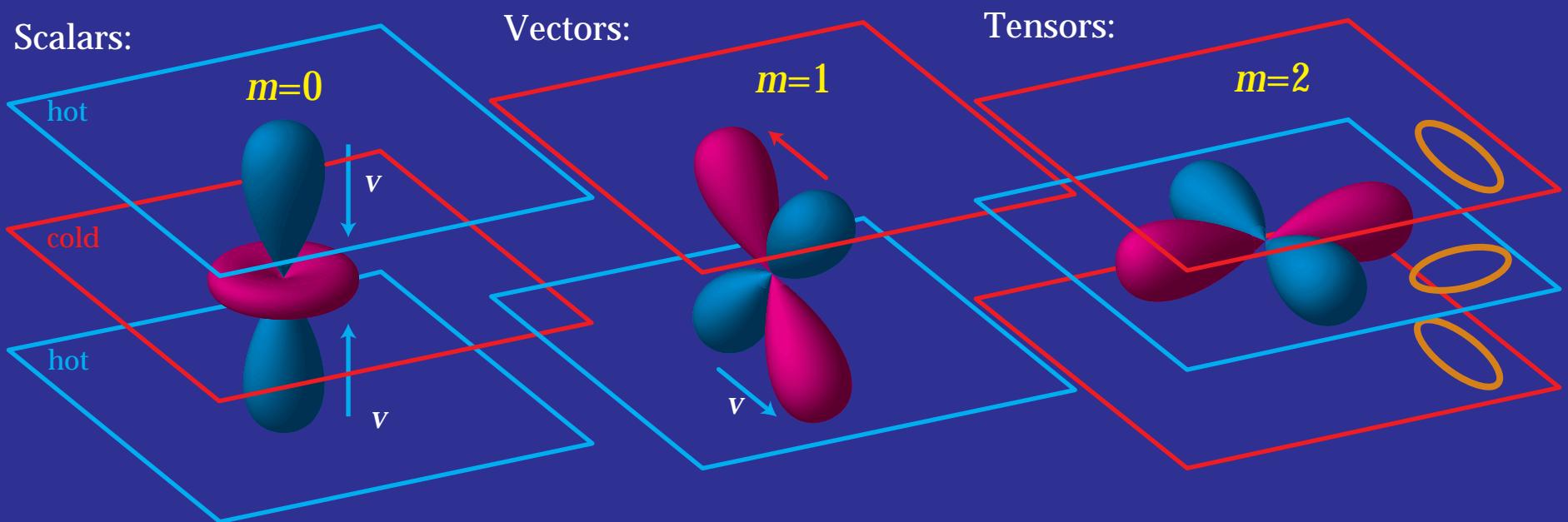
$$B_i(\mathbf{x}) = \sum_{m=-1}^1 B^{(m)}(k) Q_i^{(m)}, \quad v_i(\mathbf{x}) = \sum_{m=-1}^1 v^{(m)}(k) Q_i^{(m)},$$

and the tensors components

$$H_{Tij}(\mathbf{x}) = \sum_{m=-2}^2 H_T^{(m)}(k) Q_{ij}^{(m)}, \quad \Pi_{ij}(\mathbf{x}) = \sum_{m=-2}^2 \Pi^{(m)}(k) Q_{ij}^{(m)},$$

# Perturbations & Their Quadrupoles

- Orientation of quadrupole relative to wave ( $\mathbf{k}$ ) determines pattern
- Scalars (density)  $m=0$
- Vectors (vorticity)  $m=\pm 1$
- Tensors (gravity waves)  $m=\pm 2$



# Homogeneous Einstein Equations

- Einstein (Friedmann) equations:

$$\begin{aligned}\left(\frac{1}{a} \frac{da}{dt}\right)^2 &= \frac{8\pi G}{3} \rho \\ \frac{1}{a} \frac{d^2a}{dt^2} &= -\frac{4\pi G}{3} (\rho + 3p)\end{aligned}$$

so that  $w \equiv p/\rho < -1/3$  for acceleration

- Conservation equation  $\nabla^\mu T_{\mu\nu} = 0$  implies

$$\frac{\dot{\rho}}{\rho} = -3(1+w) \frac{\dot{a}}{a}$$

# Homogeneous Einstein Equations

- Counting exercise:

20	Variables (10 metric; 10 matter)
-17	Homogeneity and Isotropy
-2	Einstein equations
-1	Conservation equations
+1	Bianchi identities
<hr/>	
1	Degree of freedom

- without loss of generality choose ratio of homogeneous & isotropic component of the **stress tensor** to the density  $w(a) = p(a)/\rho(a)$ .

# Acceleration Implies Negative Pressure

- Role of **stresses** in the background cosmology
- Homogeneous **Einstein equations**  $G_{\mu\nu} = 8\pi G T_{\mu\nu}$  imply the two **Friedman equations** (flat universe, or associating curvature  $\rho_K = -3K/8\pi G a^2$ )

$$\begin{aligned}\left(\frac{1}{a} \frac{da}{dt}\right)^2 &= \frac{8\pi G}{3} \rho \\ \frac{1}{a} \frac{d^2a}{dt^2} &= -\frac{4\pi G}{3} (\rho + 3p)\end{aligned}$$

so that the total equation of state  $w \equiv p/\rho < -1/3$  for acceleration

- Conservation equation  $\nabla^\mu T_{\mu\nu} = 0$  implies

$$\frac{\dot{\rho}}{\rho} = -3(1+w) \frac{\dot{a}}{a}$$

- so that  $\rho$  must scale more slowly than  $a^{-2}$

# Questions regarding Dark Energy

- Coincidence: given the very different scalings of matter and dark energy with  $a$ , why are they comparable now?
- Stability: why doesn't negative pressure imply accelerated collapse? or why doesn't the vacuum suck?
- Answer: stability is associated with stress (pressure) gradients not stress (pressure) itself.
- Example: the cosmological constant  $w_\Lambda = -1$ , a constant in time and space – no gradients.
- Example: a scalar field where  $w = p/\rho \neq \delta p/\delta\rho =$  sound speed.

# Covariant Scalar Equations

- Einstein equations (suppressing 0) superscripts (Hu & Eisenstein 1999):

$$\begin{aligned}
 & (k^2 - 3K)[H_L + \frac{1}{3}H_T + \frac{\dot{a}}{a}\frac{1}{k^2}(kB - \dot{H}_T)] \\
 & = 4\pi Ga^2 \left[ \delta\rho + 3\frac{\dot{a}}{a}(\rho + p)(v - B)/k \right], \quad \text{Poisson Equation} \\
 & k^2(A + H_L + \frac{1}{3}H_T) + \left( \frac{d}{d\eta} + 2\frac{\dot{a}}{a} \right) (kB - \dot{H}_T) \\
 & = 8\pi Ga^2 p\Pi, \\
 & \frac{\dot{a}}{a}A - \dot{H}_L - \frac{1}{3}\dot{H}_T - \frac{K}{k^2}(kB - \dot{H}_T) \\
 & = 4\pi Ga^2(\rho + p)(v - B)/k, \\
 & \left[ 2\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\frac{d}{d\eta} - \frac{k^2}{3} \right] A - \left[ \frac{d}{d\eta} + \frac{\dot{a}}{a} \right] (\dot{H}_L + \frac{1}{3}kB) \\
 & = 4\pi Ga^2(\delta p + \frac{1}{3}\delta\rho).
 \end{aligned}$$

# Covariant Scalar Equations

- Conservation equations: continuity and Navier Stokes

$$\begin{aligned}\left[ \frac{d}{d\eta} + 3\frac{\dot{a}}{a} \right] \delta\rho + 3\frac{\dot{a}}{a} \delta p &= -(\rho + p)(k\textcolor{blue}{v} + 3\dot{H}_L), \\ \left[ \frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] \left[ (\rho + p) \frac{(\textcolor{blue}{v} - \textcolor{blue}{B})}{k} \right] &= \delta p - \frac{2}{3}(1 - 3\frac{K}{k^2})p\Pi + (\rho + p)A,\end{aligned}$$

- Equations are not independent since  $\nabla_\mu G^{\mu\nu} = 0$  via the Bianchi identities.
- Related to the ability to choose a coordinate system or “gauge” to represent the perturbations.

# Covariant Scalar Equations

- DOF counting exercise

8	Variables (4 metric; 4 matter)
-4	Einstein equations
-2	Conservation equations
+2	Bianchi identities
-2	Gauge (coordinate choice 1 time, 1 space)

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2 Degrees of freedom

- without loss of generality choose scalar components of the stress tensor  $\delta p, \Pi$ .

# Covariant Vector Equations

- Einstein equations

$$\begin{aligned}(1 - 2K/k^2)(k\mathbf{B}^{(\pm 1)} - \dot{\mathbf{H}}_T^{(\pm 1)}) \\ = 16\pi G a^2 (\rho + p)(\mathbf{v}^{(\pm 1)} - \mathbf{B}^{(\pm 1)})/k, \\ \left[ \frac{d}{d\eta} + 2\frac{\dot{a}}{a} \right] (k\mathbf{B}^{(\pm 1)} - \dot{\mathbf{H}}_T^{(\pm 1)}) \\ = -8\pi G a^2 p \Pi^{(\pm 1)}.\end{aligned}$$

- Conservation Equations

$$\begin{aligned}\left[ \frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] [(\rho + p)(\mathbf{v}^{(\pm 1)} - \mathbf{B}^{(\pm 1)})/k] \\ = -\frac{1}{2}(1 - 2K/k^2)p \Pi^{(\pm 1)},\end{aligned}$$

- Gravity provides no source to vorticity  $\rightarrow$  decay

# Covariant Vector Equations

- DOF counting exercise

8	Variables (4 metric; 4 matter)
-4	Einstein equations
-2	Conservation equations
+2	Bianchi identities
-2	Gauge (coordinate choice 1 time, 1 space)

---

2 Degrees of freedom

- without loss of generality choose vector components of the stress tensor  $\Pi^{(\pm 1)}$ .

# Covariant Tensor Equation

- Einstein equation

$$\left[ \frac{d^2}{d\eta^2} + 2\frac{\dot{a}}{a}\frac{d}{d\eta} + (k^2 + 2K) \right] H_T^{(\pm 2)} = 8\pi G a^2 p \Pi^{(\pm 2)}.$$

- DOF counting exercise

4	Variables (2 metric; 2 matter)
-2	Einstein equations
-0	Conservation equations
+0	Bianchi identities
-0	Gauge (coordinate choice 1 time, 1 space)
<hr/>	
2	Degrees of freedom

- wlog choose tensor components of the stress tensor  $\Pi^{(\pm 2)}$ .

# Arbitrary Dark Components

- Total stress energy tensor can be broken up into individual pieces
- Dark components interact only through gravity and so satisfy separate conservation equations
- Einstein equation source remains the sum of components.
- To specify an arbitrary dark component, give the behavior of the stress tensor: 6 components:  $\delta p$ ,  $\Pi^{(i)}$ , where  $i = -2, \dots, 2$ .
- Many types of dark components (dark matter, scalar fields, massive neutrinos,..) have simple forms for their stress tensor in terms of the energy density, i.e. described by equations of state.
- An equation of state for the background  $w = p/\rho$  is *not* sufficient to determine the behavior of the perturbations.

# Gauge Freedom & Choice

# Gauge

- Metric and matter fluctuations take on different values in different coordinate system
- No such thing as a “gauge invariant” density perturbation!
- General coordinate transformation:

$$\begin{aligned}\tilde{\eta} &= \eta + T \\ \tilde{x}^i &= x^i + L^i\end{aligned}$$

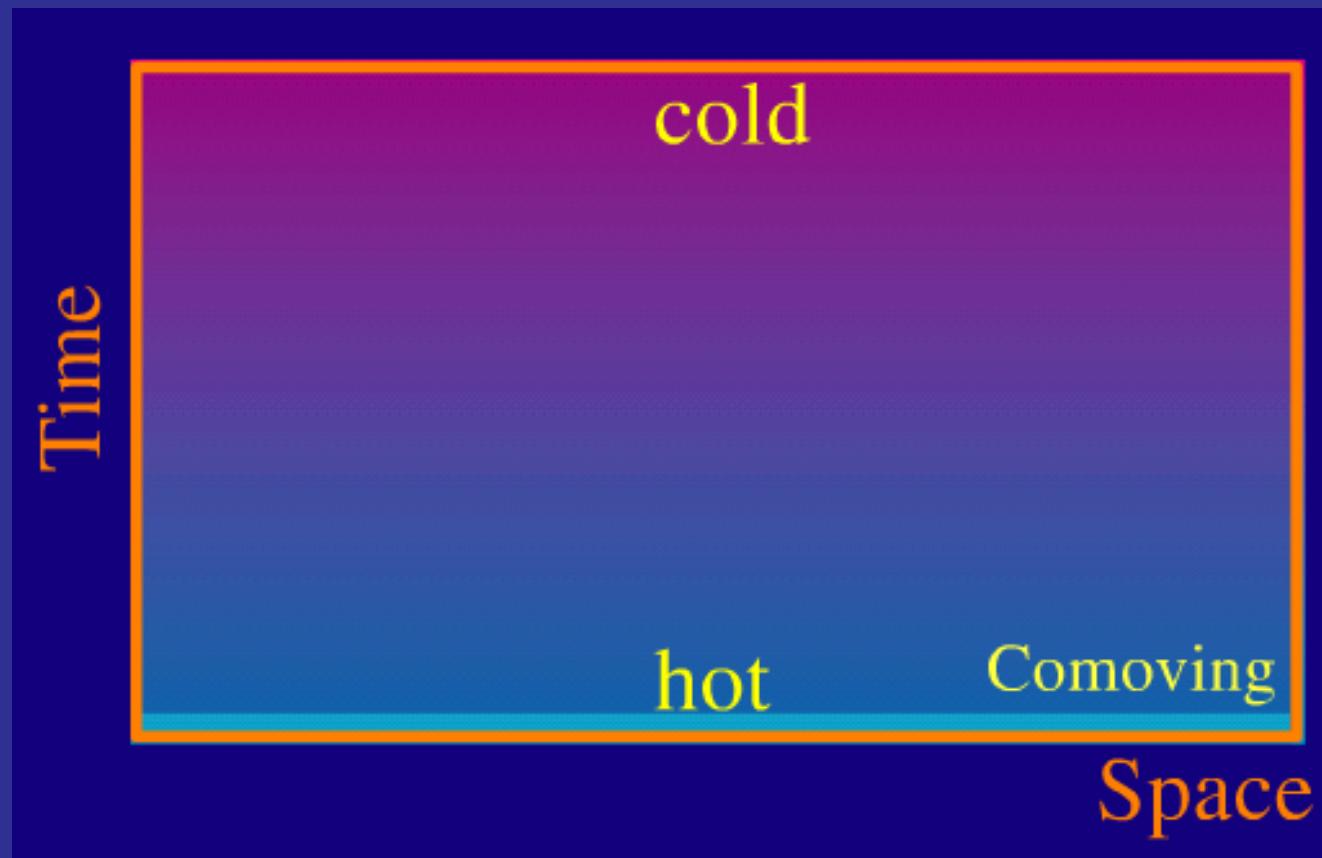
free to choose  $(T, L^i)$  to simplify equations or physics.

Decompose these into scalar and vector harmonics.

- $G_{\mu\nu}$  and  $T_{\mu\nu}$  transform as tensors, so components in different frames can be related

# Sachs-Wolfe Gauge Transformation

- Inflation: (nearly) scale-invariant curvature (potential) perturbations
- Superluminal expansion  $\rightarrow$  superhorizon scales  $\rightarrow$  "initial conditions"
- Accompanying temperature perturbations due to cosmological redshift



- Potential perturbation  $\Psi = \text{time-time metric perturbation}$   
 $\delta t/t = \Psi \quad \rightarrow \quad \delta T/T = -\delta a/a = -2/3 \delta t/t = -2/3 \Psi$

Sachs & Wolfe (1967); White & Hu (1997)

# Gauge

- Metric and matter fluctuations take on different values in different coordinate system
- No such thing as a “gauge invariant” density perturbation!
- General coordinate transformation:

$$\begin{aligned}\tilde{\eta} &= \eta + T \\ \tilde{x}^i &= x^i + L^i\end{aligned}$$

free to choose  $(T, L^i)$  to simplify equations or physics.

Decompose these into scalar and vector harmonics.

- $G_{\mu\nu}$  and  $T_{\mu\nu}$  transform as tensors, so components in different frames can be related

# Gauge Transformation

- Scalar Metric:

$$\begin{aligned}\tilde{A} &= A - \dot{T} - \frac{\dot{a}}{a}T, \\ \tilde{B} &= B + \dot{L} + kT, \\ \tilde{H}_L &= H_L - \frac{k}{3}L - \frac{\dot{a}}{a}T, \\ \tilde{H}_T &= H_T + kL,\end{aligned}$$

- Scalar Matter ( $J$ th component):

$$\begin{aligned}\delta\tilde{\rho}_J &= \delta\rho_J - \dot{\rho}_J T, \\ \delta\tilde{p}_J &= \delta p_J - \dot{p}_J T, \\ \tilde{v}_J &= v_J + \dot{L},\end{aligned}$$

- Vector:

$$\tilde{B}^{(\pm 1)} = B^{(\pm 1)} + \dot{L}^{(\pm 1)}, \quad \tilde{H}_T^{(\pm 1)} = H_T^{(\pm 1)} + kL^{(\pm 1)}, \quad \tilde{v}_J^{(\pm 1)} = v_J^{(\pm 1)} + \dot{L}^{(\pm 1)},$$

# Common Scalar Gauge Choices

- A coordinate system is **fully specified** if there is an explicit prescription for  $(T, L^i)$  or for scalars  $(T, L)$
- Newtonian:

$$\tilde{B} = \tilde{H}_T = 0$$

$$\Psi \equiv \tilde{A} \quad (\text{Newtonian potential})$$

$$\Phi \equiv \tilde{H}_L \quad (\text{Newtonian curvature})$$

$$L = -H_T/k$$

$$T = -B/k + \dot{H}_T/k^2$$

**Good:** intuitive Newtonian like gravity; matter and metric algebraically related; commonly chosen for analytic CMB and lensing work

**Bad:** numerically **unstable**

# Example: Newtonian Reduction

- In the general equations, set  $B = H_T = 0$ :

$$\begin{aligned}(k^2 - 3K)\Phi &= 4\pi G a^2 \left[ \delta\rho + 3\frac{\dot{a}}{a}(\rho + p)\mathbf{v}/k \right] \\ k^2(\Psi + \Phi) &= 8\pi G a^2 p\Pi\end{aligned}$$

so  $\Psi = -\Phi$  if anisotropic stress  $\Pi = 0$  and

$$\begin{aligned}\left[ \frac{d}{d\eta} + 3\frac{\dot{a}}{a} \right] \delta\rho + 3\frac{\dot{a}}{a} \delta p &= -(\rho + p)(k\mathbf{v} + 3\dot{\Phi}), \\ \left[ \frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] (\rho + p)\mathbf{v} &= k\delta p - \frac{2}{3}(1 - 3\frac{K}{k^2})p k\Pi + (\rho + p) k\Psi,\end{aligned}$$

- Competition between stress (pressure and viscosity) and potential gradients

# Common Scalar Gauge Choices

- Comoving:

$$\tilde{B} = \tilde{v} \quad (T_i^0 = 0)$$

$$H_T = 0$$

$$\xi = \tilde{A}$$

$$\zeta = \tilde{H}_L \quad (\text{Bardeen curvature})$$

$$\Delta = \tilde{\delta} \quad (\text{comoving density pert})$$

$$T = (v - B)/k$$

$$L = -H_T/k$$

Good: Algebraic relations between matter and metric

- Euler equation becomes an algebraic relation between stress and potential

$$(\rho + p)\xi = -\delta p + \frac{2}{3} \left(1 - \frac{3K}{k}\right) p\Pi$$

# Common Scalar Gauge Choices

- Einstein equation lacks momentum density source

$$\frac{\dot{a}}{a}\xi - \dot{\zeta} - \frac{K}{k^2}kv = 0$$

- Combine:  $\zeta$  is conserved if stress fluctuations negligible, e.g. above the horizon if  $|K| \ll H^2$

$$\dot{\zeta} + Kv/k = \frac{\dot{a}}{a} \left[ -\frac{\delta p}{\rho + p} + \frac{2}{3} \left( 1 - \frac{3K}{k^2} \right) \frac{p}{\rho + p} \Pi \right] \rightarrow 0$$

**Bad:** explicitly relativistic choice

# Common Scalar Gauge Choices

- Synchronous:

$$\begin{aligned}\tilde{A} &= \tilde{B} = 0 \\ \eta_L &\equiv -\tilde{H}_L - \frac{1}{3}\tilde{H}_T \\ h_T &= \tilde{H}_T \quad \text{or} \quad h = 6H_L \\ T &= a^{-1} \int d\eta a A + c_1 a^{-1} \\ L &= - \int d\eta (B + kT) + c_2\end{aligned}$$

**Good:** stable, the choice of numerical codes

**Bad:** residual gauge freedom in constants  $c_1, c_2$  must be specified as an initial condition, intrinsically relativistic.

# Common Scalar Gauge Choices

- Spatially Unperturbed:

$$\tilde{H}_L = \tilde{H}_T = 0$$

$$L = -H_T/k$$

$\tilde{A}, \tilde{B}$  = metric perturbations

$$T = \left(\frac{\dot{a}}{a}\right)^{-1} \left(H_L + \frac{1}{3}H_T\right)$$

Good: eliminates spatial metric in evolution equations; useful in inflationary calculations (Mukhanov et al)

Bad: intrinsically relativistic.

- Caution: perturbation evolution is governed by the behavior of stress fluctuations and an isotropic stress fluctuation  $\delta p$  is gauge dependent.

# Hybrid “Gauge Invariant” Approach

- With the gauge transformation relations, express variables of **one gauge** in terms of those in **another** – allows a mixture in the equations of motion
- Example:** Newtonian curvature and comoving density

$$(k^2 - 3K)\Phi = 4\pi G a^2 \rho \Delta$$

ordinary Poisson equation then implies  $\Phi$  approximately constant if stresses negligible.

- Example:** Exact Newtonian curvature above the horizon derived through Bardeen curvature conservation

Gauge transformation

$$\Phi = \zeta + \frac{\dot{a}}{a} \frac{v}{k}$$

# Hybrid “Gauge Invariant” Approach

Einstein equation to eliminate velocity

$$\frac{\dot{a}}{a}\Psi - \dot{\Phi} = 4\pi G a^2 (\rho + p)v/k$$

Friedman equation with no spatial curvature

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} a^2 \rho$$

With  $\dot{\Phi} = 0$  and  $\Psi \approx -\Phi$

$$\frac{\dot{a}}{a} \frac{v}{k} = -\frac{2}{3(1+w)} \Phi$$

# Hybrid “Gauge Invariant” Approach

Combining gauge transformation with velocity relation

$$\Phi = \frac{3 + 3w}{5 + 3w} \zeta$$

Usage: calculate  $\zeta$  from inflation determines  $\Phi$  for any choice of matter content or causal evolution.

- Example: Scalar field (“quintessence” dark energy) equations in comoving gauge imply a sound speed  $\delta p / \delta \rho = 1$  independent of potential  $V(\phi)$ . Solve in synchronous gauge (Hu 1998).

# Boltzmann Formalism

# Boltzmann Equation

- CMB radiation is generally described by the phase space distribution function for each polarization state  $f_a(\mathbf{x}, \mathbf{q}, \eta)$ , where  $\mathbf{x}$  is the comoving position and  $\mathbf{q}$  is the photon momentum
- Boltzmann equation describes the evolution of the distribution function under gravity and collisions
- Low order moments of the Boltzmann equation are simply the covariant conservation equations
- Higher moments provide the closure condition to the conservation law (specification of stress tensor) and the CMB observable – fine scale anisotropy
- Higher moments mainly describe the simple geometry of source projection

# Liouville Equation

- In absence of scattering, the phase space distribution of photons is conserved along the propagation path
- Rewrite variables in terms of the photon propagation direction  $\mathbf{q} = q\hat{\mathbf{n}}$ , so  $f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta)$  and

$$\begin{aligned}\frac{d}{d\eta} f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta) &= 0 \\ &= \left( \frac{\partial}{\partial\eta} + \frac{d\mathbf{x}}{d\eta} \cdot \frac{\partial}{\partial\mathbf{x}} + \frac{d\hat{\mathbf{n}}}{d\eta} \cdot \frac{\partial}{\partial\hat{\mathbf{n}}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a\end{aligned}$$

- For simplicity, assume spatially flat universe  $K = 0$  then  $d\hat{\mathbf{n}}/d\eta = 0$  and  $d\mathbf{x} = \hat{\mathbf{n}}d\eta$

$$\dot{f}_a + \hat{\mathbf{n}} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0$$

# Correspondence to Einstein Eqn.

- Geodesic equation gives the redshifting term

$$\frac{\dot{q}}{q} = -\frac{\dot{a}}{a} - \frac{1}{2} n^i n^j \dot{H}_{Tij} - \dot{H}_L + n^i \dot{B}_i - \hat{\mathbf{n}} \cdot \nabla A$$

- which is incorporated in the conservation and gauge transformation equations
- Stress energy tensor involves integrals over the distribution function the two polarization states

$$T^{\mu\nu} = \int \frac{d^3q}{(2\pi)^3} \frac{q^\mu q^\nu}{E} (f_a + f_b)$$

- Components are simply the low order angular moments of the distribution function

# Angular Moments

- Define the angularly dependent temperature perturbation

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \frac{1}{4\rho_\gamma} \int \frac{q^3 dq}{2\pi^2} (f_a + f_b) - 1$$

and likewise for the linear polarization states  $Q$  and  $U$

- Decompose into normal modes: plane waves for spatial part and spherical harmonics for angular part

$$G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$
$$\pm_2 G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} \pm_2 Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$

- In a spatially curved universe generalize the plane wave part

# Stokes Parameters

- Polarization state of radiation in direction  $\hat{\mathbf{n}}$  described by the intensity matrix  $\langle E_i(\hat{\mathbf{n}})E_j^*(\hat{\mathbf{n}}) \rangle$ , where  $\mathbf{E}$  is the electric field vector and the brackets denote time averaging.
- As a hermitian matrix, it can be decomposed into the Pauli basis

$$\begin{aligned}\mathbf{P} &= C \langle \mathbf{E}(\hat{\mathbf{n}}) \mathbf{E}^\dagger(\hat{\mathbf{n}}) \rangle \\ &= \Theta(\hat{\mathbf{n}}) \boldsymbol{\sigma}_0 + Q(\hat{\mathbf{n}}) \boldsymbol{\sigma}_3 + U(\hat{\mathbf{n}}) \boldsymbol{\sigma}_1 + V(\hat{\mathbf{n}}) \boldsymbol{\sigma}_2,\end{aligned}$$

where

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Stokes parameters recovered as  $\text{Tr}(\sigma_i \mathbf{P})/2$

# Linear Polarization

- $Q \propto \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle$ ,  $U \propto \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle$ .
- Counterclockwise rotation of axes by  $\theta = 45^\circ$

$$E_1 = (E'_1 - E'_2)/\sqrt{2}, \quad E_2 = (E'_1 + E'_2)/\sqrt{2}$$

- $U \propto \langle E'_1 E'^*_1 \rangle - \langle E'_2 E'^*_2 \rangle$ , difference of intensities at  $45^\circ$  or  $Q'$
- More generally,  $\mathbf{P}$  transforms as a tensor under rotations and

$$Q' = \cos(2\theta)Q + \sin(2\theta)U$$

$$U' = -\sin(2\theta)Q + \cos(2\theta)U$$

- or

$$Q' \pm iU' = e^{\mp 2i\theta} [Q \pm iU]$$

acquires a phase under rotation and is a spin  $\pm 2$  object

# Coordinate Independent Representation

- Two directions: orientation of polarization and change in amplitude, i.e.  $Q$  and  $U$  in the basis of the Fourier wavevector for small sections of sky are called  $E$  and  $B$  components

$$\begin{aligned} E(\mathbf{l}) \pm iB(\mathbf{l}) &= - \int d\hat{\mathbf{n}} [Q'(\hat{\mathbf{n}}) \pm iU'(\hat{\mathbf{n}})] e^{-i\mathbf{l} \cdot \hat{\mathbf{n}}} \\ &= -e^{\mp 2i\phi_l} \int d\hat{\mathbf{n}} [Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})] e^{i\mathbf{l} \cdot \hat{\mathbf{n}}} \end{aligned}$$

- For the  $B$ -mode to not vanish, the polarization must point in a direction not related to the wavevector - not possible for density fluctuations in linear theory
- Generalize to all-sky: plane waves are eigenmodes of the Laplace operator on the tensor  $\mathbf{P}$ .

# Spin Harmonics

- Laplace Eigenfunctions

$$\nabla^2_{\pm 2} Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1] = -[l(l+1) - 4]_{\pm 2} Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1]$$

- Spin  $s$  spherical harmonics: orthogonal and complete

$$\int d\hat{\mathbf{n}}_s Y_{\ell m}^*(\hat{\mathbf{n}})_s Y_{\ell m}(\hat{\mathbf{n}}) = \delta_{\ell\ell'} \delta_{mm'}$$

$$\sum_{\ell m} {}_s Y_{\ell m}^*(\hat{\mathbf{n}})_s Y_{\ell m}(\hat{\mathbf{n}}') = \delta(\phi - \phi') \delta(\cos \theta - \cos \theta')$$

where the ordinary spherical harmonics are  $Y_{\ell m} = {}_0 Y_{\ell m}$

- Given in terms of the rotation matrix

$${}_s Y_{\ell m}(\beta\alpha) = (-1)^m \sqrt{\frac{2\ell + 1}{4\pi}} D_{-ms}^\ell(\alpha\beta 0)$$

# Statistical Representation

- All-sky decomposition

$$[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})] = \sum_{\ell m} [E_{\ell m} \pm iB_{\ell m}]_{\pm 2} Y_{\ell m}(\hat{\mathbf{n}})$$

- Power spectra

$$\langle E_{\ell m}^* E_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell^{EE}$$

$$\langle B_{\ell m}^* B_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell^{BB}$$

- Cross correlation

$$\langle E_{\ell m}^* E_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell^{\Theta E}$$

others vanish if parity is conserved

# Normal Modes

- Temperature and polarization fields

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3 k}{(2\pi)^3} \sum_{\ell m} \Theta_\ell^{(m)} G_\ell^m$$

$$[Q \pm iU](\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3 k}{(2\pi)^3} \sum_{\ell m} [E_\ell^{(m)} \pm iB_\ell^{(m)}]_{\pm 2} G_\ell^m$$

- For each  $\mathbf{k}$  mode, work in coordinates where  $\mathbf{k} \parallel \mathbf{z}$  and so  $m = 0$  represents scalar modes,  $m = \pm 1$  vector modes,  $m = \pm 2$  tensor modes,  $|m| > 2$  vanishes. Since modes add incoherently and  $Q \pm iU$  is invariant up to a phase, rotation back to a fixed coordinate system is trivial.

# Scalar, Vector, Tensor

- Normalization of modes is chosen so that the lowest angular mode for scalars, vectors and tensors are normalized in the same way as the mode function

$$G_0^0 = Q^{(0)} \quad G_1^0 = n^i Q_i^{(0)} \quad G_2^0 \propto n^i n^j Q_{ij}^{(0)}$$

$$G_1^{\pm 1} = n^i Q_i^{(\pm 1)} \quad G_2^{\pm 1} \propto n^i n^j Q_{ij}^{(\pm 1)}$$

$$G_2^{\pm 2} = n^i n^j Q_{ij}^{(\pm 2)}$$

where recall

$$Q^{(0)} = \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$Q_i^{(\pm 1)} = \frac{-i}{\sqrt{2}} (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$Q_{ij}^{(\pm 2)} = -\sqrt{\frac{3}{8}} (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k} \cdot \mathbf{x})$$

# Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies
- Spatial gradient term hits plane wave:

$$\hat{\mathbf{n}} \cdot \nabla e^{i\mathbf{k} \cdot \mathbf{x}} = i\hat{\mathbf{n}} \cdot \mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x}} = i\sqrt{\frac{4\pi}{3}} k Y_1^0(\hat{\mathbf{n}}) e^{i\mathbf{k} \cdot \mathbf{x}}$$

- Dipole term adds to angular dependence through the addition of angular momentum

$$\sqrt{\frac{4\pi}{3}} Y_1^0 Y_\ell^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell+1)(2\ell-1)}} Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell+1)(2\ell+3)}} Y_{\ell+1}^m$$

where  $\kappa_\ell^m = \sqrt{\ell^2 - m^2}$  is given by Clebsch-Gordon coefficients.

# Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

$$\dot{\Theta}_\ell^{(m)} = k \left[ \frac{\kappa_\ell^m}{2\ell+1} \Theta_{\ell-1}^{(m)} - \frac{\kappa_{\ell+1}^m}{2\ell+3} \Theta_{\ell+1}^{(m)} \right] - \dot{\tau} \Theta_\ell^{(m)} + S_\ell^{(m)}$$

where  $S_\ell^{(m)}$  are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic  $\ell = 0$  temperature perturbation will eventually become a high order anisotropy by “free streaming” or simple projection
- Original CMB codes solved the full hierarchy equations out to the  $\ell$  of interest.

# Integral Solution

- Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface
- In general, the solution describes the decomposition of the source  $S_\ell^{(m)}$  with its local angular dependence as seen at a distance  $\mathbf{x} = D\hat{\mathbf{n}}$ .
- Proceed by decomposing the angular dependence of the plane wave

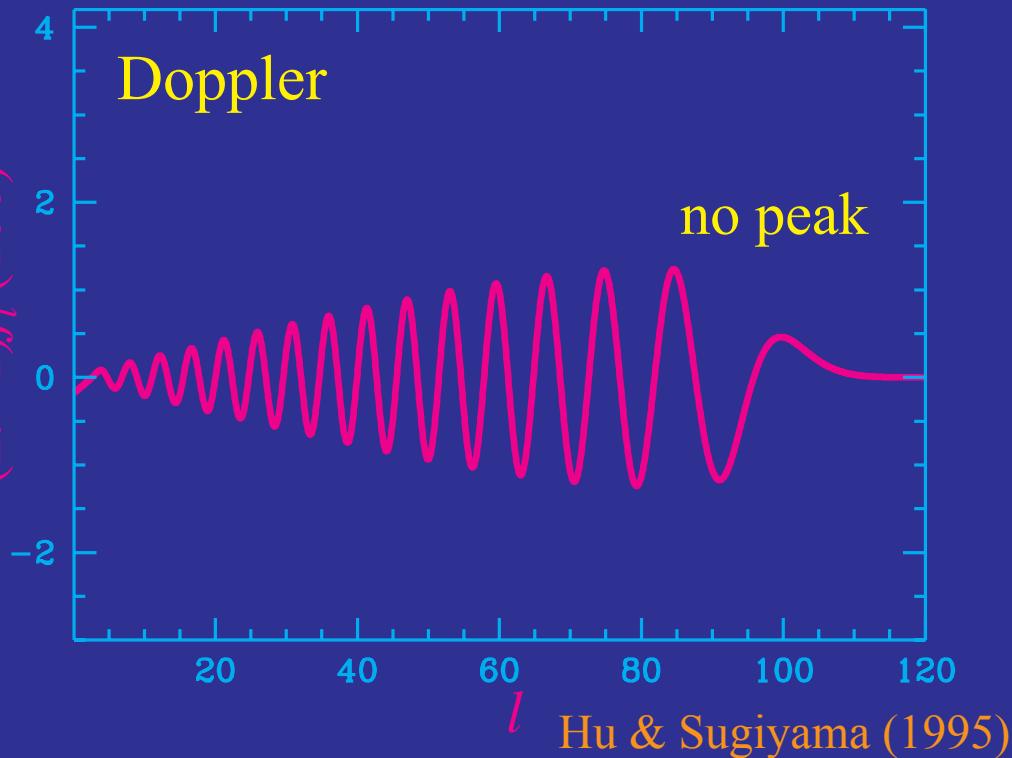
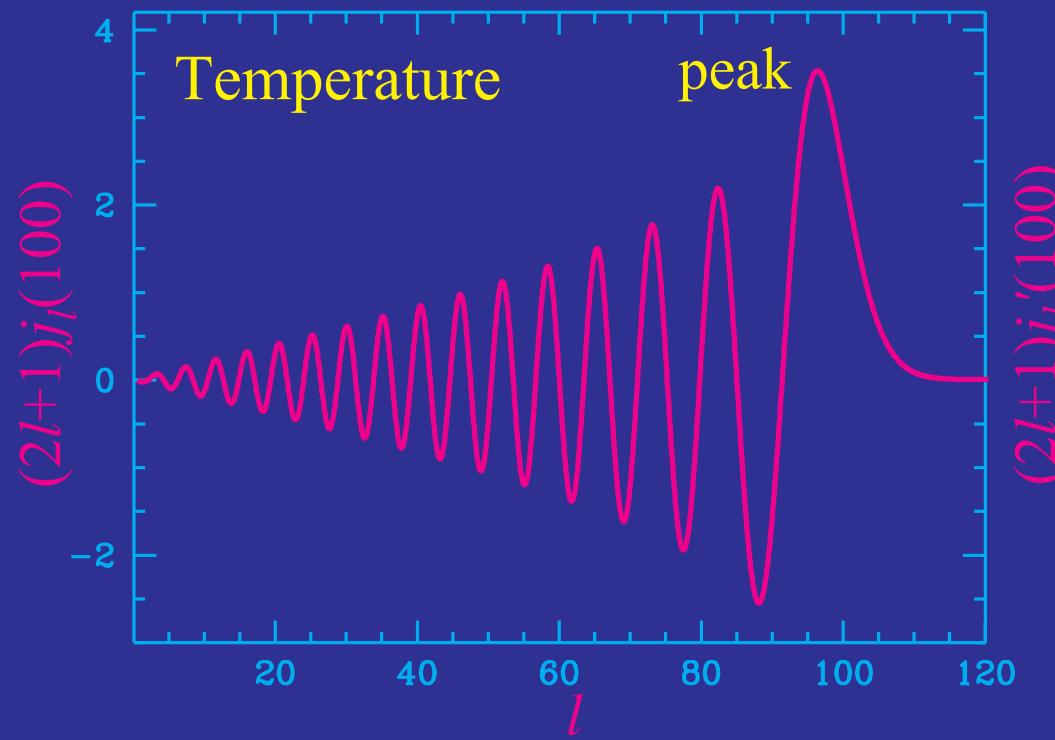
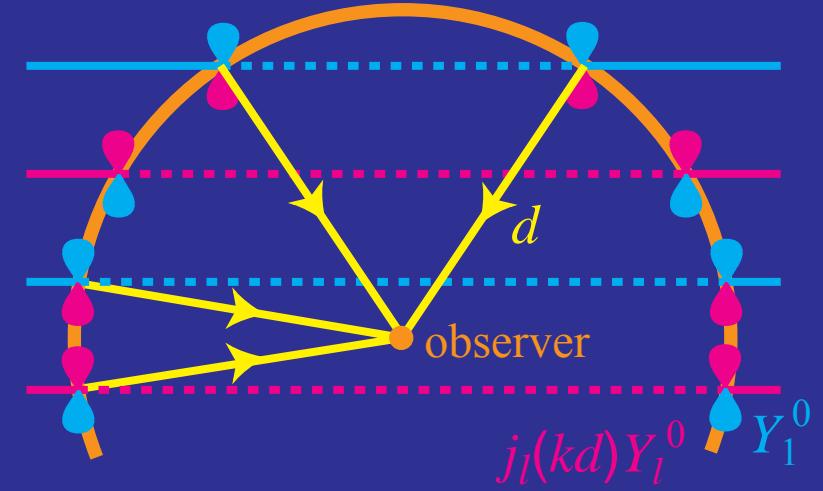
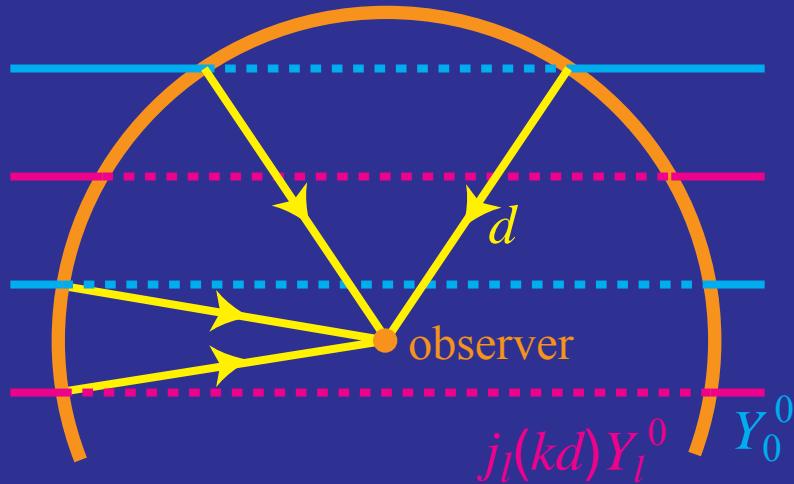
$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\ell} (-i)^\ell \sqrt{4\pi(2\ell+1)} j_\ell(kD) Y_\ell^0(\hat{\mathbf{n}})$$

- Recouple to the local angular dependence of  $G_\ell^m$

$$G_{\ell_s}^m = \sum_{\ell} (-i)^\ell \sqrt{4\pi(2\ell+1)} \alpha_{\ell_s \ell}^{(m)}(kD) Y_\ell^m(\hat{\mathbf{n}})$$

# Doppler Peaks?

- Doppler effect has lower amplitude and weak features from projection



Hu & Sugiyama (1995)

# Integral Solution

- Projection kernels:

$$\ell_s = 0, \quad m = 0 \quad \alpha_{0\ell}^{(0)} \equiv j_\ell$$

$$\ell_s = 1, \quad m = 0 \quad \alpha_{1\ell}^{(0)} \equiv j'_\ell$$

- Integral solution:

$$\frac{\Theta_\ell^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S_{\ell_s}^{(m)} \alpha_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

- Power spectrum:

$$C_\ell = \frac{2}{\pi} \int \frac{dk}{k} \sum_m \frac{k^3 \langle \Theta_\ell^{(m)*} \Theta_\ell^{(m)} \rangle}{(2\ell + 1)^2}$$

- Solving for  $C_\ell$  reduces to solving for the behavior of a handful of sources

# Polarization Integral Solution

- Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

$$\frac{E_\ell^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \epsilon_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

$$\frac{B_\ell^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \beta_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

- The only source to the polarization is from the quadrupole anisotropy so we only need  $\ell_s = 2$ , e.g. for scalars

$$\epsilon_{2\ell}^{(0)}(x) = \sqrt{\frac{3}{8}} \frac{(\ell + 2)!}{(\ell - 2)!} \frac{j_\ell(x)}{x^2} \quad \beta_{2\ell}^{(0)} = 0$$

# Polarization Hierarchy

- In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hierarchy:

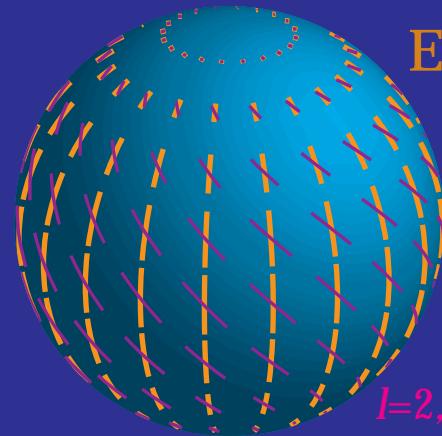
$$\dot{E}_\ell^{(m)} = k \left[ \frac{{}_2\kappa_\ell^m}{2\ell - 1} E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell + 1)} B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell + 3} \right] - \dot{\tau} E_\ell^{(m)} + \mathcal{E}_\ell^{(m)}$$
$$\dot{B}_\ell^{(m)} = k \left[ \frac{{}_2\kappa_\ell^m}{2\ell - 1} B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell + 1)} B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell + 3} \right] - \dot{\tau} E_\ell^{(m)} + \mathcal{B}_\ell^{(m)}$$

where  ${}_2\kappa_\ell^m = \sqrt{(\ell^2 - m^2)(\ell^2 - 4)/\ell^2}$  is given by the Clebsch-Gordon coefficients and  $\mathcal{E}, \mathcal{B}$  are the sources (scattering only).

- Note that for vectors and tensors  $|m| > 0$  and  $B$  modes may be generated from  $E$  modes by projection. Cosmologically  $\mathcal{B}_\ell^{(m)} = 0$

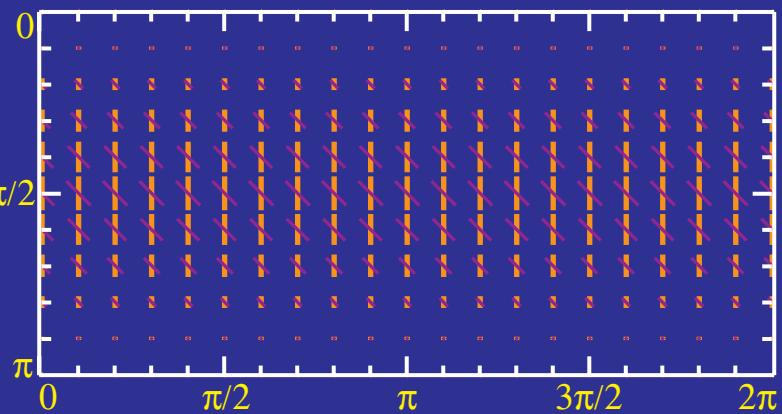
# Polarization Patterns

Scalars

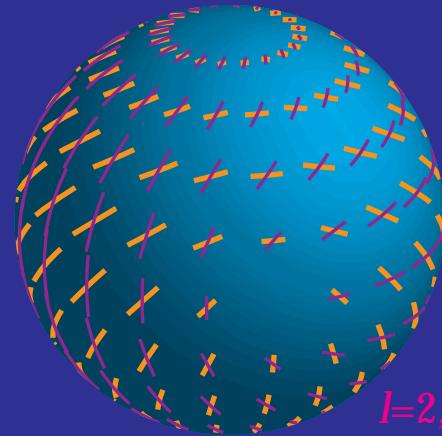


E, B

$l=2, m=0$



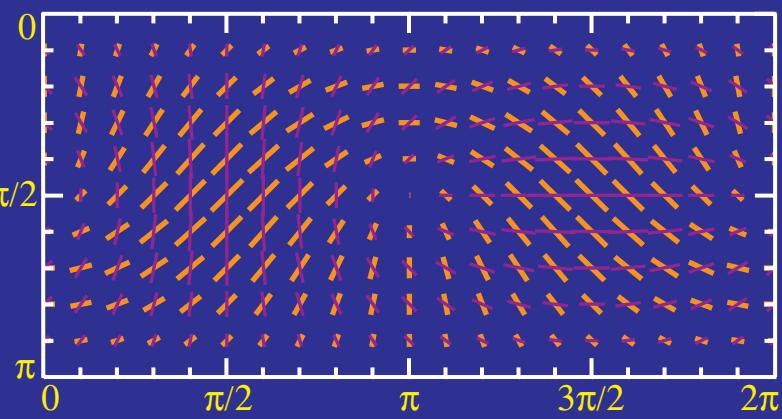
Vectors



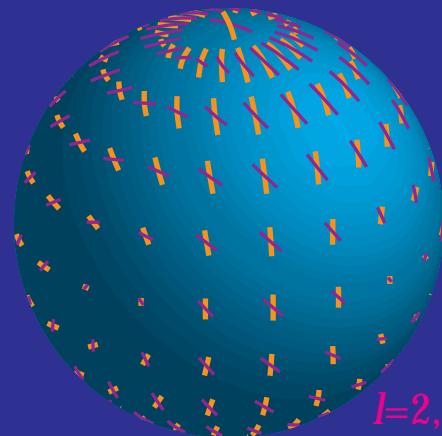
$l=2, m=1$

$\Theta$

$\phi$



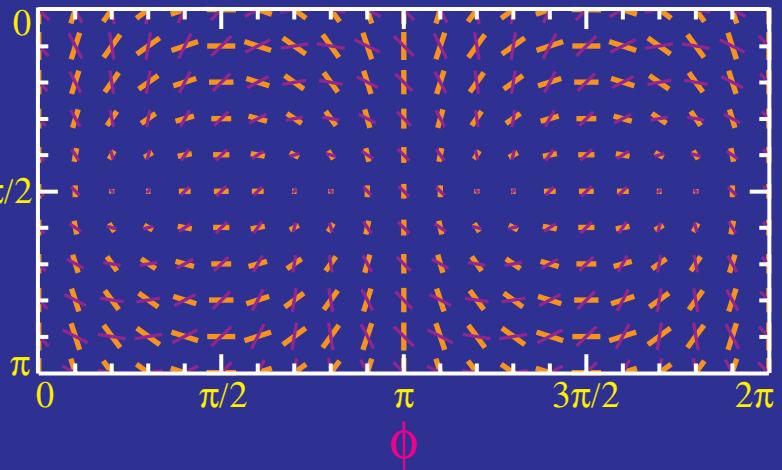
Tensors



$l=2, m=2$

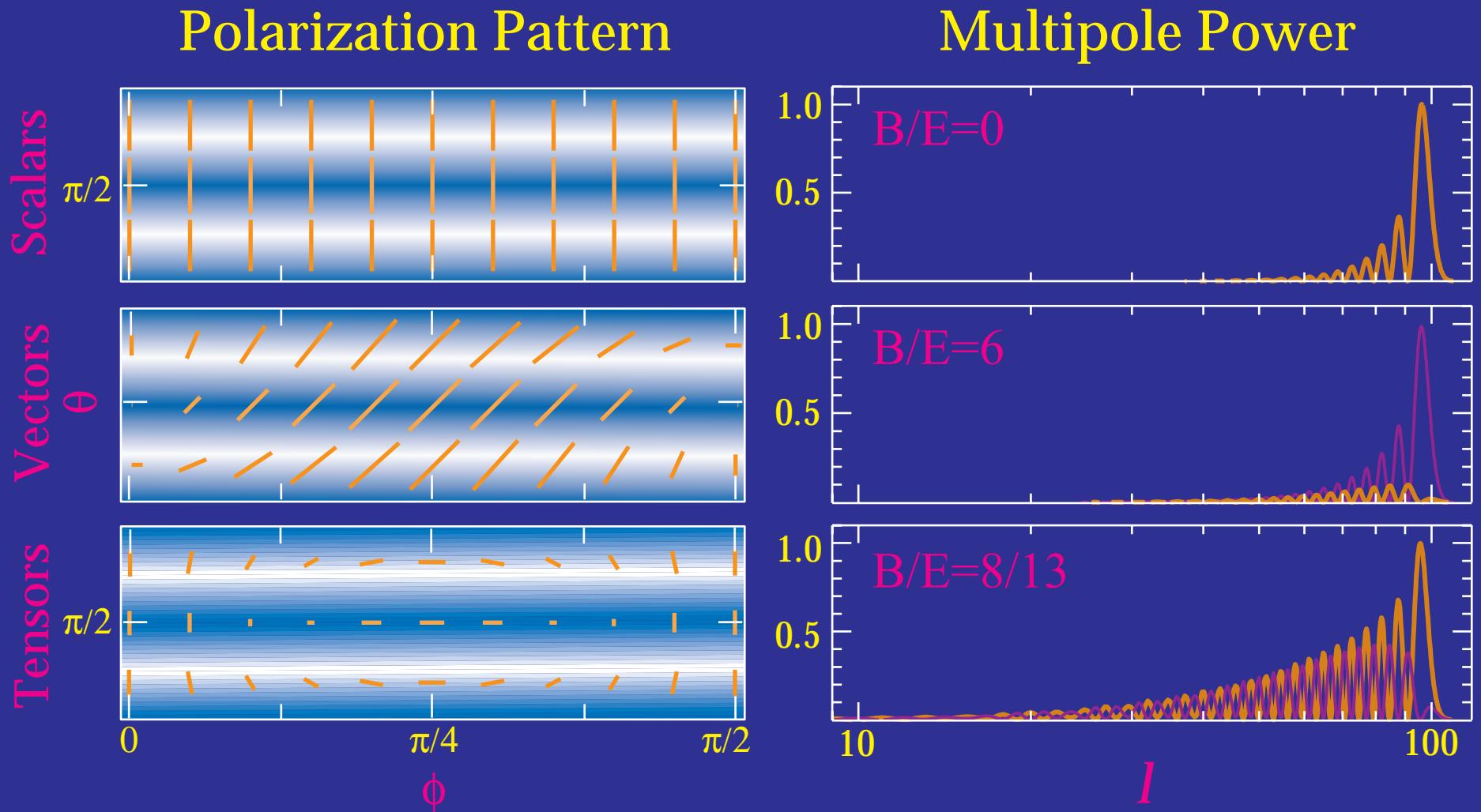
$\Theta$

$\phi$



# Patterns and Perturbation Types

- Amplitude modulated by plane wave → Principle axis
- Direction detemined by perturbation type → Polarization axis



Kamionkowski, Kosowsky, Stebbins (1997); Zaldarriaga & Seljak (1997); Hu & White (1997)

# Truncated Hierarchy

- CMBFast uses the integral solution and relies on a fast  $j_\ell$  generator
- However sources are not external to system and are defined through the Boltzmann hierarchy itself
- Solution: recall that we used this technique in the tight coupling regime by applying a closure condition from tight coupling
- CMBFast extends this idea by solving a truncated hierarchy of equations, e.g. out to  $\ell = 25$  with non-reflecting boundary conditions

# Collision Source Terms

# Thomson Collision Term

- Full Boltzmann equation

$$\frac{d}{d\eta} f_{a,b} = C[f_a, f_b]$$

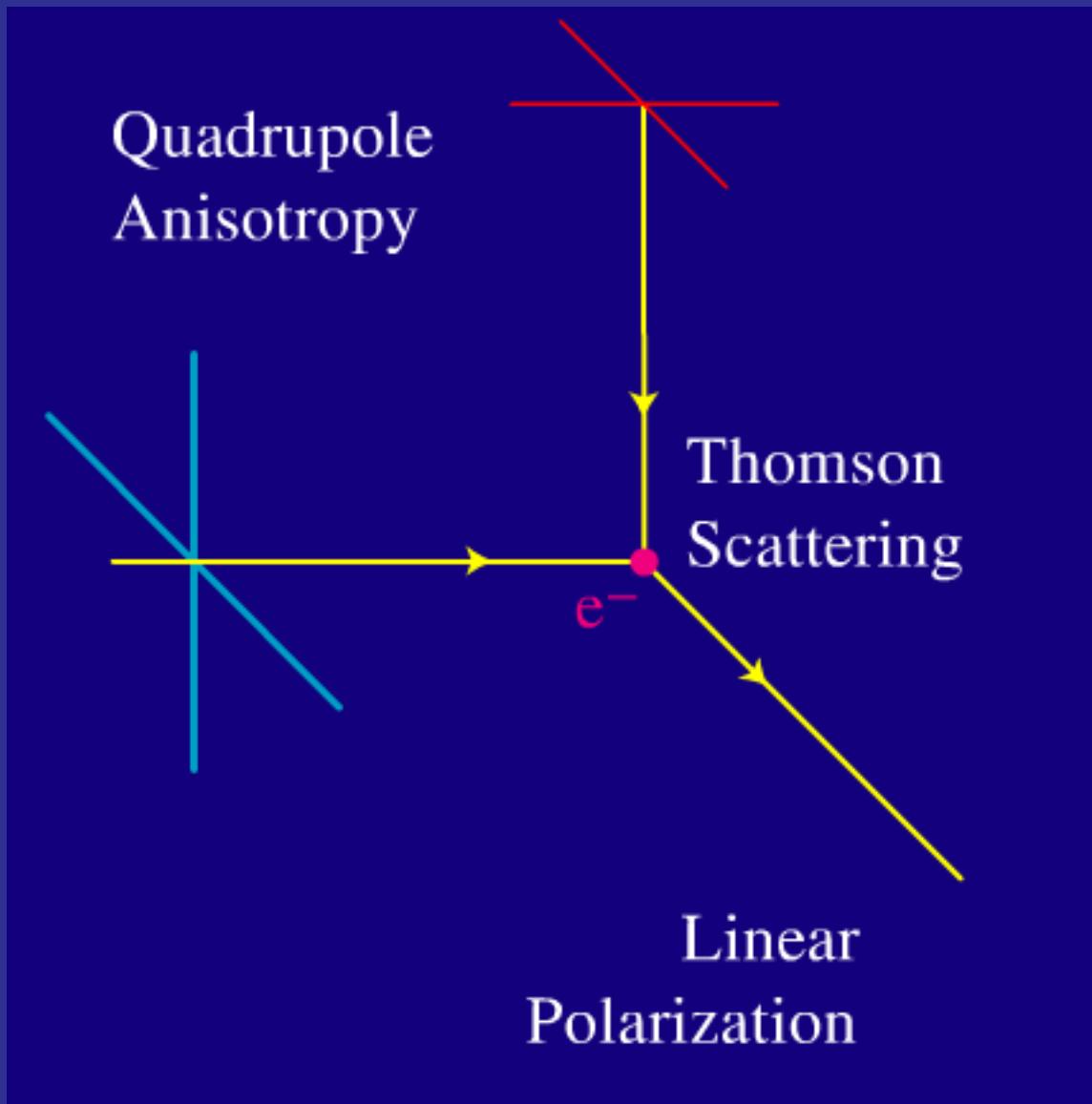
- Collision term describes the scattering out of and into a phase space element
- Thomson collision based on differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T ,$$

where  $\hat{\mathbf{E}}'$  and  $\hat{\mathbf{E}}$  denote the incoming and outgoing directions of the electric field or polarization vector.

# Polarization from Thomson Scattering

- Quadrupole anisotropies scatter into linear polarization



# Scattering Calculation

- Start in the electron rest frame and in a coordinate system fixed by the scattering plane, spanned by incoming and outgoing directional vectors  $-\hat{\mathbf{n}}' \cdot \hat{\mathbf{n}} = \cos \beta$ , where  $\beta$  is the scattering angle
- $\Theta_{\parallel}$ : in-plane polarization state;  $\Theta_{\perp}$ :  $\perp$ -plane polarization state
- Transfer probability (constant set by  $\dot{\tau}$ )

$$\Theta_{\parallel} \propto \cos^2 \beta \Theta'_{\parallel}, \quad \Theta_{\perp} \propto \Theta'_{\perp}$$

- and with the  $45^\circ$  axes as

$$\hat{\mathbf{E}}_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\parallel} + \hat{\mathbf{E}}_{\perp}), \quad \hat{\mathbf{E}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\parallel} - \hat{\mathbf{E}}_{\perp})$$

# Stokes Parameters

- Define the temperature in this basis

$$\begin{aligned}\Theta_1 &\propto |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 + |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 \\ &\propto \frac{1}{4}(\cos \beta + 1)^2 \Theta'_1 + \frac{1}{4}(\cos \beta - 1)^2 \Theta'_2 \\ \Theta_2 &\propto |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 + |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 \\ &\propto \frac{1}{4}(\cos \beta + 1)^2 \Theta'_2 + \frac{1}{4}(\cos \beta - 1)^2 \Theta'_1\end{aligned}$$

$$\text{or } \Theta_1 - \Theta_2 \propto \cos \beta (\Theta'_1 - \Theta'_2)$$

- Define  $\Theta, Q, U$  in the scattering coordinates

$$\Theta \equiv \frac{1}{2}(\Theta_{||} + \Theta_{\perp}), \quad Q \equiv \frac{1}{2}(\Theta_{||} - \Theta_{\perp}), \quad U \equiv \frac{1}{2}(\Theta_1 - \Theta_2)$$

# Scattering Matrix

- Transfer of Stokes states, e.g.

$$\Theta = \frac{1}{2}(\Theta_{\parallel} + \Theta_{\perp}) \propto \frac{1}{4}(\cos^2 \beta + 1)\Theta' + \frac{1}{4}(\cos^2 \beta - 1)Q'$$

- Transfer matrix of Stokes state  $\mathbf{T} \equiv (\Theta, Q + iU, Q - iU)$

$$\mathbf{T} \propto \mathbf{S}(\beta)\mathbf{T}'$$

$$\mathbf{S}(\beta) = \frac{3}{4} \begin{pmatrix} \cos^2 \beta + 1 & -\frac{1}{2} \sin^2 \beta & -\frac{1}{2} \sin^2 \beta \\ -\frac{1}{2} \sin^2 \beta & \frac{1}{2}(\cos \beta + 1)^2 & \frac{1}{2}(\cos \beta - 1)^2 \\ -\frac{1}{2} \sin^2 \beta & \frac{1}{2}(\cos \beta - 1)^2 & \frac{1}{2}(\cos \beta + 1)^2 \end{pmatrix}$$

normalization factor of 3 is set by photon conservation in scattering

# Scattering Matrix

- Transform to a fixed basis, by a rotation of the incoming and outgoing states  $\mathbf{T} = \mathbf{R}(\psi)\mathbf{T}$  where

$$\mathbf{R}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & e^{2i\psi} \end{pmatrix}$$

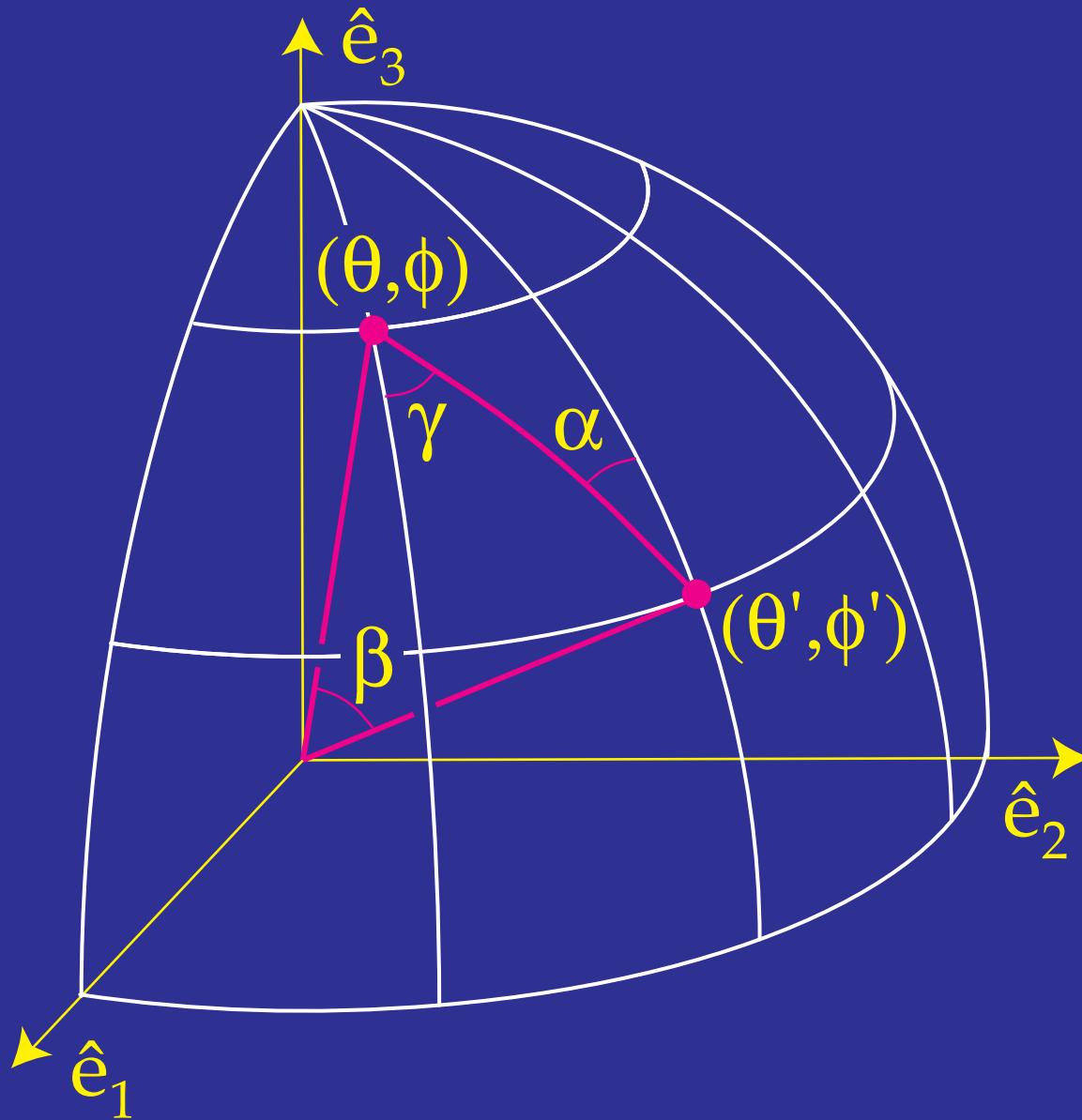
giving the scattering matrix

$$\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha) =$$

$$\frac{1}{2}\sqrt{\frac{4\pi}{5}} \begin{pmatrix} Y_2^0(\beta, \alpha) + 2\sqrt{5}Y_0^0(\beta, \alpha) & -\sqrt{\frac{3}{2}}Y_2^{-2}(\beta, \alpha) & -\sqrt{\frac{3}{2}}Y_2^2(\beta, \alpha) \\ -\sqrt{6}{}_2Y_2^0(\beta, \alpha)e^{2i\gamma} & 3{}_2Y_2^{-2}(\beta, \alpha)e^{2i\gamma} & 3{}_2Y_2^2(\beta, \alpha)e^{2i\gamma} \\ -\sqrt{6}{}_{-2}Y_2^0(\beta, \alpha)e^{-2i\gamma} & 3{}_{-2}Y_2^{-2}(\beta, \alpha)e^{-2i\gamma} & 3{}_{-2}Y_2^2(\beta, \alpha)e^{-2i\gamma} \end{pmatrix}$$

# Scattering Geometry

- Rotation from scattering frame to fixed sky basis



# Addition Theorem for Spin Harmonics

- Spin harmonics are related to rotation matrices as

$${}_s Y_\ell^m(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} \mathcal{D}_{-ms}^\ell(\phi, \theta, 0)$$

Note: for explicit evaluation sign convention differs from usual (e.g. Jackson) by  $(-1)^m$

- Multiplication of rotations

$$\sum_{m''} \mathcal{D}_{mm''}^\ell(\alpha_2, \beta_2, \gamma_2) \mathcal{D}_{m''m}^\ell(\alpha_1, \beta_1, \gamma_1) = \mathcal{D}_{mm'}^\ell(\alpha, \beta, \gamma)$$

- Implies

$$\sum_m {}_{s_1} Y_\ell^{m*}(\theta', \phi') {}_{s_2} Y_\ell^m(\theta, \phi) = (-1)^{s_1 - s_2} \sqrt{\frac{2\ell+1}{4\pi}} {}_{s_2} Y_\ell^{-s_1}(\beta, \alpha) e^{is_2 \gamma}$$

# Sky Basis

- Scattering into the state (rest frame)

$$C_{\text{in}}[\mathbf{T}] = \dot{\tau} \int \frac{d\hat{\mathbf{n}}'}{4\pi} \mathbf{R}(-\gamma) \mathbf{S}(\beta) \mathbf{R}(\alpha) \mathbf{T}(\hat{\mathbf{n}}') ,$$

$$= \dot{\tau} \int \frac{d\hat{\mathbf{n}}'}{4\pi} (\Theta', 0, 0) + \frac{1}{10} \dot{\tau} \int d\hat{\mathbf{n}}' \sum_{m=-2}^2 \mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') \mathbf{T}(\hat{\mathbf{n}}') .$$

where the quadrupole coupling term is  $\mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') =$

$$\begin{pmatrix} Y_2^{m*}(\hat{\mathbf{n}}') Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}} {}_2Y_2^{m*}(\hat{\mathbf{n}}') Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}} {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}') Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}') {}_2Y_2^m(\hat{\mathbf{n}}) & {}_3 {}_2Y_2^{m*}(\hat{\mathbf{n}}') {}_2Y_2^m(\hat{\mathbf{n}}) & {}_3 {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}') {}_2Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}') {}_{-2}Y_2^m(\hat{\mathbf{n}}) & {}_3 {}_2Y_2^{m*}(\hat{\mathbf{n}}') {}_{-2}Y_2^m(\hat{\mathbf{n}}) & {}_3 {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}') {}_{-2}Y_2^m(\hat{\mathbf{n}}) \end{pmatrix} ,$$

expression uses angle addition relation above. We call this term  $C_Q$ .

# Scattering Matrix

- Full scattering matrix involves difference of scattering into and out of state

$$C[\mathbf{T}] = C_{\text{in}}[\mathbf{T}] - C_{\text{out}}[\mathbf{T}]$$

- In the electron rest frame

$$C[\mathbf{T}] = \dot{\tau} \int \frac{d\hat{\mathbf{n}}'}{4\pi} (\Theta', 0, 0) - \dot{\tau}\mathbf{T} + C_Q[\mathbf{T}]$$

which describes isotropization in the rest frame. All moments have  $e^{-\tau}$  suppression except for isotropic temperature  $\Theta_0$ .

Transformation into the background frame simply induces a dipole term

$$C[\mathbf{T}] = \dot{\tau} \left( \hat{\mathbf{n}} \cdot \mathbf{v}_b + \int \frac{d\hat{\mathbf{n}}'}{4\pi} \Theta', 0, 0 \right) - \dot{\tau}\mathbf{T} + C_Q[\mathbf{T}]$$

# Source Terms

- Temperature source terms  $S_l^{(m)}$  (rows  $\pm|m|$ ; flat assumption)

$$\begin{pmatrix} \dot{\tau}\Theta_0^{(0)} - \dot{H}_L^{(0)} & \dot{\tau}v_b^{(0)} + \dot{B}^{(0)} & \dot{\tau}P^{(0)} - \frac{2}{3}\dot{H}_T^{(0)} \\ 0 & \dot{\tau}v_b^{(\pm 1)} + \dot{B}^{(\pm 1)} & \dot{\tau}P^{(\pm 1)} - \frac{\sqrt{3}}{3}\dot{H}_T^{(\pm 1)} \\ 0 & 0 & \dot{\tau}P^{(\pm 2)} - \dot{H}_T^{(\pm 2)} \end{pmatrix}$$

where

$$P^{(m)} \equiv \frac{1}{10}(\Theta_2^{(m)} - \sqrt{6}E_2^{(m)})$$

- Polarization source term

$$\mathcal{E}_\ell^{(m)} = -\dot{\tau}\sqrt{6}P^{(m)}\delta_{\ell,2}$$

$$\mathcal{B}_\ell^{(m)} = 0$$

# Lecture IV: Summary

- General relativistic perturbation theory: energy momentum conservation plus closure relation for stresses for scalar, vector, tensor degrees of freedom
- Choice of gauge and/or mixed “gauge invariant” conditions to simplify equations of motion
- Energy momentum conservation equations are first 2 moments of more general Boltzmann equation
- Formal integral solution to Boltzmann equation is simply a projection of sources from the first 3 moments
- Spin-spherical harmonics describe linear polarization field
- Modern codes use truncated Boltzmann hierarchy to solve for the sources on a coarse grid, interpolate and integrate