Cosmic Acceleration from Modified Gravity:

\[ f(R) \]

A Worked Example

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CalTech, December 2008
Why Study $f(R)$?

- Cosmic acceleration, like the cosmological constant, can either be viewed as arising from

  Missing, or dark energy, with $w \equiv \bar{p}/\bar{\rho} < -1/3$

  Modification of gravity on large scales

  $$G_{\mu\nu} = 8\pi G \left( T_{\mu\nu}^M + T_{\mu\nu}^{\text{DE}} \right)$$

  $$F(g_{\mu\nu}) + G_{\mu\nu} = 8\pi G T_{\mu\nu}^M$$

- Proof of principle models for both exist: quintessence, k-essence; DGP braneworld acceleration, $f(R)$ modified action

- Compelling models for either explanation lacking

- Study models as illustrative toy models whose features can be generalized
Two Gravitational Potentials

- **Newtonian potential**: \( \Psi = \delta g_{00} / 2g_{00} \) which non-relativistic particles feel

- **Space curvature**: \( \Phi = \delta g_{ii} / 2g_{ii} \)

- **Combination**: \( \Phi_- = (\Phi - \Psi) / 2 \) for gravitational **lensing** + **redshift**

Three Regimes

- Three regimes defined by $\gamma = -\Phi/\Psi$ BUT with different dynamics
- Examples $f(R)$ and DGP braneworld acceleration
- Parameterized Post-Friedmann description

- Non-linear regime follows a halo paradigm but a full parameterization still lacking and theoretical, examples few: $f(R)$ now fully worked

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General Relativistic Non-Linear Regime  Scalar-Tensor Regime  Conserved-Curvature Regime

$r_*$  $r_c$  $r$

halos, galaxy  large scale structure  CMB

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Outline

- $f(R)$ Basics and Background
- Linear Theory Predictions
- N-body Simulations and the Chameleon

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- Marcos Lima
- Hiro Oyaizu
- Hiranya Peiris
- Iggy Sawicki
- Fabian Schmidt
- Yong-Seon Song
$f(R)$ Basics
Cast of $f(R)$ Characters

- $R$: Ricci scalar or “curvature”
- $f(R)$: modified action (Starobinsky 1980; Carroll et al 2004)

$$S = \int d^4 x \sqrt{-g} \left[ \frac{R + f(R)}{16\pi G} + \mathcal{L}_m \right]$$
Cast of $f(R)$ Characters

- $R$: Ricci scalar or "curvature"
- $f(R)$: modified action (Starobinsky 1980; Carroll et al 2004)

$$ S = \int d^4x \sqrt{-g} \left[ \frac{R + f(R)}{16\pi G} + \mathcal{L}_m \right] $$

- $f_R \equiv df/dR$: additional propagating scalar degree of freedom (metric variation)
- $f_{RR} \equiv d^2f/dR^2$: Compton wavelength of $f_R$ squared, inverse mass squared
- $B$: Compton wavelength of $f_R$ squared in units of the Hubble length

$$ B \equiv \frac{f_{RR}}{1 + f_R} \frac{R'}{H} \frac{H}{H'} $$

- $\equiv d/d \ln a$: scale factor as time coordinate
Modified Einstein Equation

- In the **Jordan frame**, gravity becomes 4th order but matter remains minimally coupled and separately conserved

\[
G_{\alpha\beta} + f_R R_{\alpha\beta} - \left( \frac{f}{2} - \Box f_R \right) g_{\alpha\beta} - \nabla_\alpha \nabla_\beta f_R = 8\pi G T_{\alpha\beta}
\]

- Trace can be interpreted as a **scalar field equation** for \(f_R\) with a density-dependent effective potential \((\rho = 0)\)

\[
3\Box f_R + f_R R - 2f = R - 8\pi G \rho
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- Trace can be interpreted as a scalar field equation for \( f_R \) with a density-dependent effective potential (\( \rho = 0 \))

\[ 3\Box f_R + f_R R - 2f = R - 8\pi G \rho \]

- For small deviations, \( |f_R| \ll 1 \) and \( |f/R| \ll 1 \),

\[ \Box f_R \approx \frac{1}{3} (R - 8\pi G \rho) \]

the field is sourced by the deviation from GR relation between curvature and density and has a mass

\[ m_{f_R}^2 \approx \frac{1}{3} \frac{\partial R}{\partial f_R} = \frac{1}{3 f_{RR}} \]
Effective Potential

- Scalar $f_R$ rolls in an effective potential that depends on density.
- At high density, extrema is at GR $R=8\pi G \rho$.
- Minimum for $B>0$, pinning field to $|f_R| \ll 1$, maximum for $B<0$.

Sawicki & Hu (2007)
$f(R)$ Expansion History
Modified Friedmann Equation

- Expansion history parameterization: Friedmann equation becomes

\[ H^2 - f_R(2HH' + H^2) + \frac{1}{6} f + H^2 f_{RR} R' = \frac{8\pi G \rho}{3} \]

- Reverse engineering \( f(R) \) from the expansion history: for any desired \( H \), solve a 2nd order diffeq to find \( f(R) \)

- Allows a family of \( f(R) \) models, parameterized in terms of the Compton wavelength parameter \( B \)
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- Allows a family of \( f(R) \) models, parameterized in terms of the Compton wavelength parameter \( B \)

- Formally includes models where \( B < 0 \), such as \( f(R) = -\mu^4/R \), leading to confusion as to whether such models provide viable expansion histories

- Answer: no these have short-time scale tachyonic instabilities at high curvature and limit as \( B \to 0 \) from below is not GR

- \( B > 0 \) family has very different implications for structure formation but with identical distance-redshift relations
Expansion History Family of $f(R)$

- Each expansion history, matched by dark energy model $[w(z), \Omega_{DE}, H_0]$ corresponds to a family of $f(R)$ models due to its 4th order nature.
- Parameterized by $B \propto f_{RR} = \frac{d^2 f}{dR^2}$ evaluated at $z=0$.

Song, Hu & Sawicki (2006)
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Song, Hu & Sawicki (2006)
Instability at High Curvature

- **Tachyonic instability** for negative mass squared $B<0$ makes high curvature regime increasingly **unstable**: high density ≠ high curvature
- Linear metric **perturbations** immediately drop the expansion history to low curvature solution

![Graph showing curvature $R$ vs. $\alpha$ for $B>0$ and $B<0$.](image)

Sawicki & Hu (2007)
$f(R)$ Linear Theory
Three Regimes

- Three regimes defined by $\gamma = -\Phi/\Psi$ BUT with different dynamics
- Examples $f(R)$ and DGP braneworld acceleration
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General Relativistic Non-Linear Regime

Scalar-Tensor Regime

Conserved-Curvature Regime

$\gamma$ $r_*$ $r_c$

halos, galaxy large scale structure CMB
Linear Theory for $f(R)$

- In $f(R)$ model, “superhorizon” behavior persists until Compton wavelength smaller than fluctuation wavelength $B^{1/2}(k/aH) < 1$

- Once Compton wavelength becomes larger than fluctuation

$$B^{1/2}(k/aH) > 1$$

perturbations are in scalar-tensor regime described by $\gamma = 1/2$. (Chiba 2003; Erickcek, Smith, Kamionkowski 2006)

- Small scale density growth enhanced and

$$8\pi G \rho > R$$

low curvature regime with order unity deviations from GR

- Transitions in the non-linear regime where the Compton wavelength can shrink via chameleon mechanism

- Given $k_{NL}/aH \gg 1$, even very small $f_R$ have scalar-tensor regime
Curvature Conservation

- On superhorizon scales, energy momentum conservation and expansion history constrain the evolution of metric fluctuations (Bertschinger 2006)

- For adiabatic perturbations in a flat universe, conservation of comoving curvature applies $\zeta' = 0$ where $' \equiv d/d \ln a$ (Bardeen 1980)

- Gauge transformation to Newtonian gauge

$$ds^2 = -(1 + 2\Psi) dt^2 + a^2 (1 + 2\Phi) dx^2$$

yields (Hu & Eisenstein 1999)

$$\Phi'' - \Psi' - \frac{H''}{H'} \Phi' - \left( \frac{H'}{H} - \frac{H''}{H'} \right) \Psi = 0$$

- Modified gravity theory supplies the closure relationship $\Phi = -\gamma (\ln a) \Psi$ between and expansion history $H = \dot{a}/a$ supplies rest.
Deviation Parameter

- Express the 4th order nature of equations as a deviation parameter

\[ \Phi'' - \Psi' - \frac{H''}{H'} \Phi' - \left( \frac{H'}{H} - \frac{H''}{H'} \right) \Psi = \left( \frac{k}{aH} \right)^2 B \epsilon \]

- Einstein equation become a second order equation for \( \epsilon \)
Deviation Parameter

- Express the 4th order nature of equations as a deviation parameter

\[
\Phi'' - \Psi' - \frac{H''}{H'} \Phi' - \left( \frac{H'}{H} - \frac{H''}{H'} \right) \Psi = \left( \frac{k}{aH} \right)^2 B\epsilon
\]

- Einstein equation become a second order equation for \( \epsilon \)

- In high redshift, high curvature \( R \) limit this is

\[
\epsilon'' + \left( \frac{7}{2} + 4 \frac{B'}{B} \right) \epsilon' + \frac{2}{B} \epsilon = \frac{1}{B} \times \text{metric sources}
\]

\[
B = \frac{f_{RR}}{1 + f_R} R' \frac{H}{H'}
\]

- \( R \to \infty, B \to 0 \) and for \( B < 0 \) short time-scale **tachyonic instability** appears making previous models not cosmologically viable

\[
f(R) = -M^{2+2n}/R^n
\]
Potential Growth

- On the stable $B>0$ branch, potential evolution reverses from decay to growth as wavelength becomes smaller than Compton scale.
- Quasistatic equilibrium reached in linear theory with $\gamma=-\Phi/\Psi=1/2$ until non-linear effects restore $\gamma=1$.
Integrated Sachs-Wolfe Effect

- CMB photons transit gravitational potentials of large-scale structure
- If potential decays during transit, gravitational blueshift of infall not cancelled by gravitational redshift of exit
- Spatial curvature of gravitational potential leads to additional effect $\Delta T/T = -\Delta(\Phi-\Psi)$
Integrated Sachs-Wolfe Effect

- CMB photons transit gravitational potentials of large-scale structure.
- If potential decays during transit, gravitational blueshift of infall not cancelled by gravitational redshift of exit.
- Spatial curvature of gravitational potential leads to additional effect \( \Delta T/T = -\Delta(\Phi-\Psi) \).
ISW Quadrupole

- **Reduction of large angle anisotropy** for $B_0 \sim 1$ for same expansion history and distances as $\Lambda$CDM
- **Well-tested** small scale anisotropy unchanged

Song, Hu & Sawicki (2006)
ISW-Galaxy Correlation

- **Decaying potential**: galaxy positions correlated with CMB
- **Growing potential**: galaxy positions anticorrelated with CMB
- **Observations** indicate correlation
Galaxy-ISW Anti-Correlation

- Large Compton wavelength $B^{1/2}$ creates potential growth which can anti-correlate galaxies and the CMB
- In tension with detections of positive correlations across a range of redshifts

Song, Peiris & Hu (2007)
Linear Power Spectrum

- Linear real space power spectrum enhanced on scales below Compton scale in the background
- Scale-dependent growth rate and potentially large deviations on small scales
Redshift Space Distortion

- Relationship between velocity and density field given by continuity with modified growth rate \( f_v = \frac{d\ln D}{d\ln a} \)
- Redshift space power spectrum further distorted by Kaiser effect
$f(R)$ Non-Linear Evolution
Three Regimes

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Diagram:

- General Relativistic Non-Linear Regime
- Scalar-Tensor Regime
- Conserved-Curvature Regime

- $r_*$: halos, galaxy
- $r_c$: large scale structure
- $r$: CMB
Non-Linear Chameleon

• For $f(R)$ the field equation

$$\nabla^2 f_R \approx \frac{1}{3} (\delta R(f_R) - 8\pi G\delta\rho)$$

is the non-linear equation that returns general relativity

• High curvature implies short Compton wavelength and suppressed deviations but requires a change in the field from the background value $\delta R(f_R)$

• Change in field is generated by density perturbations just like gravitational potential so that the chameleon appears only if

$$\Delta f_R \leq \frac{2}{3}\Phi,$$

else required field gradients too large despite $\delta R = 8\pi G\delta\rho$ being the local minimum of effective potential
Non-Linear Dynamics

- Supplement that with the modified Poisson equation
  \[
  \nabla^2 \Psi = \frac{16 \pi G}{3} \delta \rho - \frac{1}{6} \delta R(f_R)
  \]

- Matter evolution given metric unchanged: usual motion of matter in a gravitational potential \( \Psi \)

- Prescription for \( N \)-body code

- Particle Mesh (PM) for the Poisson equation

- Field equation is a non-linear Poisson equation: relaxation method for \( f_R \)

- Initial conditions set to GR at high redshift
Environment Dependent Force

- Chameleon suppresses extra force (scalar field) in high density, deep potential regions

Environment Dependent Force

- For **large background field**, gradients in the scalar **prevent** the chameleon from appearing.

<table>
<thead>
<tr>
<th>Density: max[ln(1+δ)]</th>
<th>Potential: min[Ψ]</th>
<th>Field: min[f_R/f_{R0}]</th>
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512^3 PM-relaxation code resolves the chameleon transition to GR: greatly reduced non-linear effect
• Artificially turning off the chameleon mechanism restores much of enhancement

\[ P(k)/P_{GR}(k)-1 \]

\[ f_{R0} = 10^{-6} \]

N-body Power Spectrum

- Models where the chameleon absent today (large field models) show residual effects from a high redshift chameleon

\[
P(k)/P_{GR}(k) - 1
\]

With smooth dark energy, distance predicts scale-invariant growth to a few percent - a falsifiable prediction.
Scaling Relations

- Fitting functions based on normal gravity fail to capture chameleon and effect of extra forces on dark matter halos

\[ \frac{P(k)}{P_{GR}(k)} = 1 \]

• Models where the chameleon absent today (large field models) show residual effects from a high redshift chameleon

$P(k)/P_{GR}(k)$

$|f_{R0}| = 10^{-4}$

$|f_{R0}| = 10^{-6}$

$k (h/\text{Mpc})$
Enhanced abundance of rare dark matter halos (clusters) with extra force

Halo Bias

- Halos at a fixed mass less rare and less highly biased

Halo Mass Correlation

• Enhanced forces vs lower bias

Halo Model

- Power spectrum trends also consistent with halos and modified collapse

$f(R)$ Solar System Tests
• Density profile of Sun is not a constant density sphere - interior photosphere, chromosphere, corona

• Density drops by \(~25\) orders of magnitude - does curvature follow?

Hu & Sawicki (2007)
Solar System Constraint

- **Cassini** constraint on PPN $|\gamma - 1| < 2.3 \times 10^{-5}$
- Easily satisfied if **galactic field** is at potential minimum $|f_{Rg}| < 4.9 \times 10^{-11}$
- Allows even **order unity** cosmological fields

Hu & Sawicki (2007)
Field Solution

- **Field** solution smoothly relaxes from exterior value to high curvature interior value $f_R \sim 0$, minimizing potential + kinetic.
- **Juncture** is where thin-shell criterion is satisfied $|\Delta f_R| \sim \Delta \Phi$.

Hu & Sawicki (2007)
Solar Curvature

- Curvature drops suddenly as field moves slightly from zero
- Enters into low curvature regime where $R < 8\pi G\rho$

Hu & Sawicki (2007)
Galactic Thin Shell

- Galaxy must have a thin shell for interior to remain at high curvature
- Rotation curve $v/c \sim 10^{-3}$, $\Phi \sim 10^{-6} \sim |\Delta f_R|$ limits cosmological field
- Has the low cosmological curvature propagated through local group and galactic exterior?

![Graph showing density and $f_R$ versus radius](image)

Hu & Sawicki (2007)
Summary

- General lessons from $f(R)$ example – 3 regimes:
  - large scales: conservation determined
  - intermediate scales: scalar-tensor
  - small scales: GR in high density regions, modified in low
- Given fixed expansion history $f(R)$ has additional continuous parameter: Compton wavelength
- Enhanced gravitational forces below environment-dependent Compton scale affect growth of structure
- Enhancement hidden by non-linear chameleon mechanism at high curvature $\neq$ high density)
- $N$-body (PM-relaxation) simulations show potentially observable differences in the power spectrum and mass function