

Lecture 2: Linear Perturbation Theory



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Outline

- Covariant Perturbation Theory
- Scalar, Vector, Tensor Decomposition
- Linearized Einstein-Conservation Equations
- Dark (Multi) Components
- Gauge
- Applications:

Bardeen Curvature

Baryonic wiggles

Scalar Fields

Parameterizing dark components

Transfer function

Massive neutrinos

Sachs-Wolfe Effect

Dark energy

COBE normalization

Covariant Perturbation Theory

- **Covariant** = takes same **form** in all coordinate systems
- **Invariant** = takes the same **value** in all coordinate systems
- Fundamental equations: **Einstein equations**, covariant **conservation** of stress-energy tensor:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$
$$\nabla_\mu T^{\mu\nu} = 0$$

- Preserve general covariance by keeping all **degrees of freedom**: 10 for each symmetric 4×4 tensor

1	2	3	4
	5	6	7
		8	9
			10

Metric Tensor

- Expand the metric tensor around the **general FRW metric**

$$g_{00} = -a^2, \quad g_{ij} = a^2 \gamma_{ij}.$$

where the “0” component is **conformal time** $\eta = dt/a$ and γ_{ij} is a **spatial metric of constant curvature** $K = H_0^2(\Omega_{\text{tot}} - 1)$.

- Add in a general perturbation (**Bardeen 1980**)

$$\begin{aligned} g^{00} &= -a^{-2}(1 - 2A), \\ g^{0i} &= -a^{-2}B^i, \\ g^{ij} &= a^{-2}(\gamma^{ij} - 2H_L\gamma^{ij} - 2H_T^{ij}). \end{aligned}$$

- (1) $A \equiv$ a scalar **potential**; (3) B^i a vector **shift**, (1) H_L a perturbation to the spatial **curvature**; (6) H_T^{ij} a **trace-free** distortion to spatial metric = (10)

Matter Tensor

- Likewise expand the matter **stress energy** tensor around a homogeneous density ρ and pressure p :

$$T^0_0 = -\rho - \delta\rho,$$

$$T^0_i = (\rho + p)(v_i - B_i),$$

$$T^i_0 = -(\rho + p)v^i,$$

$$T^i_j = (p + \delta p)\delta^i_j + p\Pi^i_j,$$

- (1) $\delta\rho$ a **density perturbation**; (3) v_i a vector **velocity**, (1) δp a **pressure perturbation**; (5) Π_{ij} an **anisotropic stress** perturbation
- So far this is **fully general** and applies to any type of matter or coordinate choice including non-linearities in the matter, e.g. cosmological defects.

Counting DOF's

20	Variables (10 metric; 10 matter)
−10	Einstein equations
−4	Conservation equations
+4	Bianchi identities
−4	Gauge (coordinate choice 1 time, 3 space)
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6	Degrees of freedom

- Without loss of generality these can be taken to be the 6 components of the matter stress tensor
- For the background, specify $p(a)$ or equivalently $w(a) \equiv p(a)/\rho(a)$ the equation of state parameter.

Scalar, Vector, Tensor

- In **linear perturbation theory**, perturbations may be separated by their **transformation properties** under rotation and translation.
- The eigenfunctions of the **Laplacian operator** form a complete set

$$\begin{aligned}\nabla^2 Q^{(0)} &= -k^2 Q^{(0)} && \text{S}, \\ \nabla^2 Q_i^{(\pm 1)} &= -k^2 Q_i^{(\pm 1)} && \text{V}, \\ \nabla^2 Q_{ij}^{(\pm 2)} &= -k^2 Q_{ij}^{(\pm 2)} && \text{T},\end{aligned}$$

and functions built out of covariant derivatives and the metric

$$\begin{aligned}Q_i^{(0)} &= -k^{-1} \nabla_i Q^{(0)}, \\ Q_{ij}^{(0)} &= (k^{-2} \nabla_i \nabla_j - \frac{1}{3} \gamma_{ij}) Q^{(0)}, \\ Q_{ij}^{(\pm 1)} &= -\frac{1}{2k} [\nabla_i Q_j^{(\pm 1)} + \nabla_j Q_i^{(\pm 1)}],\end{aligned}$$

Spatially Flat Case

- For a spatially flat background metric, harmonics are related to plane waves:

$$\begin{aligned}Q^{(0)} &= \exp(i\mathbf{k} \cdot \mathbf{x}) \\Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k} \cdot \mathbf{x}) \\Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k} \cdot \mathbf{x})\end{aligned}$$

where $\hat{\mathbf{e}}_3 \parallel \mathbf{k}$.

- For vectors, the harmonic points in a direction orthogonal to \mathbf{k} suitable for the vortical component of a vector
- For tensors, the harmonic is transverse and traceless as appropriate for the decomposition of gravitational waves

Perturbation k -Modes

- For the k th eigenmode, the scalar components become

$$\begin{aligned} A(\mathbf{x}) &= A(k) Q^{(0)}, & H_L(\mathbf{x}) &= H_L(k) Q^{(0)}, \\ \delta\rho(\mathbf{x}) &= \delta\rho(k) Q^{(0)}, & \delta p(\mathbf{x}) &= \delta p(k) Q^{(0)}, \end{aligned}$$

the vectors components become

$$B_i(\mathbf{x}) = \sum_{m=-1}^1 B^{(m)}(k) Q_i^{(m)}, \quad v_i(\mathbf{x}) = \sum_{m=-1}^1 v^{(m)}(k) Q_i^{(m)},$$

and the tensors components

$$\begin{aligned} H_{Tij}(\mathbf{x}) &= \sum_{m=-2}^2 H_T^{(m)}(k) Q_{ij}^{(m)}, \\ \Pi_{ij}(\mathbf{x}) &= \sum_{m=-2}^2 \Pi^{(m)}(k) Q_{ij}^{(m)}, \end{aligned}$$

Homogeneous Einstein Equations

- Einstein (Friedmann) equations:

$$\left(\frac{1}{a} \frac{da}{dt}\right)^2 = \frac{8\pi G}{3} \rho$$
$$\frac{1}{a} \frac{d^2 a}{dt^2} = -\frac{4\pi G}{3} (\rho + 3p)$$

so that $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot{\rho}}{\rho} = -3(1 + w) \frac{\dot{a}}{a}$$

Homogeneous Einstein Equations

- Counting exercise:

20	Variables (10 metric; 10 matter)
−17	Homogeneity and Isotropy
−2	Einstein equations
−1	Conservation equations
+1	Bianchi identities
<hr/>	
1	Degree of freedom

- without loss of generality choose ratio of homogeneous & isotropic component of the stress tensor to the density $w(a) = p(a)/\rho(a)$.

Covariant Scalar Equations

- Einstein equations (suppressing 0) superscripts (Hu & Eisenstein 1999):

$$(k^2 - 3K)[H_L + \frac{1}{3}H_T + \frac{\dot{a}}{a}\frac{1}{k^2}(kB - \dot{H}_T)] \\ = 4\pi Ga^2 \left[\delta\rho + 3\frac{\dot{a}}{a}(\rho + p)(v - B)/k \right], \quad \text{Poisson Equation}$$

$$k^2(A + H_L + \frac{1}{3}H_T) + \left(\frac{d}{d\eta} + 2\frac{\dot{a}}{a} \right) (kB - \dot{H}_T) \\ = 8\pi Ga^2 p\Pi,$$

$$\frac{\dot{a}}{a}A - \dot{H}_L - \frac{1}{3}\dot{H}_T - \frac{K}{k^2}(kB - \dot{H}_T) \\ = 4\pi Ga^2(\rho + p)(v - B)/k,$$

$$\left[2\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\frac{d}{d\eta} - \frac{k^2}{3} \right] A - \left[\frac{d}{d\eta} + \frac{\dot{a}}{a} \right] (\dot{H}_L + \frac{1}{3}kB) \\ = 4\pi Ga^2(\delta p + \frac{1}{3}\delta\rho).$$

Covariant Scalar Equations

- Conservation equations: **continuity** and **Navier Stokes**

$$\left[\frac{d}{d\eta} + 3\frac{\dot{a}}{a} \right] \delta\rho + 3\frac{\dot{a}}{a} \delta p = -(\rho + p)(kv + 3\dot{H}_L),$$
$$\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] \left[(\rho + p) \frac{(v - B)}{k} \right] = \delta p - \frac{2}{3} \left(1 - 3\frac{K}{k^2} \right) p\Pi + (\rho + p)A,$$

- Equations are not independent since $\nabla_\mu G^{\mu\nu} = 0$ via the **Bianchi identities**.
- Related to the ability to choose a **coordinate system** or “**gauge**” to represent the perturbations.

Covariant Scalar Equations

- DOF counting exercise

8 Variables (4 metric; 4 matter)

−4 Einstein equations

−2 Conservation equations

+2 Bianchi identities

−2 Gauge (coordinate choice 1 time, 1 space)

2 Degrees of freedom

- without loss of generality choose scalar components of the stress tensor $\delta p, \Pi$.

Covariant Vector Equations

- Einstein equations

$$\begin{aligned}(1 - 2K/k^2)(kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) \\ = 16\pi Ga^2(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k, \\ \left[\frac{d}{d\eta} + 2\frac{\dot{a}}{a} \right] (kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) \\ = -8\pi Ga^2 p \Pi^{(\pm 1)}.\end{aligned}$$

- Conservation Equations

$$\begin{aligned}\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] [(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k] \\ = -\frac{1}{2}(1 - 2K/k^2)p\Pi^{(\pm 1)},\end{aligned}$$

- Gravity provides **no source** to vorticity \rightarrow **decay**

Covariant Vector Equations

- DOF counting exercise

8 Variables (4 metric; 4 matter)

−4 Einstein equations

−2 Conservation equations

+2 Bianchi identities

−2 Gauge (coordinate choice 1 time, 1 space)

2 Degrees of freedom

- without loss of generality choose vector components of the stress tensor $\Pi^{(\pm 1)}$.

Covariant Tensor Equation

- Einstein equation

$$\left[\frac{d^2}{d\eta^2} + 2\frac{\dot{a}}{a} \frac{d}{d\eta} + (k^2 + 2K) \right] H_T^{(\pm 2)} = 8\pi G a^2 p \Pi^{(\pm 2)}.$$

- DOF counting exercise

4 Variables (2 metric; 2 matter)

−2 Einstein equations

−0 Conservation equations

+0 Bianchi identities

−0 Gauge (coordinate choice 1 time, 1 space)

2 Degrees of freedom

- wlog choose tensor components of the stress tensor $\Pi^{(\pm 2)}$.

Arbitrary Dark Components

- Total stress energy tensor can be broken up into **individual pieces**
- **Dark components** interact only through gravity and so satisfy **separate conservation equations**
- Einstein equation source remains the sum of components.
- To specify an arbitrary dark component, give the behavior of the **stress tensor: 6 components**: $\delta p, \Pi^{(i)}$, where $i = -2, \dots, 2$.
- Many types of dark components (dark matter, scalar fields, massive neutrinos,..) have **simple forms** for their stress tensor in terms of the energy density, i.e. described by **equations of state**.
- An equation of state for the background $w = p/\rho$ is **not sufficient** to determine the behavior of the perturbations.

Gauge

- Metric and matter fluctuations take on **different values** in different coordinate system
- No such thing as a “gauge invariant” density perturbation!
- General **coordinate transformation**:

$$\begin{aligned}\tilde{\eta} &= \eta + T \\ \tilde{x}^i &= x^i + L^i\end{aligned}$$

free to choose (T, L^i) to simplify equations or physics.
Decompose these into scalar and vector harmonics.

- $G_{\mu\nu}$ and $T_{\mu\nu}$ transform as **tensors**, so components in different frames can be related

Gauge Transformation

- Scalar Metric:

$$\begin{aligned}\tilde{A} &= A - \dot{T} - \frac{\dot{a}}{a}T, \\ \tilde{B} &= B + \dot{L} + kT, \\ \tilde{H}_L &= H_L - \frac{k}{3}L - \frac{\dot{a}}{a}T, \\ \tilde{H}_T &= H_T + kL,\end{aligned}$$

- Scalar Matter (J th component):

$$\begin{aligned}\delta\tilde{\rho}_J &= \delta\rho_J - \dot{\rho}_J T, \\ \delta\tilde{p}_J &= \delta p_J - \dot{p}_J T, \\ \tilde{v}_J &= v_J + \dot{L},\end{aligned}$$

- Vector:

$$\tilde{B}^{(\pm 1)} = B^{(\pm 1)} + \dot{L}^{(\pm 1)}, \tilde{H}_T^{(\pm 1)} = H_T^{(\pm 1)} + kL^{(\pm 1)}, \tilde{v}_J^{(\pm 1)} = v_J^{(\pm 1)} + \dot{L}^{(\pm 1)},$$

Gauge Dependence of Density

- Background evolution of the density induces a density fluctuation from a shift in the time coordinate

Common Scalar Gauge Choices

- A coordinate system is **fully specified** if there is an explicit prescription for (T, L^i) or for scalars (T, L)
- Newtonian:

$$\tilde{B} = \tilde{H}_T = 0$$

$$\Psi \equiv \tilde{A} \quad (\text{Newtonian potential})$$

$$\Phi \equiv \tilde{H}_L \quad (\text{Newtonian curvature})$$

$$L = -H_T/k$$

$$T = -B/k + \dot{H}_T/k^2$$

Good: intuitive Newtonian like gravity; matter and metric algebraically related; commonly chosen for **analytic CMB** and **lensing** work

Bad: numerically **unstable**

Example: Newtonian Reduction

- In the general equations, set $B = H_T = 0$:

$$\begin{aligned}(k^2 - 3K)\Phi &= 4\pi G a^2 \left[\delta\rho + 3\frac{\dot{a}}{a}(\rho + p)v/k \right] \\ k^2(\Psi + \Phi) &= 8\pi G a^2 p\Pi\end{aligned}$$

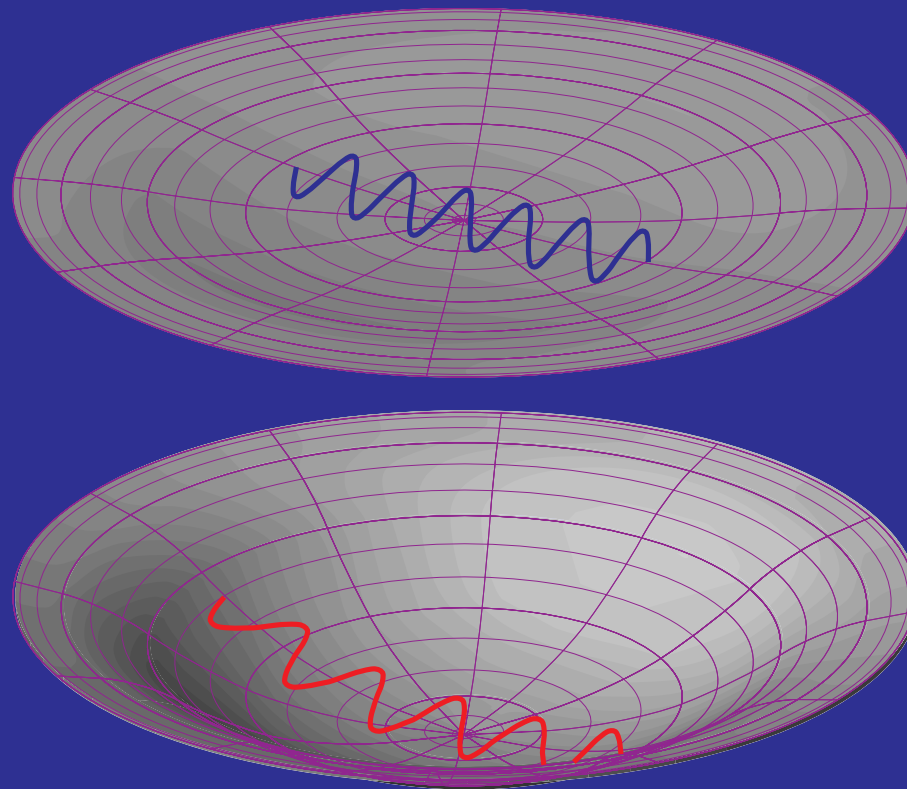
so $\Psi = -\Phi$ if anisotropic stress $\Pi = 0$ and

$$\begin{aligned}\left[\frac{d}{d\eta} + 3\frac{\dot{a}}{a} \right] \delta\rho + 3\frac{\dot{a}}{a} \delta p &= -(\rho + p)(kv + 3\dot{\Phi}), \\ \left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] (\rho + p)v &= k\delta p - \frac{2}{3}\left(1 - 3\frac{K}{k^2}\right)p k\Pi + (\rho + p) k\Psi,\end{aligned}$$

- Competition between **stress** (pressure and viscosity) and **potential** gradients

Relativistic Term in Continuity

- Continuity equation contains **relativistic term** from changes in the **spatial curvature** – perturbation to the **scale factor**
- For $w=0$ (matter), simply **density dilution**; for $w=1/3$ (radiation) density dilution plus (cosmological) **redshift**



- a.k.a. **ISW effect** – photon redshift from change in grav. potential

Common Scalar Gauge Choices

- Comoving:

$$\tilde{B} = \tilde{v} \quad (T_i^0 = 0)$$

$$H_T = 0$$

$$\xi = \tilde{A}$$

$$\zeta = \tilde{H}_L \quad (\text{Bardeen curvature})$$

$$T = (v - B)/k$$

$$L = -H_T/k$$

Good: ζ is conserved if stress fluctuations negligible, e.g. above the horizon if $|K| \ll H^2$

$$\dot{\zeta} + K v/k = \frac{\dot{a}}{a} \left[-\frac{\delta p}{\rho + p} + \frac{2}{3} \left(1 - \frac{3K}{k^2} \right) \frac{p}{\rho + p} \Pi \right] \rightarrow 0$$

Bad: explicitly relativistic choice

Common Scalar Gauge Choices

- Synchronous:

$$\tilde{A} = \tilde{B} = 0$$

$$\eta_L \equiv -\tilde{H}_L - \frac{1}{3}\tilde{H}_T$$

$$h_T = \tilde{H}_T \quad \text{or} \quad h = 6H_L$$

$$T = a^{-1} \int d\eta a A + c_1 a^{-1}$$

$$L = - \int d\eta (B + kT) + c_2$$

Good: stable, the choice of numerical codes

Bad: residual **gauge freedom** in constants c_1, c_2 must be specified as an initial condition, intrinsically relativistic.

Common Scalar Gauge Choices

- Spatially Unperturbed:

$$\tilde{H}_L = \tilde{H}_T = 0$$

$$L = -H_T/k$$

$$\tilde{A}, \tilde{B} = \text{metric perturbations}$$

$$T = \left(\frac{\dot{a}}{a}\right)^{-1} \left(H_L + \frac{1}{3}H_T\right)$$

Good: eliminates spatial metric in evolution equations; useful in **inflationary calculations** (Mukhanov et al)

Bad: intrinsically relativistic.

- **Caution:** perturbation evolution is governed by the behavior of stress fluctuations and an isotropic stress fluctuation δp is gauge dependent.

Hybrid “Gauge Invariant” Approach

- With the gauge transformation relations, express variables of **one gauge** in terms of those in **another** – allows a mixture in the equations of motion
- **Example:** Newtonian curvature above the horizon. Conservation of the Bardeen-curvature $\zeta = \text{const.}$ implies:

$$\Phi = \frac{3 + 3w}{5 + 3w} \zeta$$

e.g. calculate ζ from inflation determines Φ for any choice of matter content or causal evolution.

- **Example:** Scalar field (“quintessence” dark energy) equations in comoving gauge imply a **sound speed** $\delta p / \delta \rho = 1$ independent of potential $V(\phi)$. Solve in synchronous gauge (Hu 1998).

Transfer Function Example

- **Example:** Transfer function transfers the initial Newtonian curvature to its value today (linear response theory)

$$T(k) = \frac{\Phi(k, a = 1)}{\Phi(k, a_{\text{init}})}$$

- Conservation of Bardeen curvature: Newtonian curvature is a **constant** when **stress perturbations are negligible**: above the horizon during radiation and dark energy domination, on all scales during matter domination
- When stress fluctuations dominate, perturbations are stabilized by the **Jeans mechanism**
- Hybrid **Poisson equation**: Newtonian curvature, comoving density perturbation $\Delta \equiv (\delta\rho/\rho)_{\text{com}}$ implies Φ decays

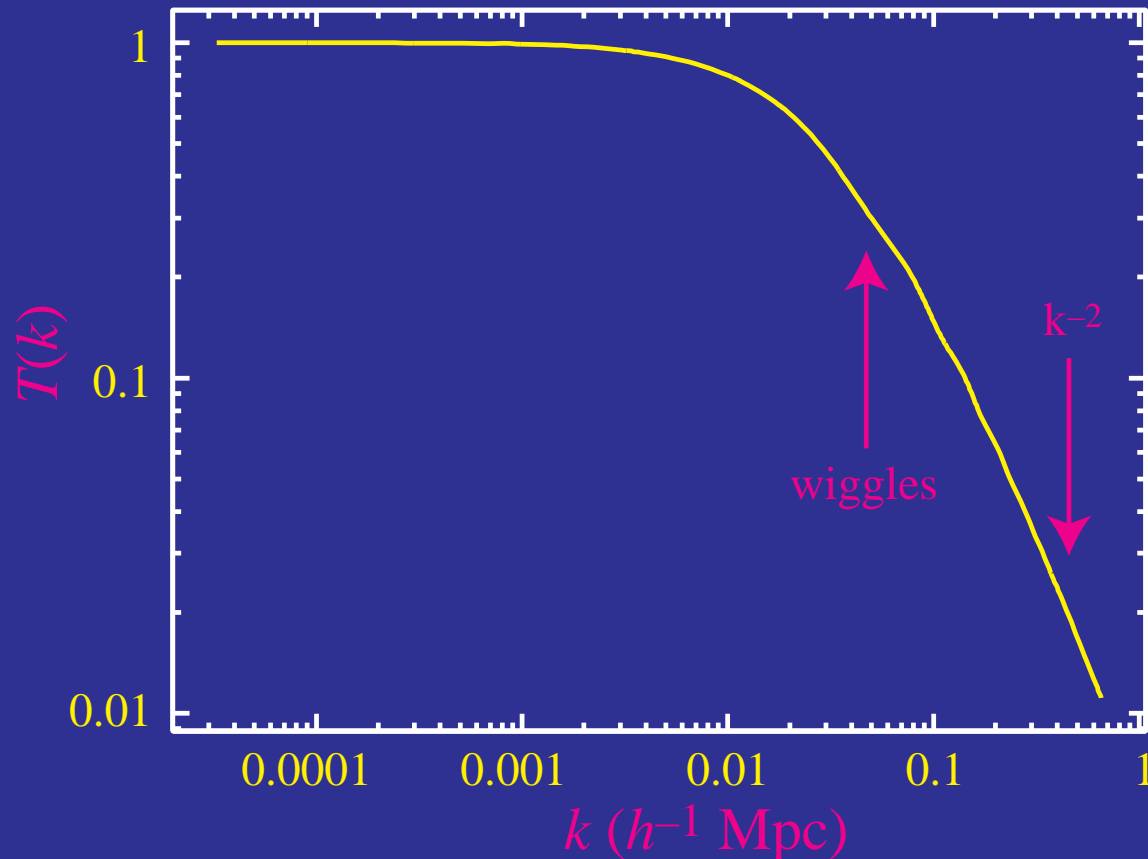
$$(k^2 - 3K)\Phi = 4\pi G\rho\Delta \sim \eta^{-2}\Delta$$

Transfer Function Example

- Freezing of Δ stops at η_{eq}

$$\Phi \sim (k\eta_{\text{eq}})^{-2} \Delta_H \sim (k\eta_{\text{eq}})^{-2} \Phi_{\text{init}}$$

- Transfer function has a k^{-2} fall-off beyond $k_{\text{eq}} \sim \eta_{\text{eq}}^{-1}$



Gauge and the Sachs-Wolfe Effect

- Going from **comoving gauge**, where the CMB temperature perturbation is initially negligible by the Poisson equation, to the **Newtonian** gauge involves a temporal shift

$$\frac{\delta t}{t} = \Psi$$

- Temporal shift implies a shift in the **scale factor** during matter domination

$$\frac{\delta a}{a} = \frac{2}{3} \frac{\delta t}{t}$$

- CMB temperature is **cooling** as $T \propto a$
- Induced **temperature fluctuation**

$$\frac{\delta T}{T} = -\frac{\delta a}{a} = -\frac{2}{3}\Psi$$

Gauge and the Sachs-Wolfe Effect

- Add the **gravitational redshift** a photon suffers climbing out of the gravitational potential

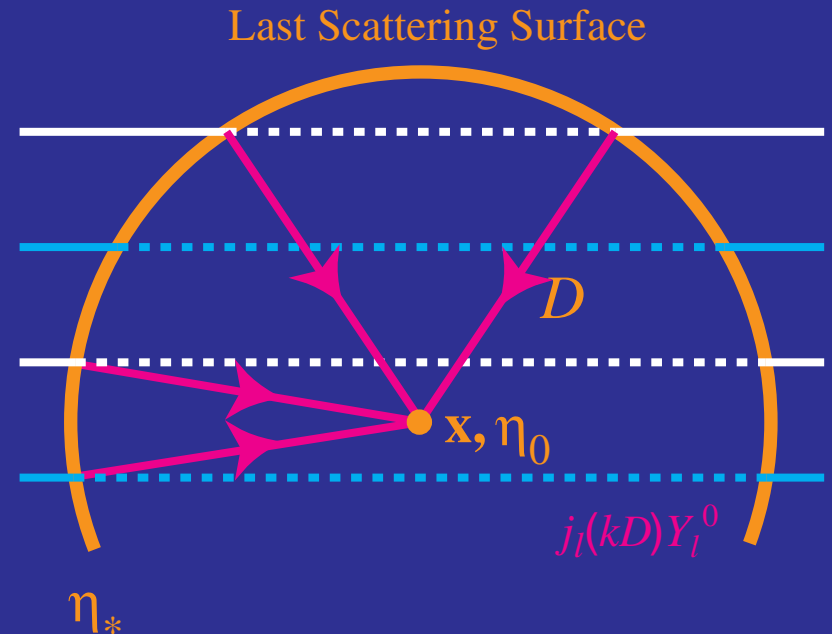
$$\left(\frac{\delta T}{T}\right)_{\text{obs}} = \Psi + \frac{\delta T}{T} = \frac{1}{3}\Psi$$

COBE Normalization

- Sachs-Wolfe Effect relates the COBE detection to the gravitational potential on the last scattering surface

$$[\Theta + \Psi](\hat{\mathbf{n}}, \mathbf{x}) = \frac{1}{3}\Psi(\mathbf{x} + D\hat{\mathbf{n}}, \eta_*)$$

$$D = \eta_0 - \eta$$



- Decompose the angular and spatial information into normal modes: spherical harmonics for angular, plane waves for spatial

$$G_\ell^m(\hat{\mathbf{n}}, \mathbf{x}, \mathbf{k}) = (-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} Y_\ell^m(\hat{\mathbf{n}}) e^{i\mathbf{k} \cdot \mathbf{x}}.$$

COBE Normalization *cont.*

- Multipole moment decomposition for each \mathbf{k}

$$\Theta(\mathbf{n}, \mathbf{x}) = \int \frac{d^3 k}{(2\pi)^3} \sum_{\ell m} \Theta_{\ell}^{(m)}(k) G_{\ell}^m(\mathbf{x}, \mathbf{k}, \mathbf{n})$$

- Power spectrum is the integral over \mathbf{k} modes

$$C_{\ell} = 4\pi \int \frac{d^3 k}{(2\pi)^3} \sum_m \frac{\langle \Theta_{\ell}^{(m)*} \Theta_{\ell}^{(m)} \rangle}{(2\ell + 1)^2}$$

- Fourier transform Sachs-Wolfe source

$$[\Theta + \Psi](\hat{\mathbf{n}}, \mathbf{x}) = \frac{1}{3} \int \frac{d^3 k}{(2\pi)^3} \Psi(k, \eta_*) e^{i\mathbf{k} \cdot (D \hat{\mathbf{n}} + \mathbf{x})}$$

- Decompose plane wave

$$\exp(i\mathbf{k} D \cdot \hat{\mathbf{n}}) = \sum_{\ell} (-i)^{\ell} \sqrt{4\pi(2\ell + 1)} j_{\ell}(kD) Y_{\ell}^0(\mathbf{n}),$$

COBE Normalization *cont.*

- Extract **multipole moment**, assume a constant **potential**

$$\begin{aligned}\frac{\Theta_{\ell}^{(0)}}{2\ell + 1} &= \frac{1}{3} \Psi(k, \eta_*) j_{\ell}(kD) \\ &= \frac{1}{3} \Psi(k, \eta_0) j_{\ell}(kD)\end{aligned}$$

- Construct angular **power spectrum**

$$C_{\ell} = 4\pi \int \frac{dk}{k} j_{\ell}^2(kD) \frac{1}{9} \Delta_{\Psi}^2$$

- For **scale invariant potential** ($n=1$), integral reduces to

$$\int_0^{\infty} \frac{dx}{x} j_{\ell}^2(x) = \frac{1}{2\ell(\ell + 1)}$$

- Log power spectrum = Log potential spectrum / 9

$$\frac{\ell(\ell + 1)}{2\pi} C_{\ell} = \frac{1}{9} \Delta_{\Psi}^2 \quad (n = 1)$$

COBE Normalization *cont.*

- Relate to **density fluctuations**: **Poisson** equation and **Friedmann** eqn.

$$\begin{aligned} k^2 \Psi &= -4\pi G a^2 \delta \rho \\ &= -\frac{3}{2} H_0^2 \Omega_m^2 \delta \end{aligned}$$

- Power spectra relation

$$\Delta_{\Psi}^2 = \frac{9}{4} \left(\frac{H_0}{k} \right)^4 \Omega_m^2 \Delta_{\delta}^2$$

- In terms of density fluctuation at **horizon** and **transfer function**

$$\Delta_{\delta}^2 \equiv \delta_H^2 \left(\frac{k}{H_0} \right)^{n+3} T^2(k)$$

- For scale invariant potential

$$\frac{\ell(\ell+1)}{2\pi} C_{\ell} = \frac{1}{4} \Omega_m^2 \delta_H^2 \quad (n=1)$$

COBE Normalization *cont.*

- Some numbers

$$\begin{aligned}\frac{\ell(\ell+1)}{2\pi}C_\ell &= \frac{1}{4}\Omega_m^2\delta_H^2 \quad (n=1) \\ &= \left(\frac{28\mu\text{K}}{2.726 \times 10^6\mu\text{K}}\right)^2 \approx 10^{-10}\end{aligned}$$

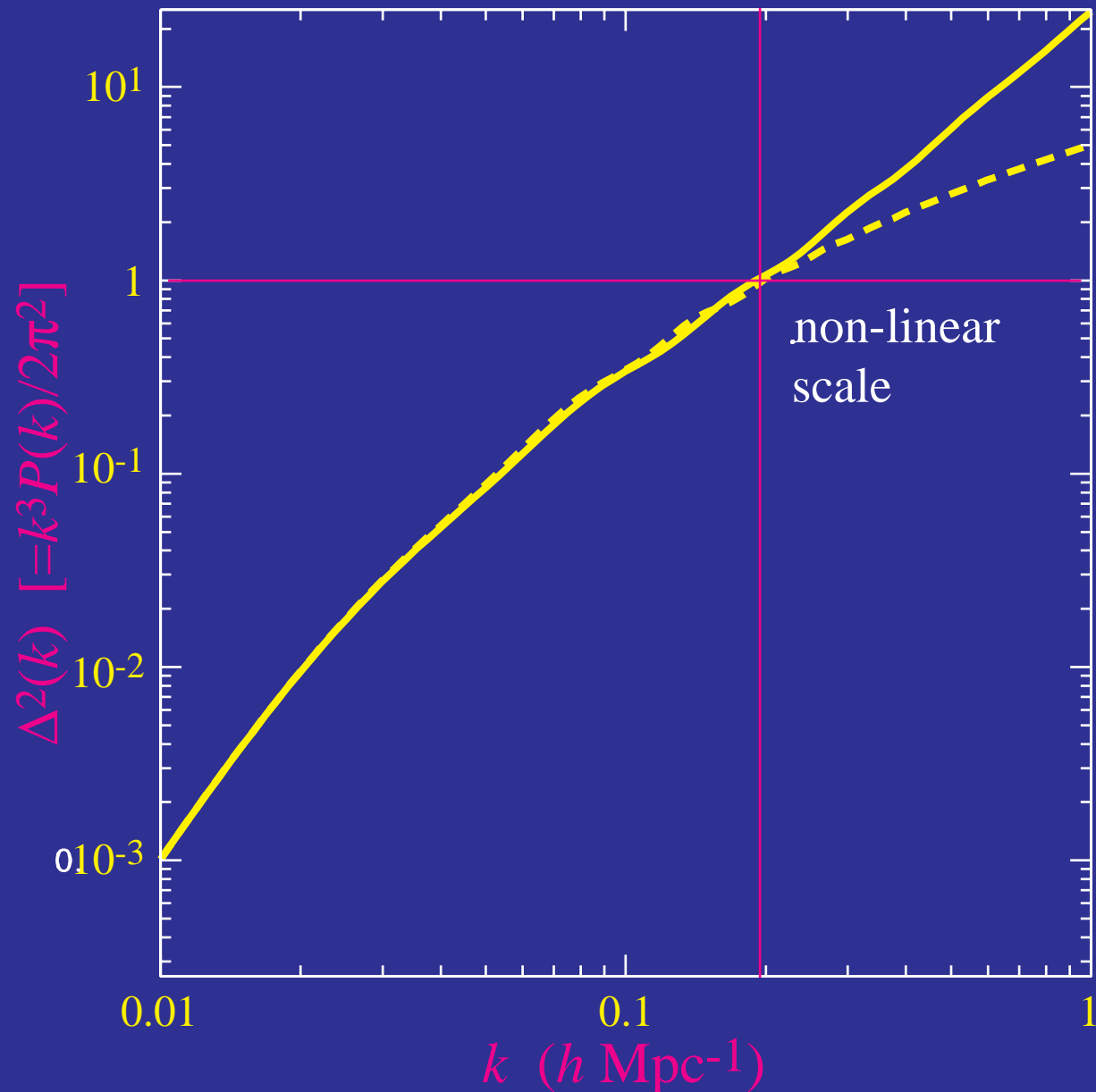
$$\delta_H \approx (2 \times 10^{-5})\Omega_m^{-1}$$

- Detailed calculation from [Bunn & White \(1997\)](#) including decay of potential in low density universe and tilt

$$\delta_H = 1.94 \times 10^{-5} \Omega_m^{-0.785-0.05 \ln \Omega_m} e^{-0.95(n-1)-0.169(n-1)^2}$$

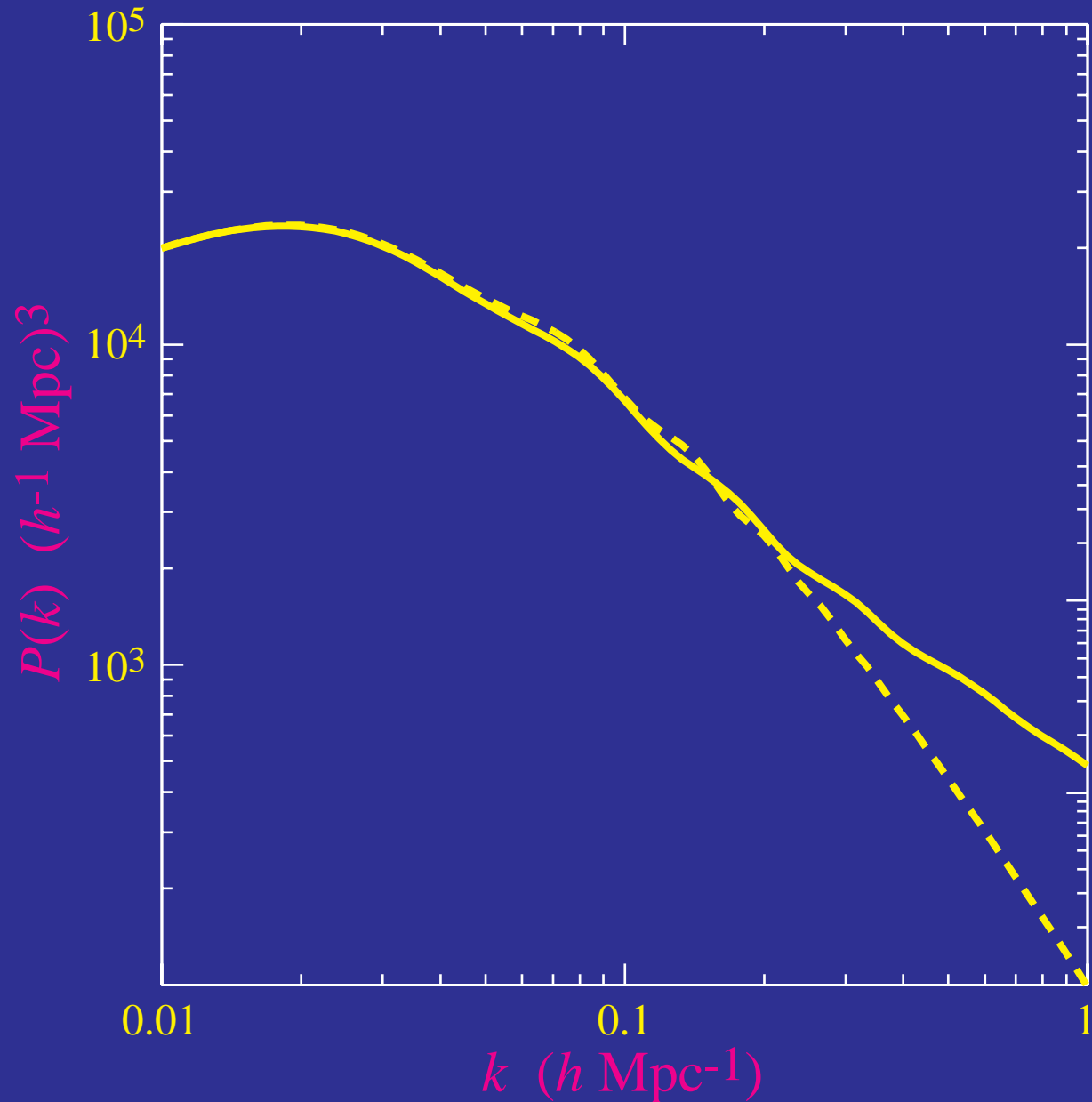
Matter Power Spectrum

- Combine the transfer function with the COBE normalization



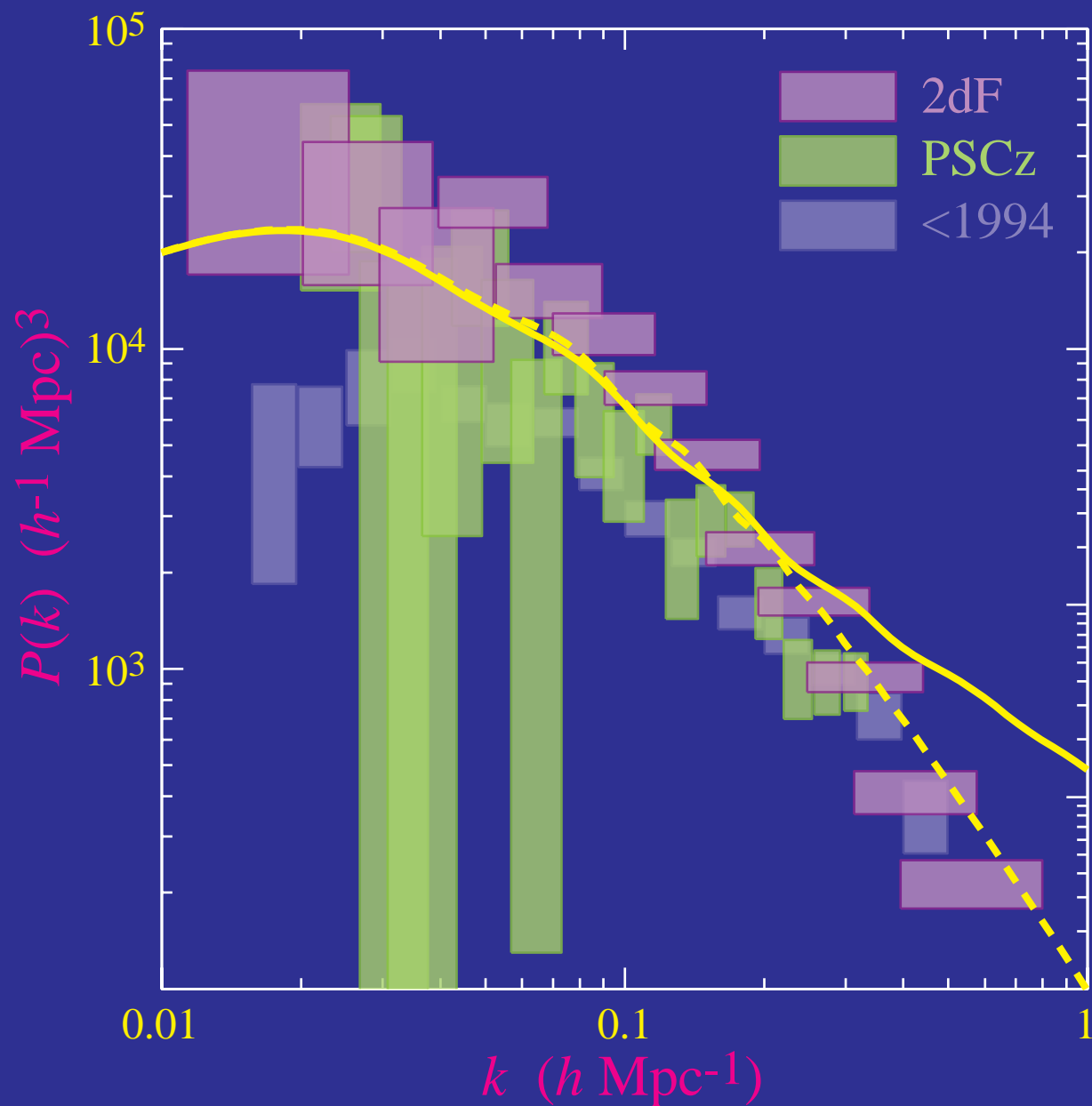
Matter Power Spectrum

- Usually plotted as the **power spectrum**, not log-power spectrum



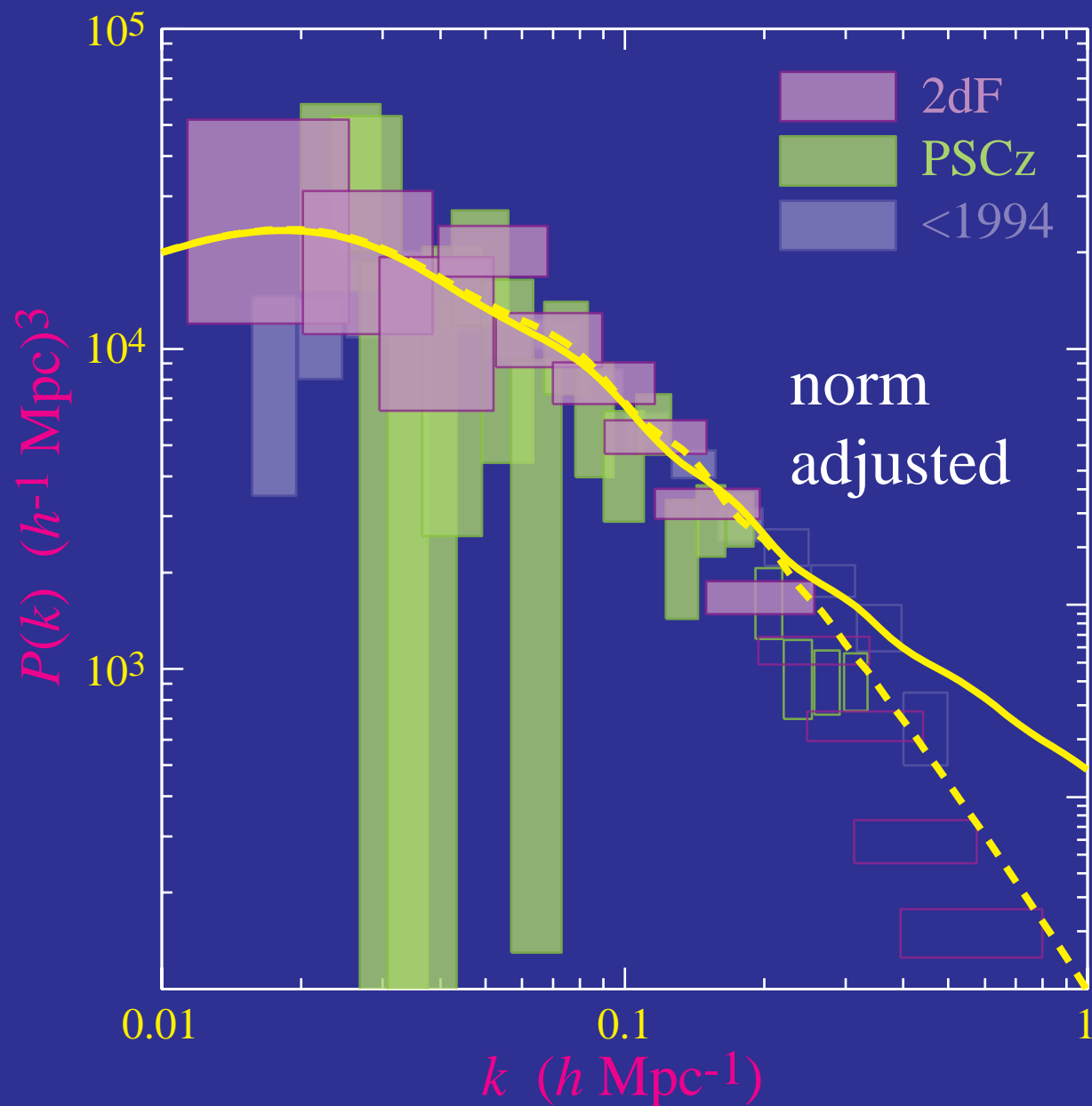
Galaxy Power Spectrum Data

- Galaxy clustering tracks the dark matter – but as a **biased** tracer



Galaxy Power Spectrum Data

- Each galaxy population has **different linear bias** in linear regime



Acoustic Oscillations

- **Example:** Stabilization accompanied by **acoustic oscillations**
- **Photon-baryon** system - under rapid scattering

$$\left[\frac{d}{d\eta} + 3\frac{\dot{a}}{a} \right] \delta\rho_{\gamma b} + 3\frac{\dot{a}}{a} \delta p_{\gamma b} = -(\rho_{\gamma b} + p_{\gamma b})(k v_{\gamma b} + 3\dot{\Phi}),$$
$$\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] [(\rho_{\gamma b} + p_{\gamma b})v_{\gamma b}/k] = \delta p_{\gamma b} + (\rho + p)\Psi,$$

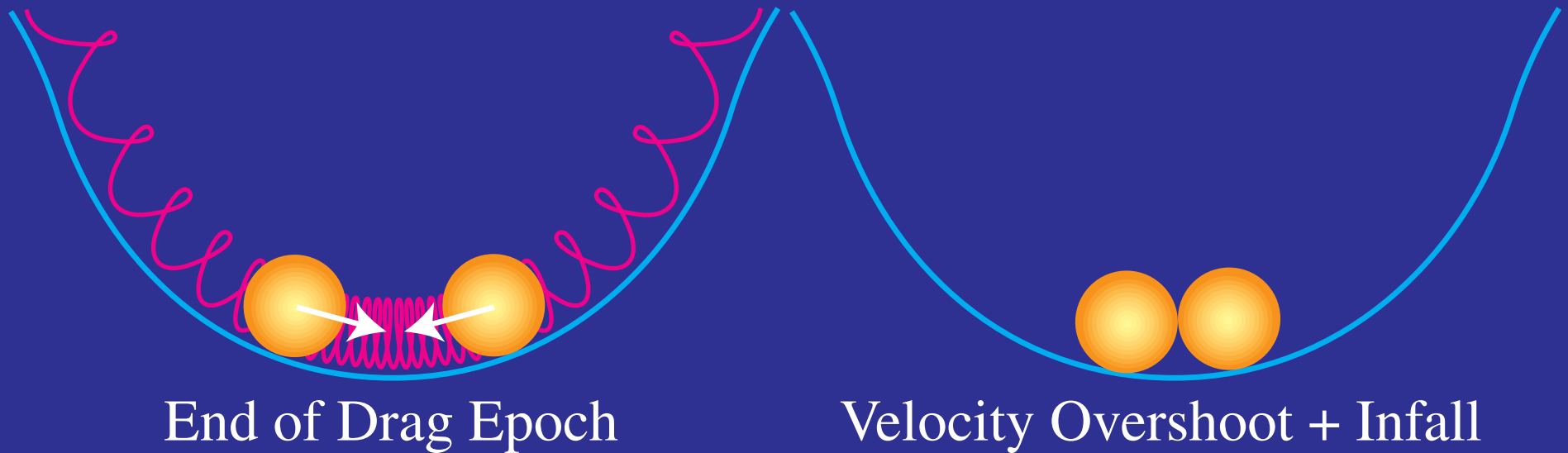
or with $\Theta = \delta\rho_{\gamma}/4\rho_{\gamma}$ and $R = 3\rho_b/4\rho_{\gamma}$

$$\dot{\Theta} = -\frac{k}{3}v_{\gamma b} - \dot{\Phi}$$
$$\dot{v}_{\gamma b} = -\frac{\dot{R}}{1+R}v_{\gamma b} + \frac{1}{1+R}k\Theta + k\Psi$$

- **Forced oscillator** equation – see Zaldarriaga's talks
- **Anisotropic stresses** and entropy generation through **non-adiabatic stress** Silk **damps** fluctuations **Silk 1968**

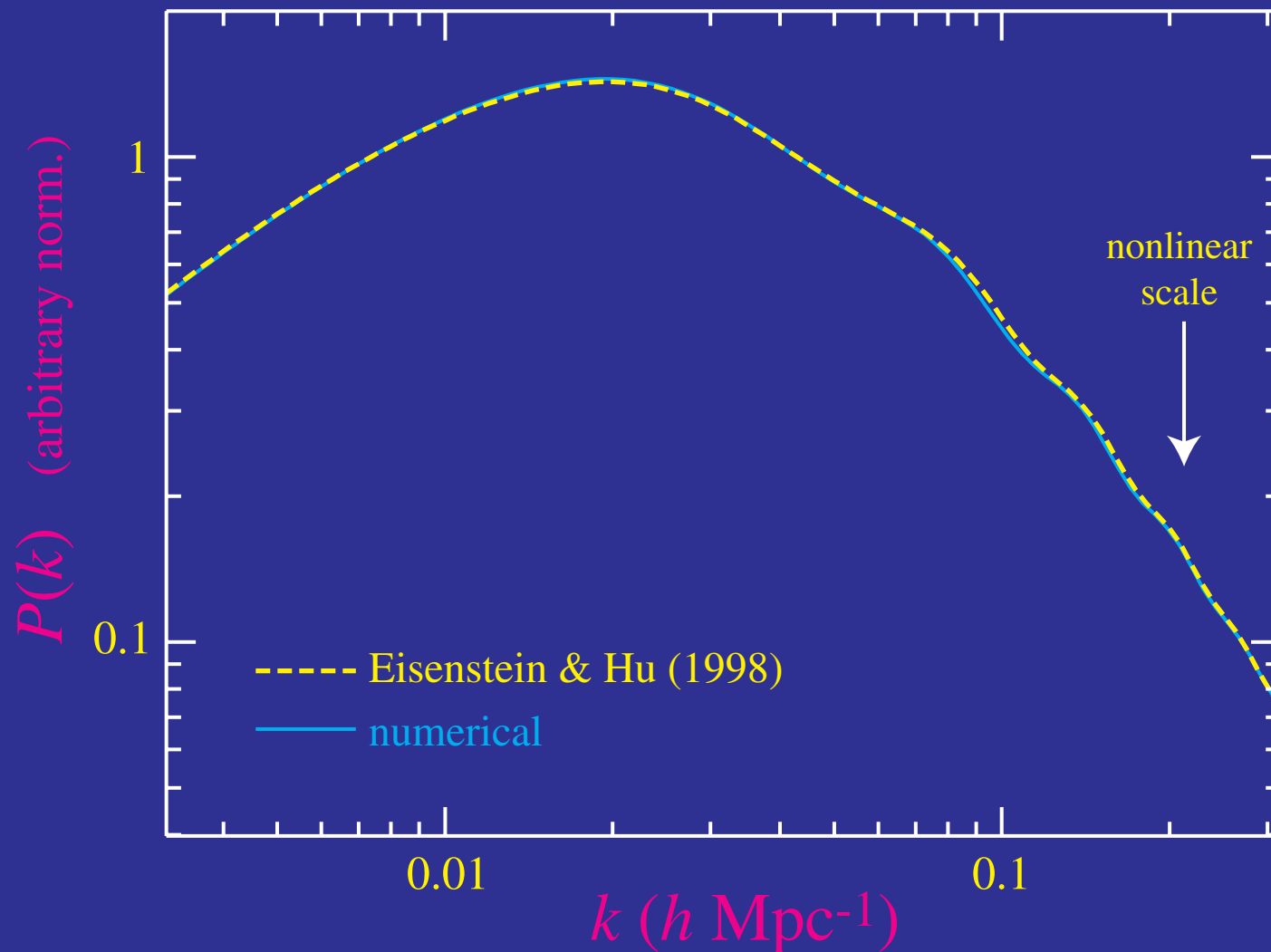
Acoustic Peaks in the Matter

- Baryon density & velocity oscillates with CMB
- Baryons decouple at $\tau/R \sim 1$, the end of Compton drag epoch
- Decoupling: $\delta_b(\text{drag}) \sim V_b(\text{drag})$, but not frozen
- Continuity: $\dot{\delta}_b = -kV_b$
- **Velocity Overshoot Dominates**: $\delta_b \sim V_b(\text{drag}) k\eta \gg \delta_b(\text{drag})$
- Oscillations $\pi/2$ out of phase with CMB
- **Infall** into potential wells (**DC component**)



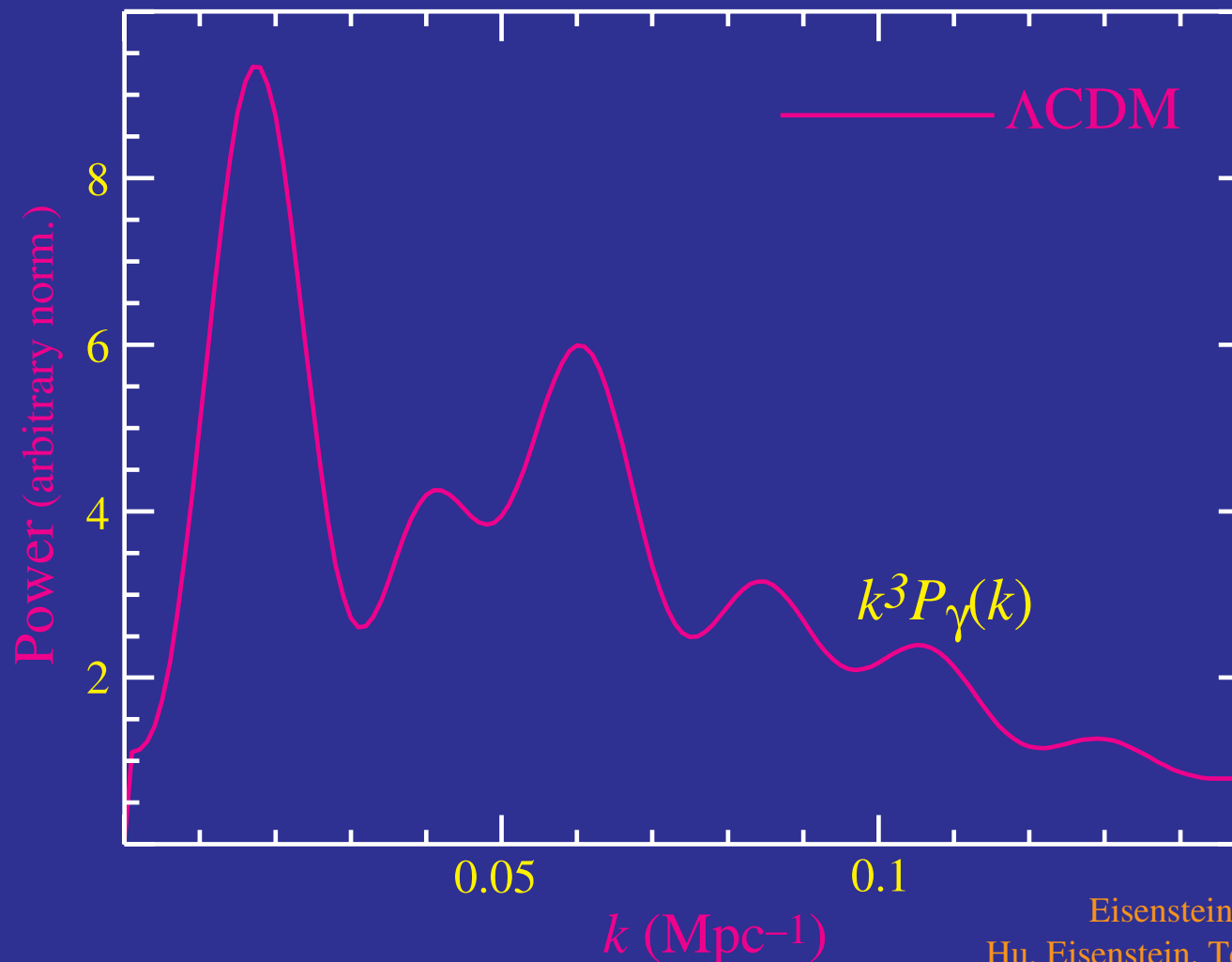
Features in the Power Spectrum

- **Features** in the linear power spectrum
- **Break** at sound horizon
- **Oscillations** at small scales; washed out by nonlinearities



Combining Features in LSS + CMB

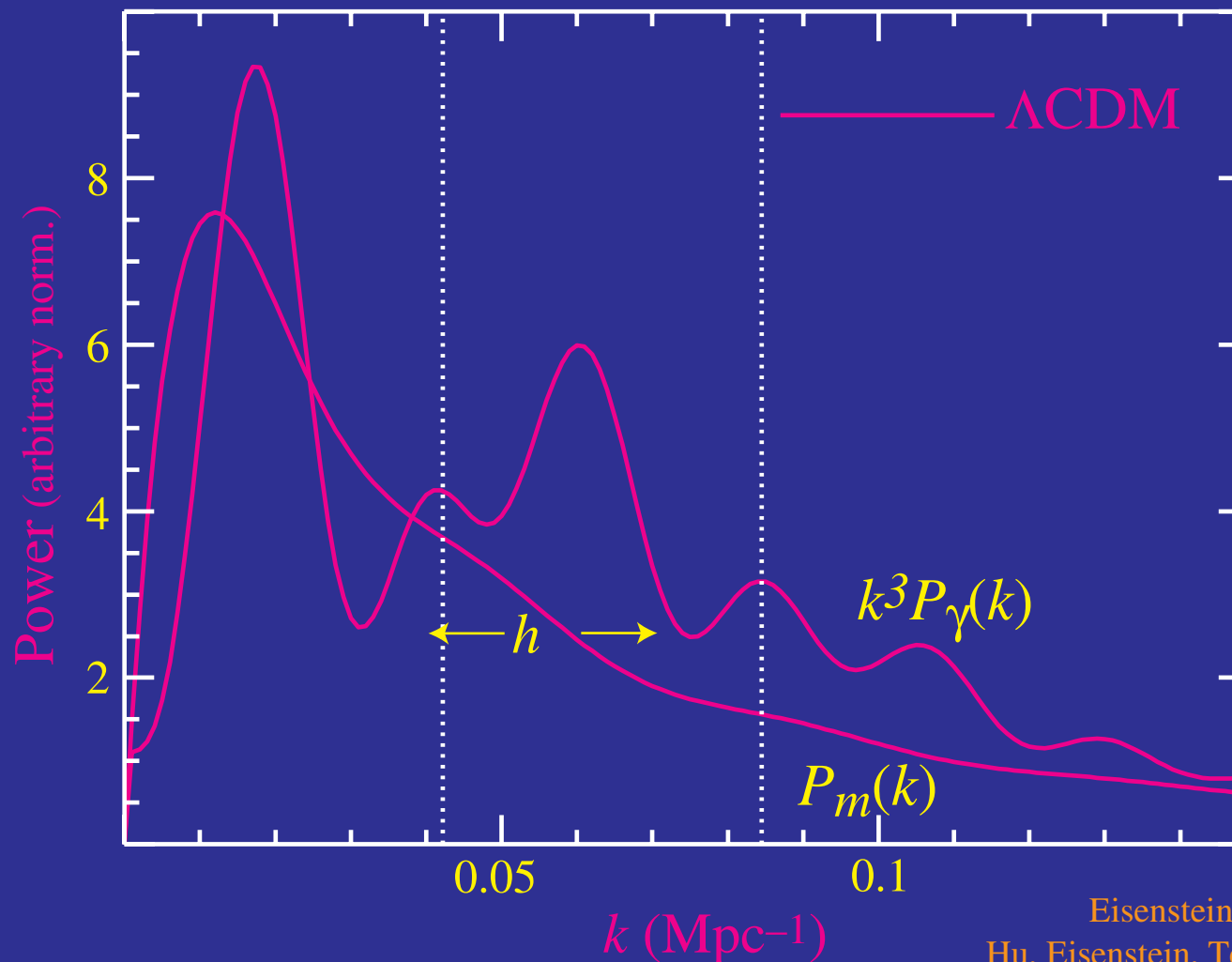
- Consistency check on thermal history and photon–baryon ratio
- Infer physical scale $l_{\text{peak}}(\text{CMB}) \rightarrow k_{\text{peak}}(\text{LSS})$ in Mpc^{-1}



Eisenstein, Hu & Tegmark (1998)
Hu, Eisenstein, Tegmark & White (1998)

Combining Features in LSS + CMB

- Consistency check on thermal history and photon–baryon ratio
- Infer physical scale $l_{\text{peak}}(\text{CMB}) \rightarrow k_{\text{peak}}(\text{LSS})$ in Mpc^{-1}
- Measure in redshift survey $k_{\text{peak}}(\text{LSS})$ in $h \text{ Mpc}^{-1} \rightarrow h$



Eisenstein, Hu & Tegmark (1998)
Hu, Eisenstein, Tegmark & White (1998)

Parameterizing Dark Components

Prototypes:

- Cold dark matter (WIMPs)
- Hot dark matter (light neutrinos)
- Cosmological constant (vacuum energy)

equation of state w_g sound speed c_{eff}^2 viscosity c_{vis}^2

0	0	0
	$1/3 \rightarrow 0$	
-1	arbitrary	arbitrary

Exotica:

- Quintessence (slowly-rolling scalar field)
- Decaying dark matter (massive neutrinos)
- Radiation backgrounds (rapidly-rolling scalar field, NBR)
- Ultra-light fuzzy dark mat.

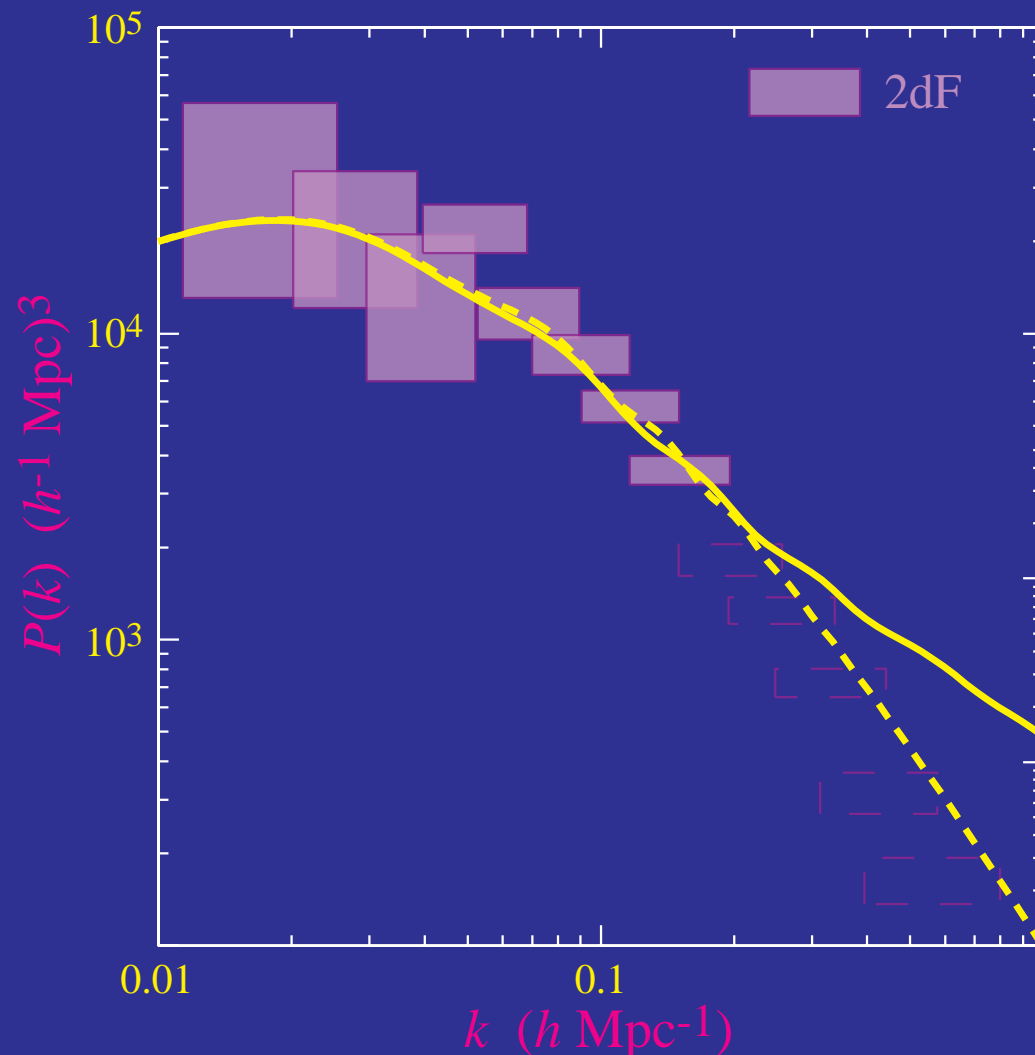
variable	1	0
	$1/3 \rightarrow 0 \rightarrow 1/3$	
$1/3$	$1/3$	$0 \rightarrow 1/3$
0	scale dependent	0

Massive Neutrinos

- Relativistic stresses of a light neutrino slow the growth of structure
- Neutrino species with cosmological abundance contribute to matter as $\Omega_\nu h^2 = m_\nu / 94 \text{eV}$, suppressing power as $\Delta P / P \approx -8\Omega_\nu / \Omega_m$

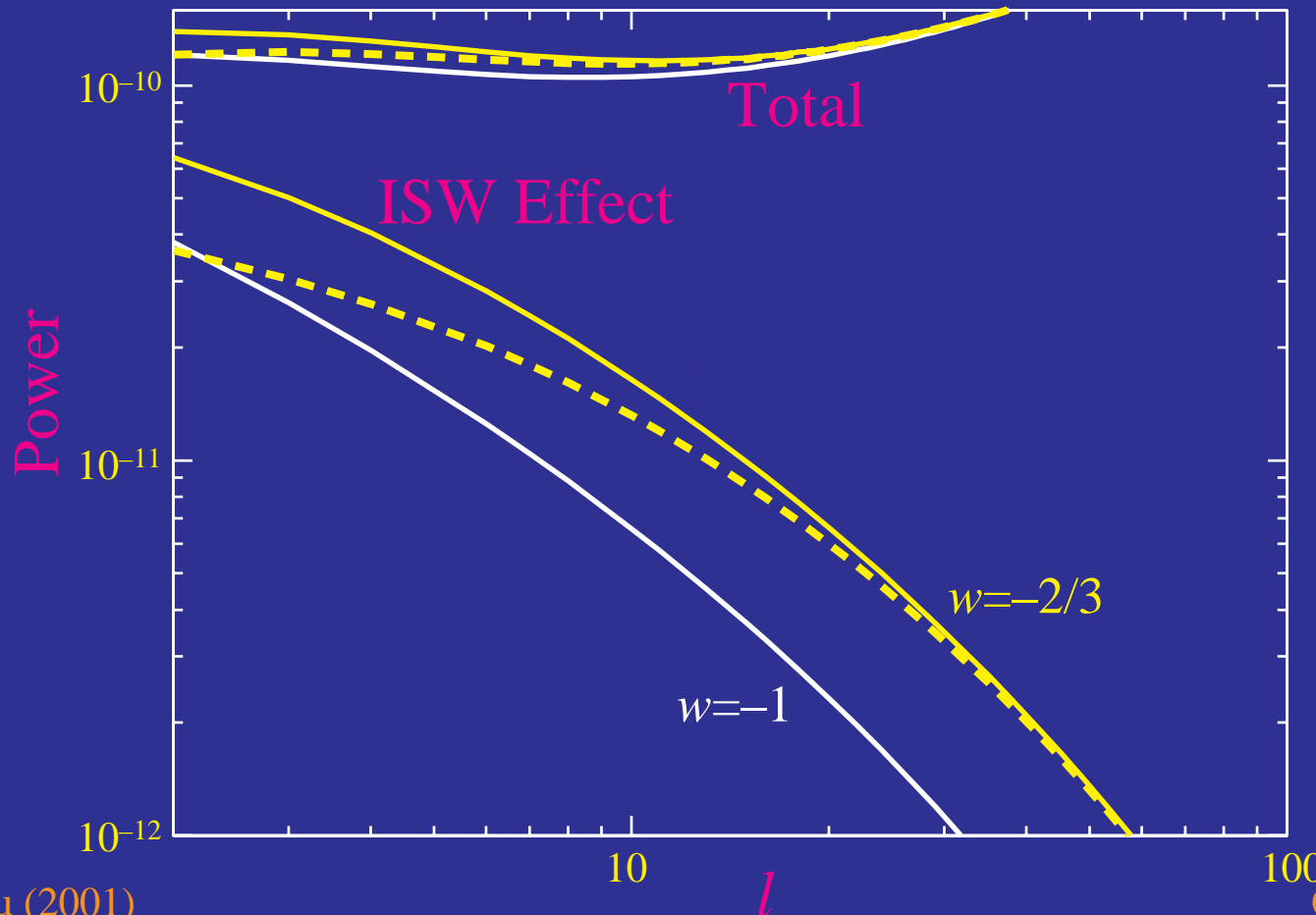
Massive Neutrinos

- Current data from 2dF galaxy survey indicates $m_\nu < 1.8 \text{ eV}$ assuming a ΛCDM model with parameters constrained by the CMB.



Dark Energy Stress & Smoothness

- Raising **equation of state** increases redshift of dark energy domination and **raises** large scale anisotropies
- Lowering the **sound speed** increases clustering and **reduces** ISW effect at large angles

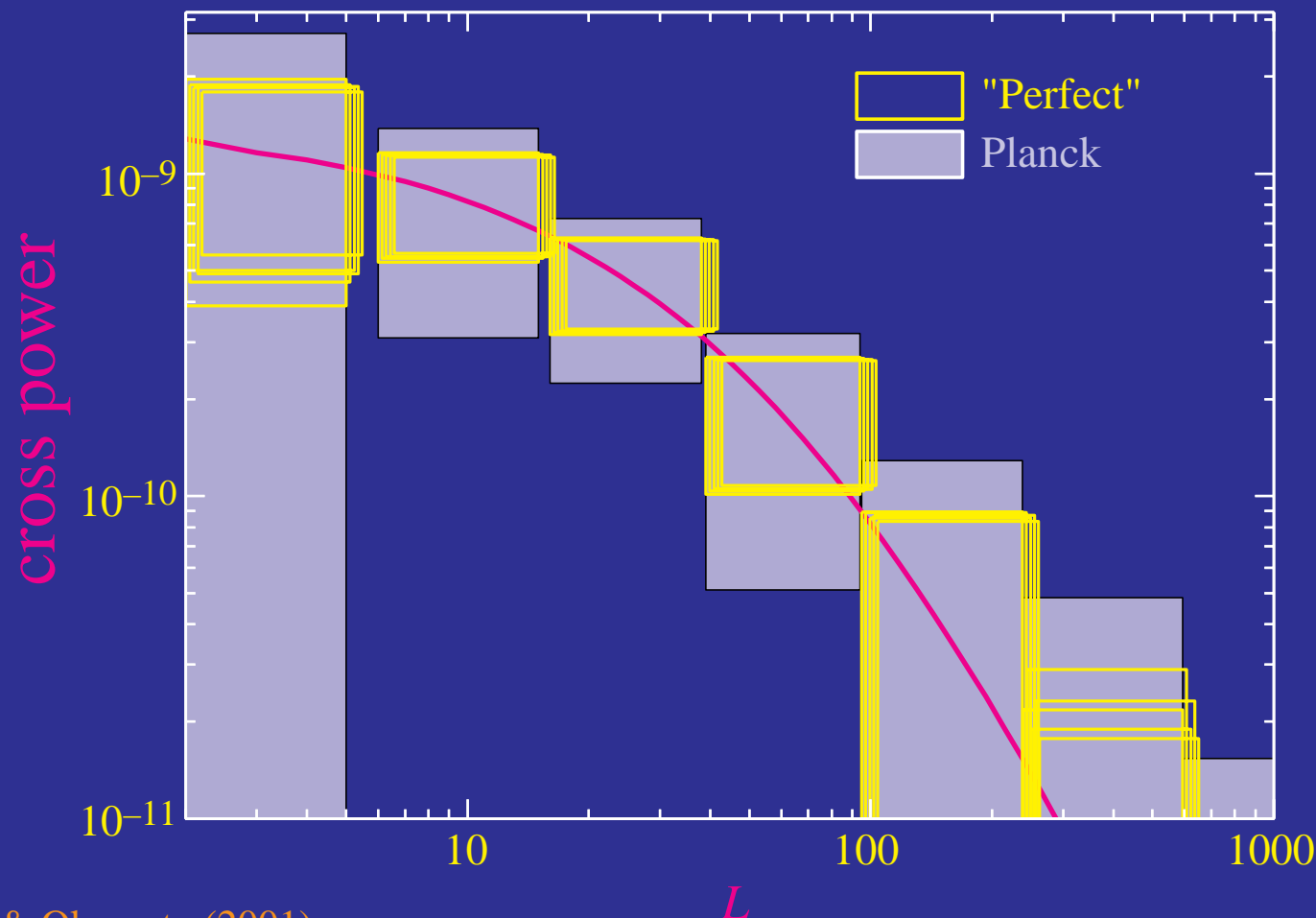


Hu (1998); Hu (2001)

Coble et al. (1997)
Caldwell et al. (1998)

Lensing–CMB Temperature Correlation

- Any correlation is a **direct detection** of a **smooth energy density** component through the **ISW effect**
- Show dark energy smooth **>5-6 Gpc** scale, **test quintessence**



Summary

- In linear theory, evolution of fluctuations is completely defined once the **stresses** in the matter fields are specified.
- Stresses and their effects take on simple forms in particular coordinate or **gauge choices**, e.g. the comoving gauge.
- Gauge **covariant equations** can be used to take advantage of these simplifications in an arbitrary frame.
- **Curvature** (potential) fluctuations remain **constant** in the absence of stresses.
- Evolution can be used to test the nature of the **dark components**, e.g. **massive neutrinos** and the **dark energy** by measuring the matter power spectrum.
- **Problem:** luminous tracers of the matter clustering are **biased** – next lecture.