Outline

- Spherical Collapse

- Mass Function

    Press-Schechter Formalism

    Extended Press-Schechter Formalism

    Halo Abundance

- Halo Bias

- Halo Profile

- Halo Model

    Density Field

    Baryonic Gas

    Galaxies

Closed Universe

- A spherical perturbation of radius r behaves as a closed universe

- Radius $r \propto a \rightarrow 0$, collapse in finite time

- Friedman equation in a closed universe

$$\frac{1}{a}\frac{da}{dt} = H_0 \left(\Omega_m a^{-3} + (1 - \Omega_m)a^{-2} \right)^{1/2}$$

- Parametric solution in terms of a development angle
$\theta = H_0 \eta (\Omega_m - 1)^{1/2}$, scaled conformal time η
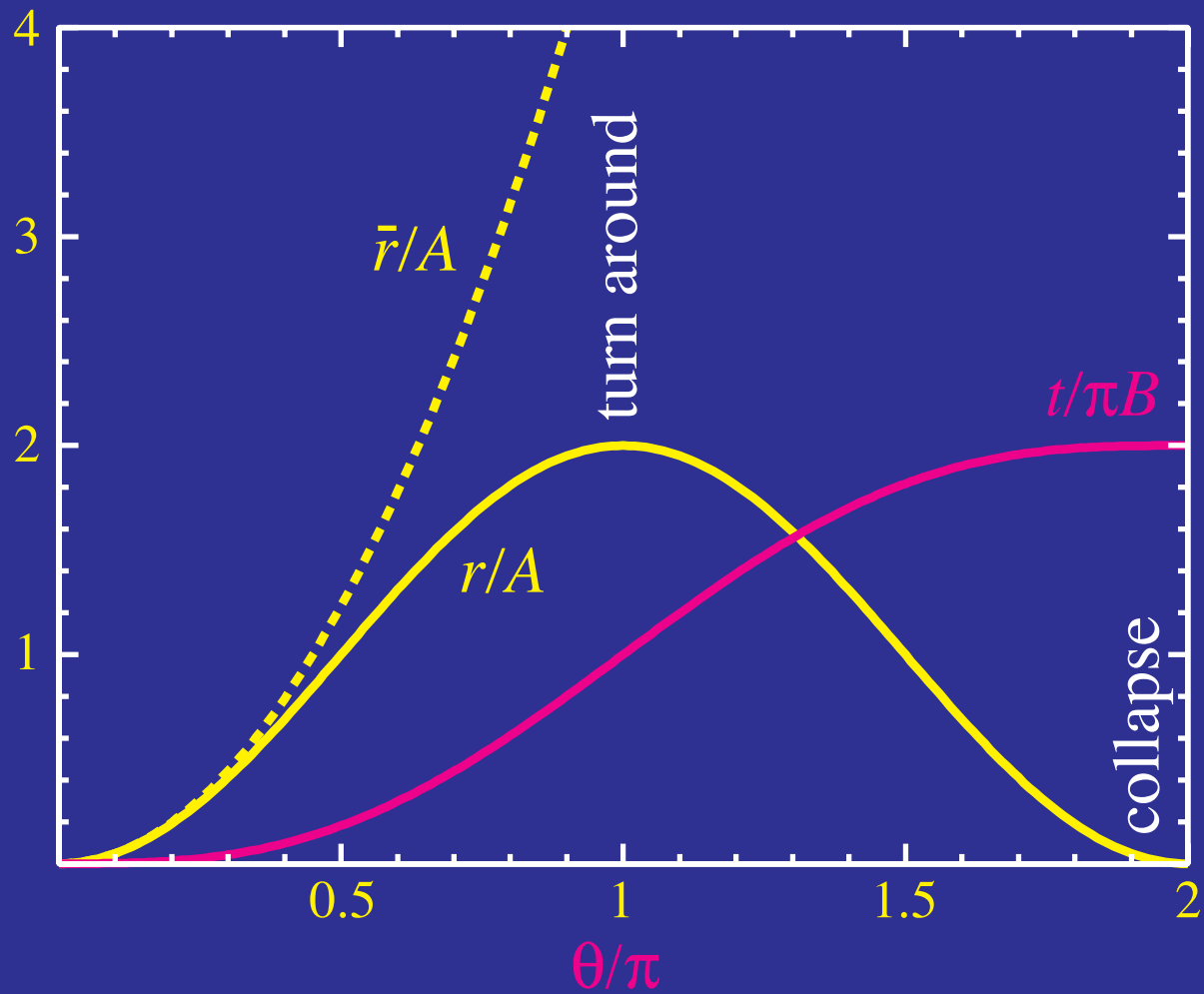
$$r(\theta) = A(1 - \cos\theta)$$
$$t(\theta) = B(\theta - \sin\theta)$$

where $A = r_0 \Omega_m / 2(\Omega_m - 1)$, $B = H_0^{-1}\Omega_m / 2(\Omega_m - 1)^{3/2}$.

- Turn around at $\theta = \pi$, $r = 2A$, $t = B\pi$.

- Collapse at $\theta = 2\pi$, $r \rightarrow 0$, $t = 2\pi B$

Spherical Collapse

- Parametric Solution:

Correspondence

- Eliminate cosmological correspondence in A and B in terms of enclosed mass M

$$M = \frac{4\pi}{3} r_0^3 \Omega_m \rho_c = \frac{4\pi}{3} r_0^3 \Omega_m \frac{3H_0^2}{8\pi G}$$

- Related as $A^3 = GMB^2$, and to initial perturbation

$$\lim_{\theta \to 0} r(\theta) = A\left(\frac{1}{2}\theta^2 - \frac{1}{4}\theta^4\right)$$

$$\lim_{\theta \to 0} t(\theta) = B\left(\frac{1}{6}\theta^3 - \frac{1}{120}\theta^5\right)$$

- Leading Order: $r = A\theta^2/2$, $t = B\theta^3/6$

$$r = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}$$

- Unperturbed matter dominated expansion $r \propto a \propto t^{2/3}$

Next Order

- Iterate r and t solutions

$$\lim_{\theta \to 0} t(\theta) = \frac{\theta^3}{6} B \left[1 - \frac{1}{20} \left(\frac{6t}{B} \right)^{2/3} \right]$$

$$\theta \approx \left(\frac{6t}{B} \right)^{1/3} \left[1 + \frac{1}{60} \left(\frac{6t}{B} \right)^{2/3} \right]$$

- Substitute back into $r(\theta)$

$$
\begin{aligned}
r(\theta) &= A \frac{\theta^2}{2} \left(1 - \frac{\theta^2}{12} \right) \\
&= A \left(\frac{6t}{B} \right)^{2/3} \left[1 - \frac{1}{20} \left(\frac{6t}{B} \right)^{2/3} \right] \\
&= (6t)^{2/3} (GM)^{1/3} \left[1 - \frac{1}{20} \left(\frac{6t}{B} \right)^{2/3} \right]
\end{aligned}
$$

Density Correspondence

- Density

$$\rho_m \ = \ \frac{M}{\frac{4}{3}\pi r^3}$$

$$= \ \frac{1}{6\pi t^2 G}\left[1 + \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}\right]$$

- Density perturbation

$$\delta \equiv \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m} \approx \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}$$

- Time $\rightarrow$ scale factor

$$t \ = \ \frac{2}{3H_0\Omega_m^{1/2}}a^{3/2}$$

$$\delta \ = \ \frac{3}{20}a\left(4BH_0\Omega_m^{1/2}\right)^{2/3}$$

Spherical Collapse Relations

- A and B constants $\rightarrow$ initial cond.

$$B = \frac{1}{2H_0\Omega_m^{1/2}}\left(\frac{3}{5}\frac{a_i}{\delta_i}\right)^{3/2}$$

$$A = \frac{3}{10}\frac{r_i}{\delta_i}$$

- Scale factor $a \propto t^{2/3}$

$$a = \left(\frac{3}{4}\right)^{2/3}\left(\frac{3}{5}\frac{a_i}{\delta_i}\right)(\theta - \sin\theta)^{2/3}$$

- At collapse $\theta = 2\pi$

$$a_{\mathrm{col}} = \left(\frac{3}{4}\right)^{2/3}\left(\frac{3}{5}\frac{a_i}{\delta_i}\right)(2\pi)^{2/3} \approx 1.686\frac{a_i}{\delta_i}$$

- Perturbation collapses when linear theory predicts $\delta_c \equiv 1.686$

Virialization

- A real density perturbation is neither spherical nor homogeneous

- Shell crossing if δ_i doesn't monotonically decrease

- Collapse does not proceed to a point but reaches virial equilibrium

$$
\begin{aligned}
U &= -2K \\
E &= U + K = U(r_{\mathrm{max}}) = \frac{1}{2}U(r_{\mathrm{vir}}) \\
r_{\mathrm{vir}} &= \frac{1}{2}r_{\mathrm{max}}
\end{aligned}
$$

since $U \propto r^{-1}$. Thus $\theta_{\mathrm{vir}} = \frac{3}{2}\pi$

- Overdensity at virialization

$$
\frac{\rho_m(\theta = 3\pi/2)}{\bar{\rho}_m(\theta = 2\pi)} = 18\pi^2 \approx 178
$$

- Threshold $\Delta_v = 178$ often used to define a collapsed object

Virialization

- Schematic Picture:
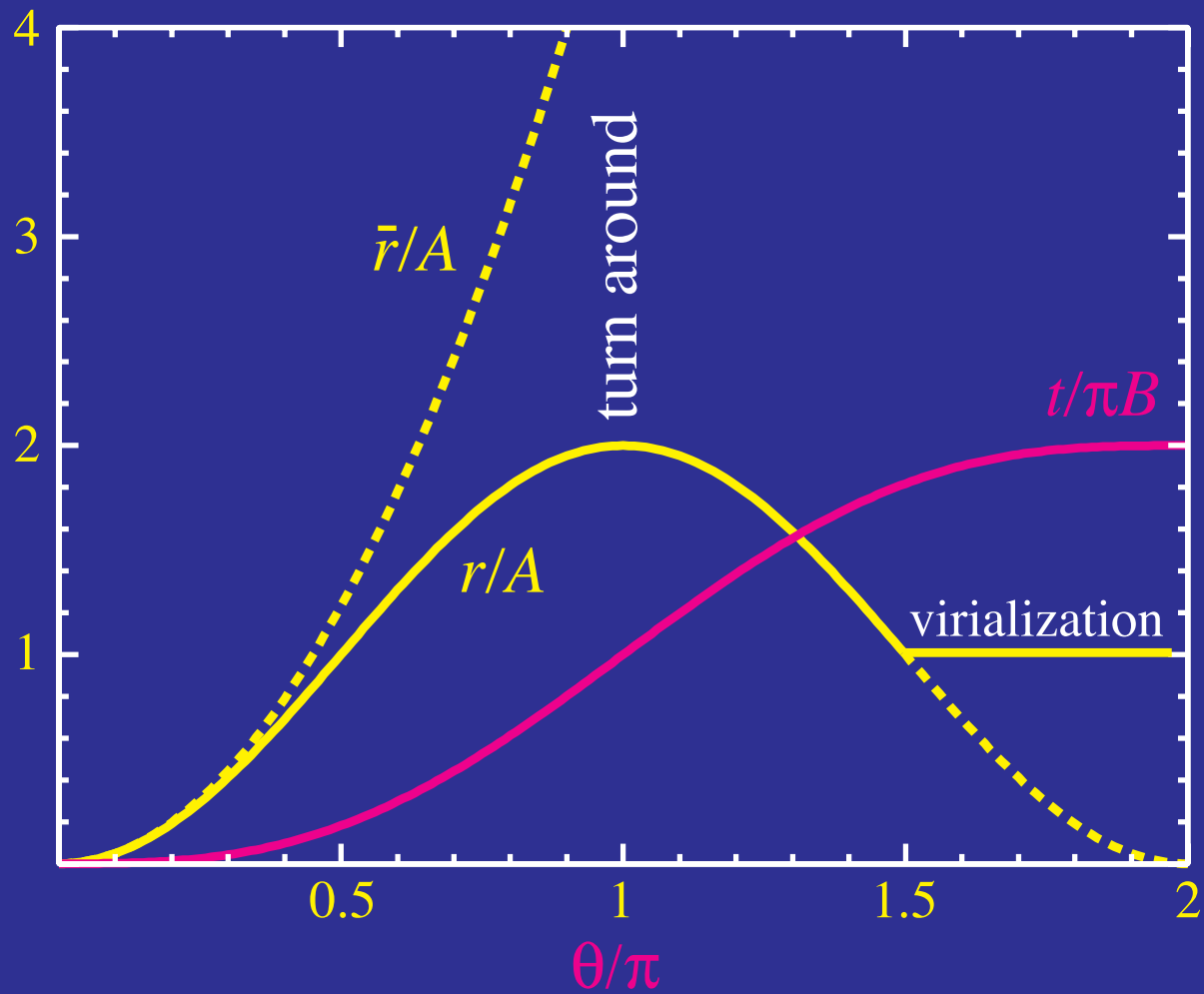
The Mass Function

- **Spherical collapse** predicts the end state as virialized **halos** given an initial density perturbation

- Initial density perturbation is a **Gaussian random field**

- Compare the variance in the linear density field to **threshold** $\delta_c = 1.686$ to determine collapse fraction

- Combine to form the **mass function**, the number density of halos in a range dM around M.

- Halo density defined entirely by linear theory

- Fudge the result to get the right answer compared with simulations (a la Press-Schechter)!

Press-Schechter Formalism

- Smooth linear density density field on mass scale M with tophat

$$R = \left(\frac{3M}{4\pi}\right)^{1/3}$$

- Result is a Gaussian random field with variance $\sigma^2(M)$

- Fluctuations above the threshold δ_c correspond to collapsed regions. The fraction in halos $> M$ becomes

$$\frac{1}{\sqrt{2\pi}\sigma(M)} \int_{\delta_c}^{\infty} d\delta \exp\left(-\frac{\delta^2}{2\sigma^2(M)}\right) = \frac{1}{2}\text{erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$

where $\nu \equiv \delta_c/\sigma(M)$

- Problem: even as $\sigma(M) \to \infty$, $\nu \to 0$, collapse fraction $\to 1/2$ – only overdense regions participate in spherical collapse.

- Multiply by an ad hoc factor of 2!

Press-Schechter Mass Function

- **Differentiate** in M to find fraction in range dM and multiply by ρ_m/M the number density of halos if all of the mass were composed of such halos $\rightarrow$ **differential number density** of halos

$$\begin{aligned}
\frac{dn}{d\ln M} &= \frac{\rho_m}{M}\frac{d}{d\ln M}\text{erfc}\left(\frac{\nu}{\sqrt{2}}\right) \\
&= \sqrt{\frac{2}{\pi}}\frac{\rho_m}{M}\frac{d\ln\sigma^{-1}}{d\ln M}\nu\exp(-\nu^2/2)
\end{aligned}$$

- High mass: **exponential cut off** above M_* where $\sigma(M_*) = \delta_c$

$$M_* \sim 10^{13}h^{-1}M_\odot \quad \text{today}$$

- Low mass **divergence**: (**too many** for the observations?)

$$\frac{dn}{d\ln M} \propto\sim M^{-1}$$

Observational Mass Functions

- SDSS optically identified clusters (assuming M/L; Bahcall et al 2002)
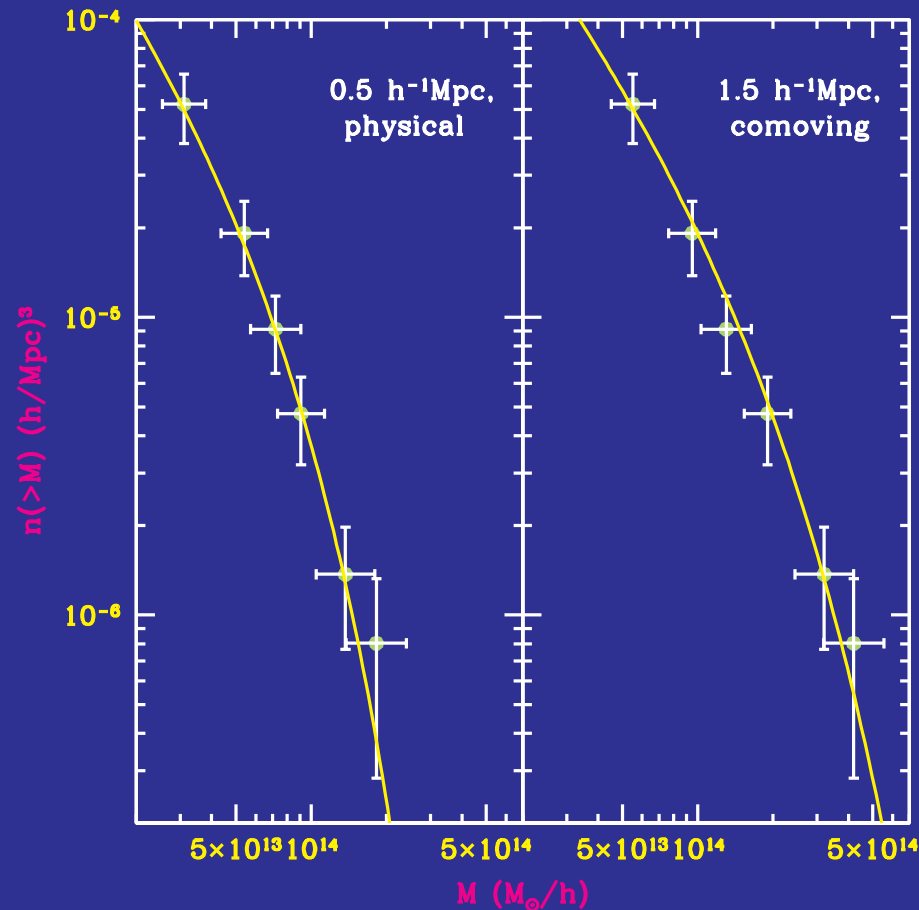
with cluster X-ray temp. function, sensitive to power amplitude σ_8^2

Counting Halos → Dark Energy

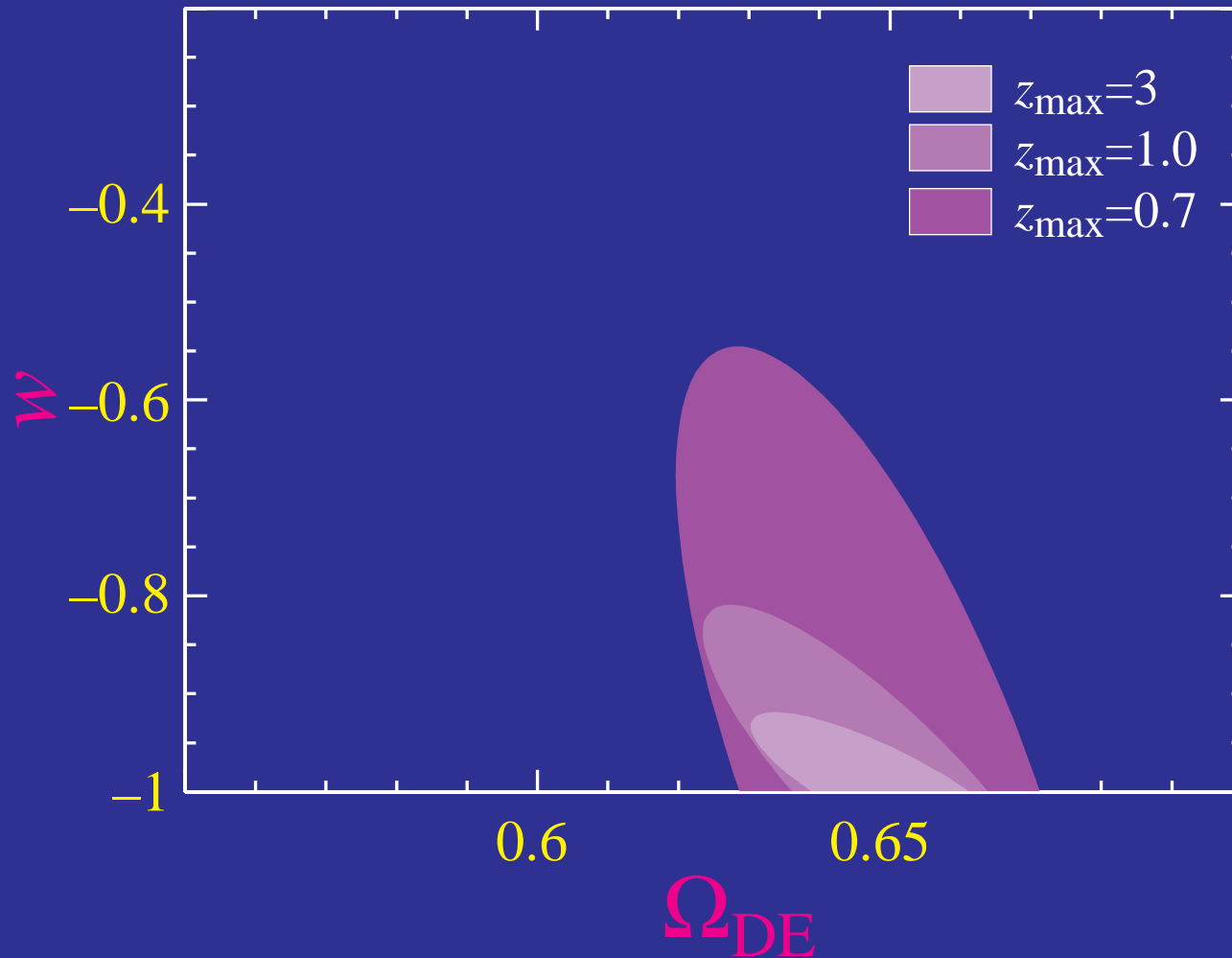- Halo abundance exponentially sensitive to growth rate

Projected Constraints

- Studies of $M > 2.5 \times 10^{14} \, M_\odot$ (Haiman et al. 2000; Hu & Kravstov)
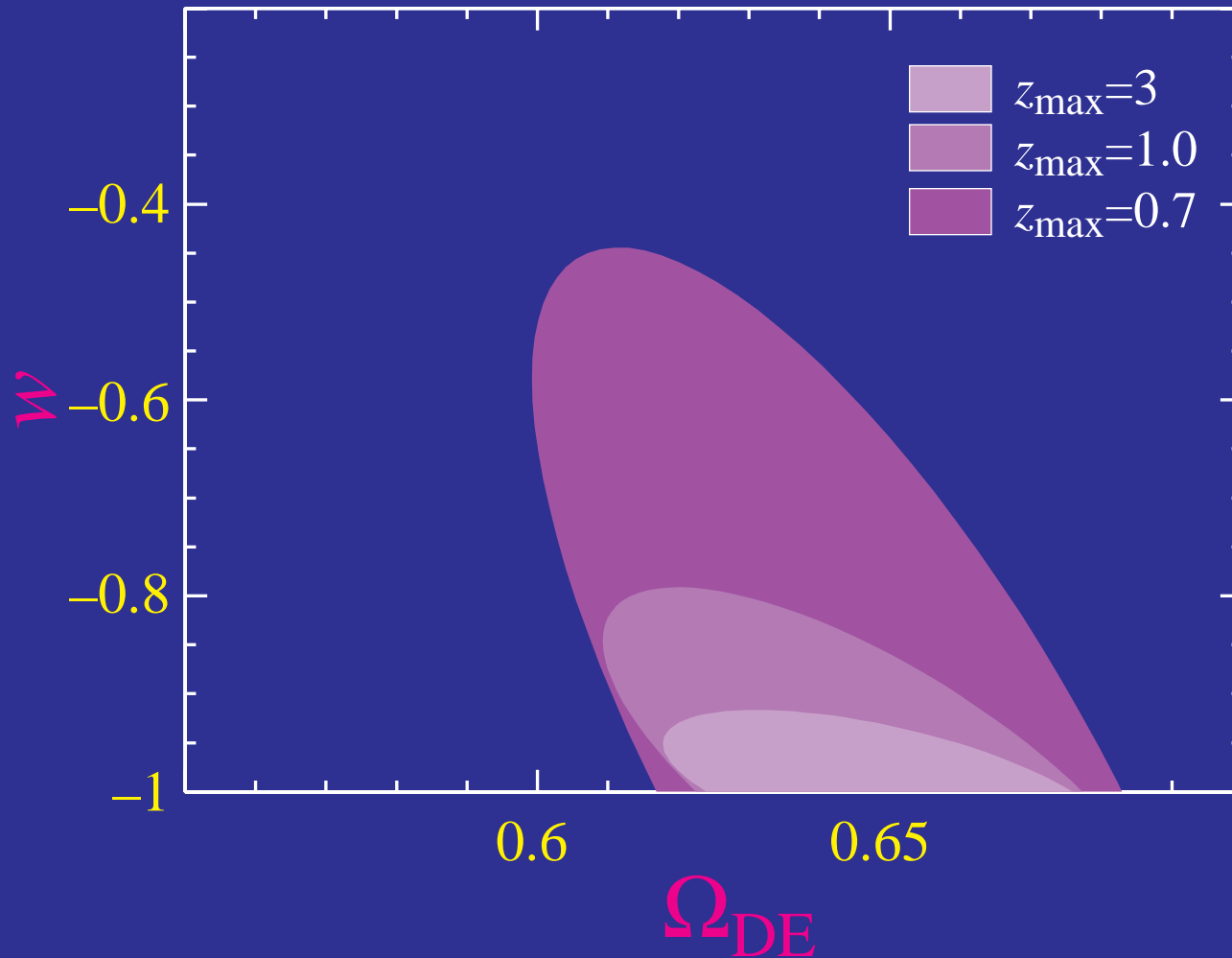
- All other parameters known
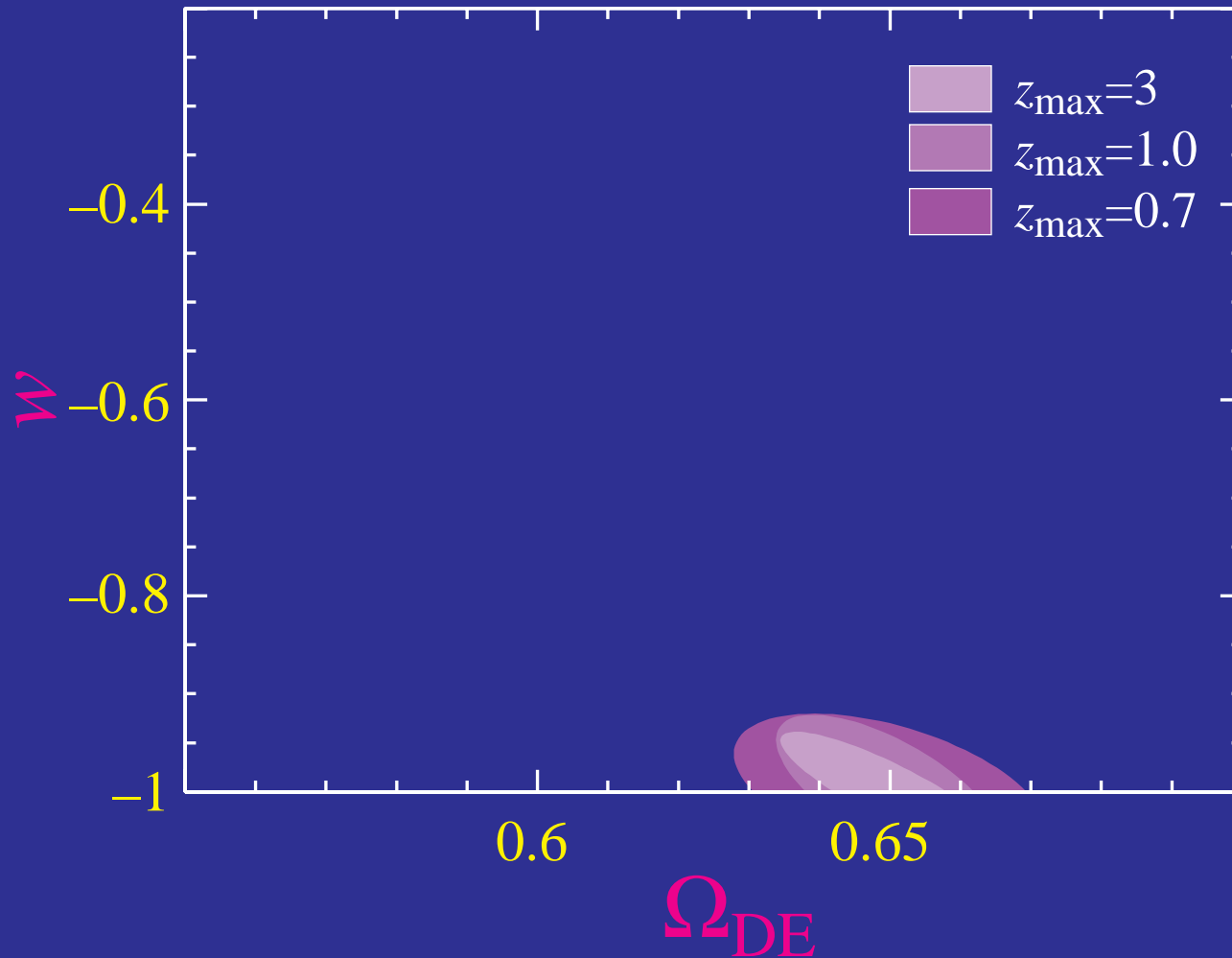
Projected Constraints

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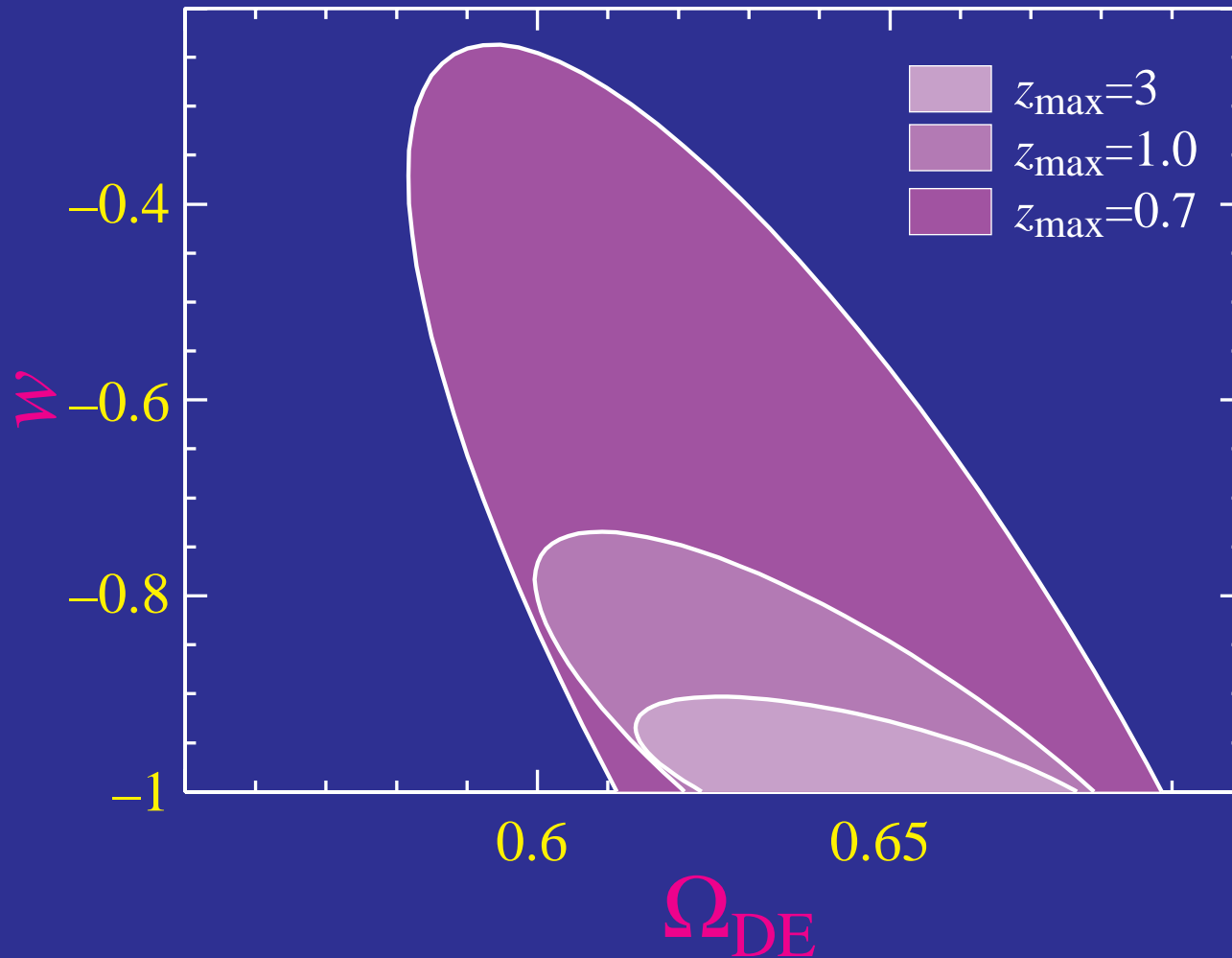- Local halo abundance known + present day cosmological params

Projected Constraints

- Studies of $M > 2.5 \times 10^{14}\,M_{\odot}$ (Haiman et al. 2000; Hu & Kravstov)

- Present day cosmological parameters

Projected Constraints

- Studies of $M > 2.5 \times 10^{14}\ M_\odot$ (Haiman et al. 2000; Hu & Kravstov)

- Present day cosmological parameters + sample variance

Extended Press-Schechter Formalism

- A region that is underdense when smoothed on the scale M may be overdense on a scale of a larger M

- If smoothing is a tophat in k-space, independence of k-modes implies fluctuation executes a random walk
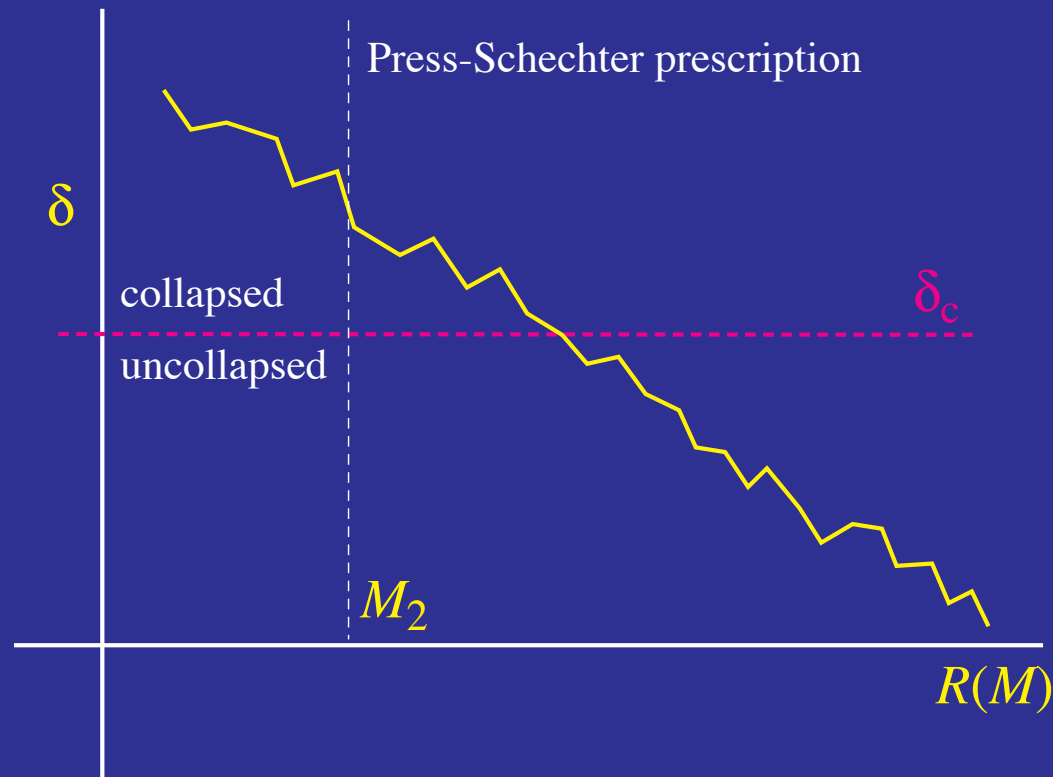
Extended Press-Schechter Formalism

- For each trajectory that lies above threshold at M_2, there is an equivalent trajectory that is its mirror image reflected around δ_c

- Press-Schechter ignored this branch. It supplies the missing factor of 2
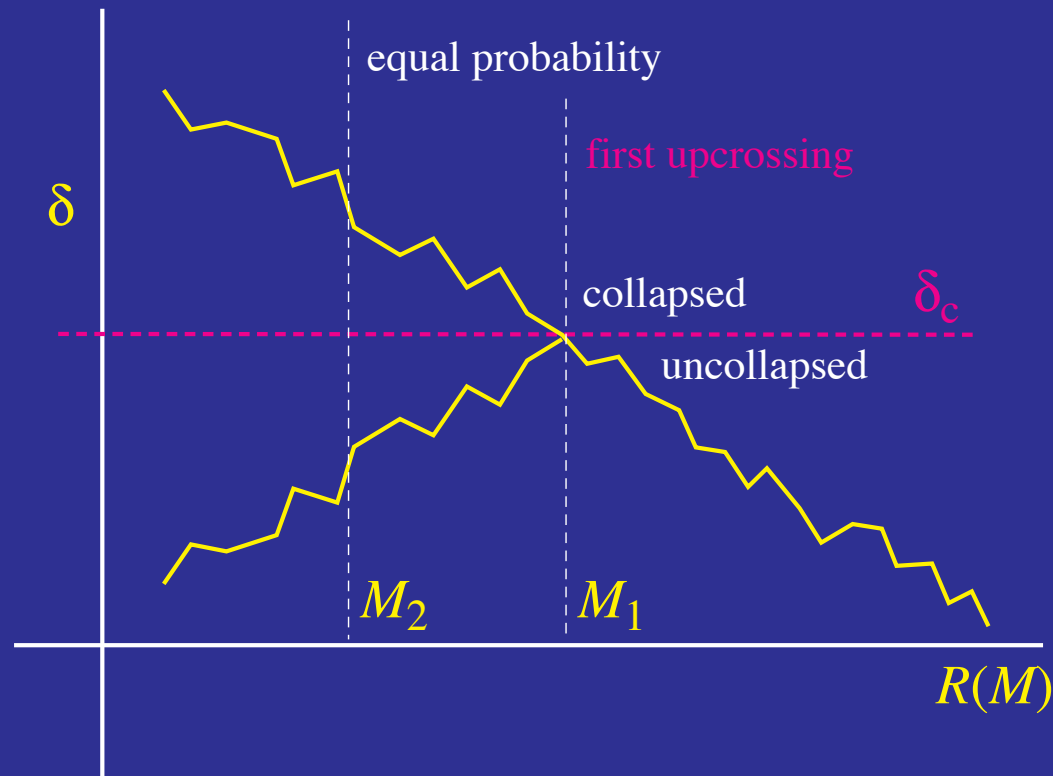
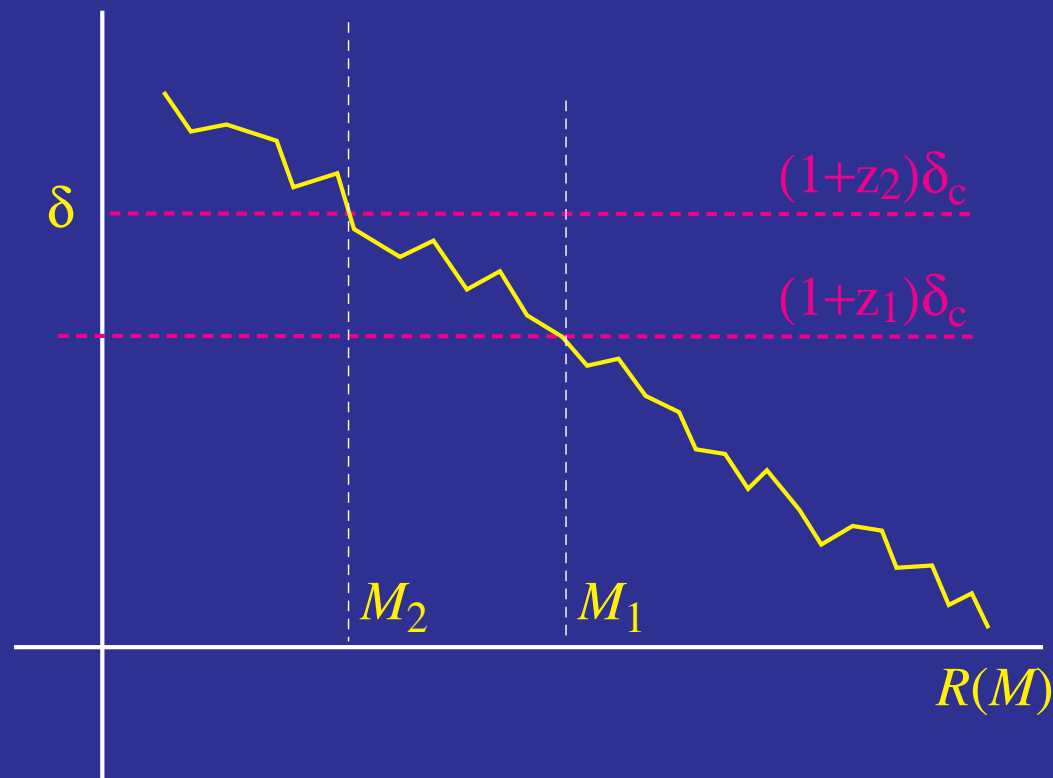# Conditional Mass Function

- Extended Press-Schechter also gives the conditional mass function, useful for merger histories.

- Given a halo of mass M_1 exists at z_1, what is the probability that it was part of a halo of mass M_2 at z_2
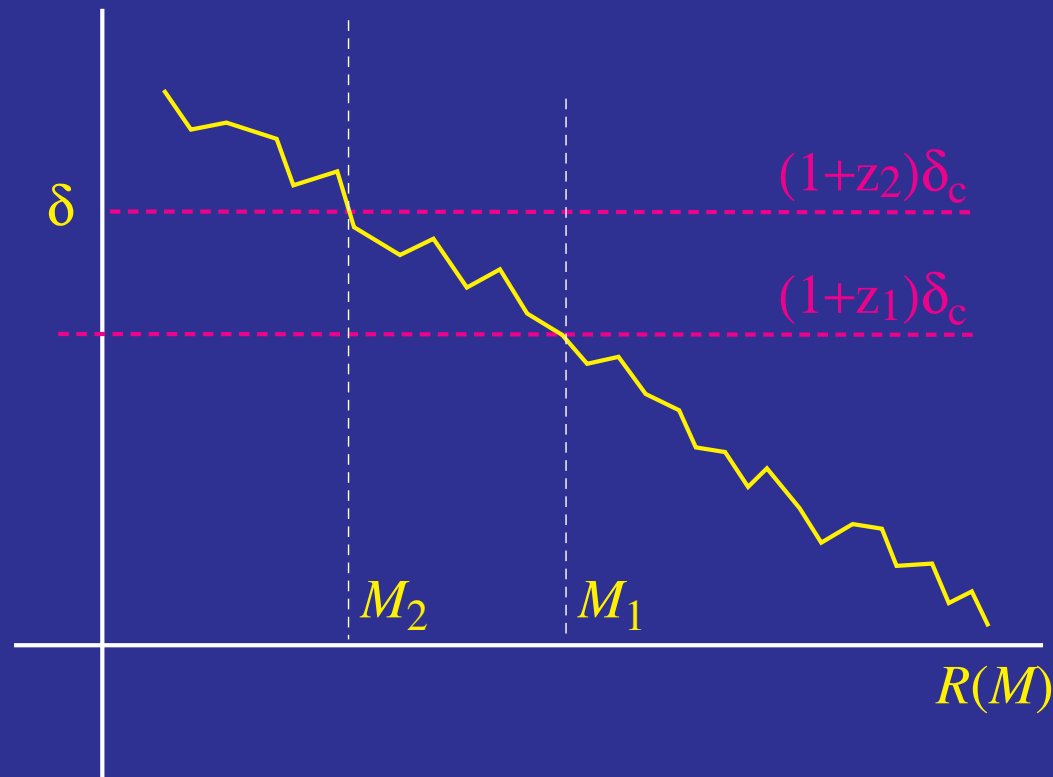
Conditional Mass Function

- Same as before but with the origin translated.

- Conditional mass function is mass function with δ_c and $\sigma^2(M)$ shifted

Merger Simulation

- Simulation by Andrey Kravstov

Magic "2" resolved?

- Spherical collapse is defined for a real-space not k-space smoothing. Random walk is only a qualitative explanation.

- Modern approach: think of spherical collapse as motivating a fitting form for the mass function

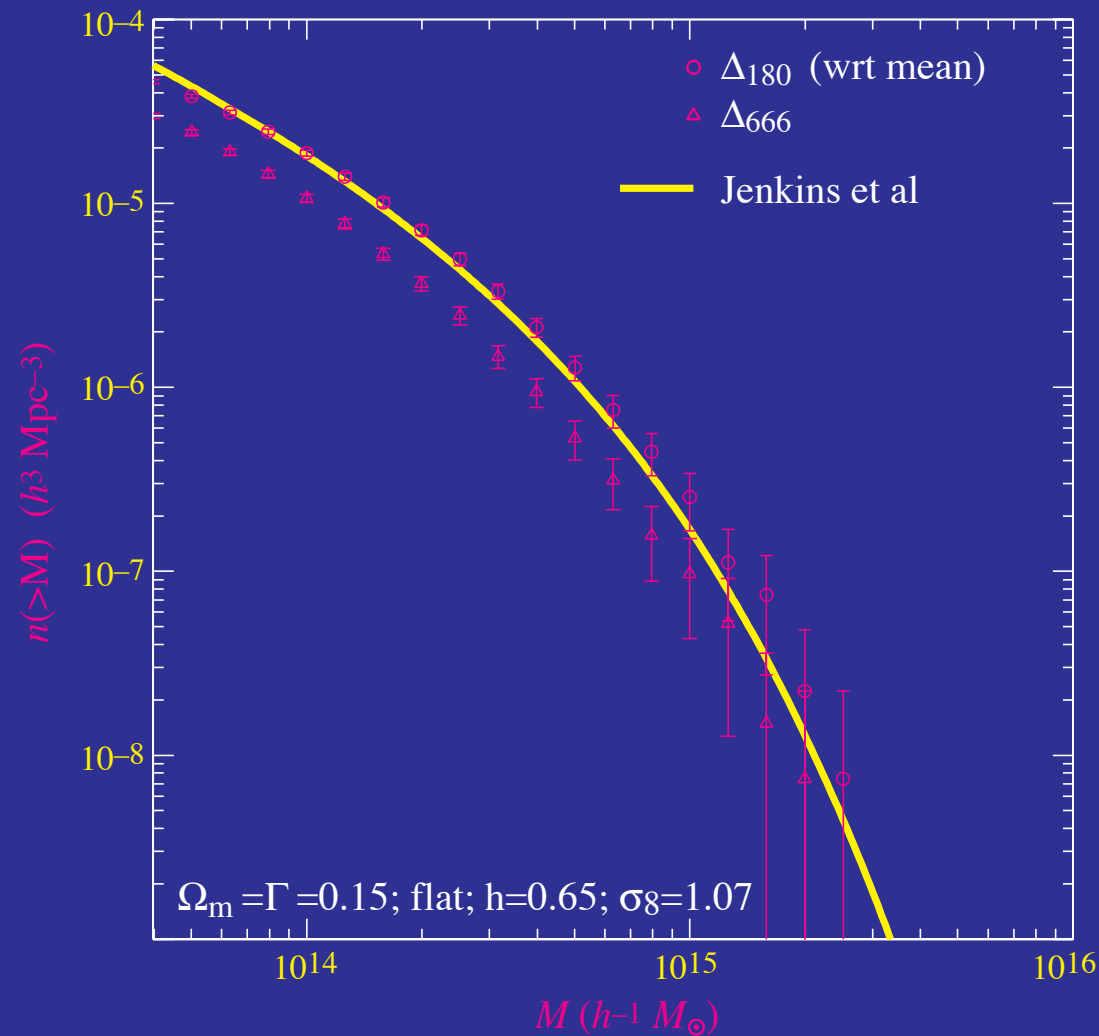$$\nu \exp(-\nu^2/2) \rightarrow A[1 + (a\nu^2)^{-p}]\sqrt{a\nu^2}\exp(-a\nu^2/2)$$

Sheth-Torman 1999, $a = 0.75$, $p = 0.3$. or a completely empirical fitting

$$\frac{dn}{d\ln M} = 0.301\frac{\rho_m}{M}\frac{d\ln\sigma^{-1}}{d\ln M}\exp[-|\ln\sigma^{-1} + 0.64|^{3.82}]$$

Jenkins et al 2001. Choice is tied up with the question: what is the mass of a halo?

Numerical Mass Function

- Example of difference in mass definition (from Hu & Kravstov 2002)

Plot with axes $n(>M)$ $(h^3 \, Mpc^{-3})$ versus M $(h^{-1} \, M_\odot)$.

Legend:
- Δ_{180} (wrt mean)
- Δ_{666}
- Jenkins et al

$\Omega_m = \Gamma = 0.15$; flat; h=0.65; $\sigma_8 = 1.07$

Halo Bias

- If halos are formed without regard to the underlying density fluctuation and move under the gravitational field then their number density is an unbiased tracer of the dark matter density fluctuation

$$\left(\frac{\delta n}{n}\right)_{\text{halo}} = \left(\frac{\delta \rho}{\rho}\right)$$

- However spherical collapse says the probability of forming a halo depends on the initial density field

- Large scale density field acts as "background" enhancement of probability of forming a halo or "peak"

- Peak-Background Split (Mo & White 1997)

Peak-Background Split

- Schematic Picture:
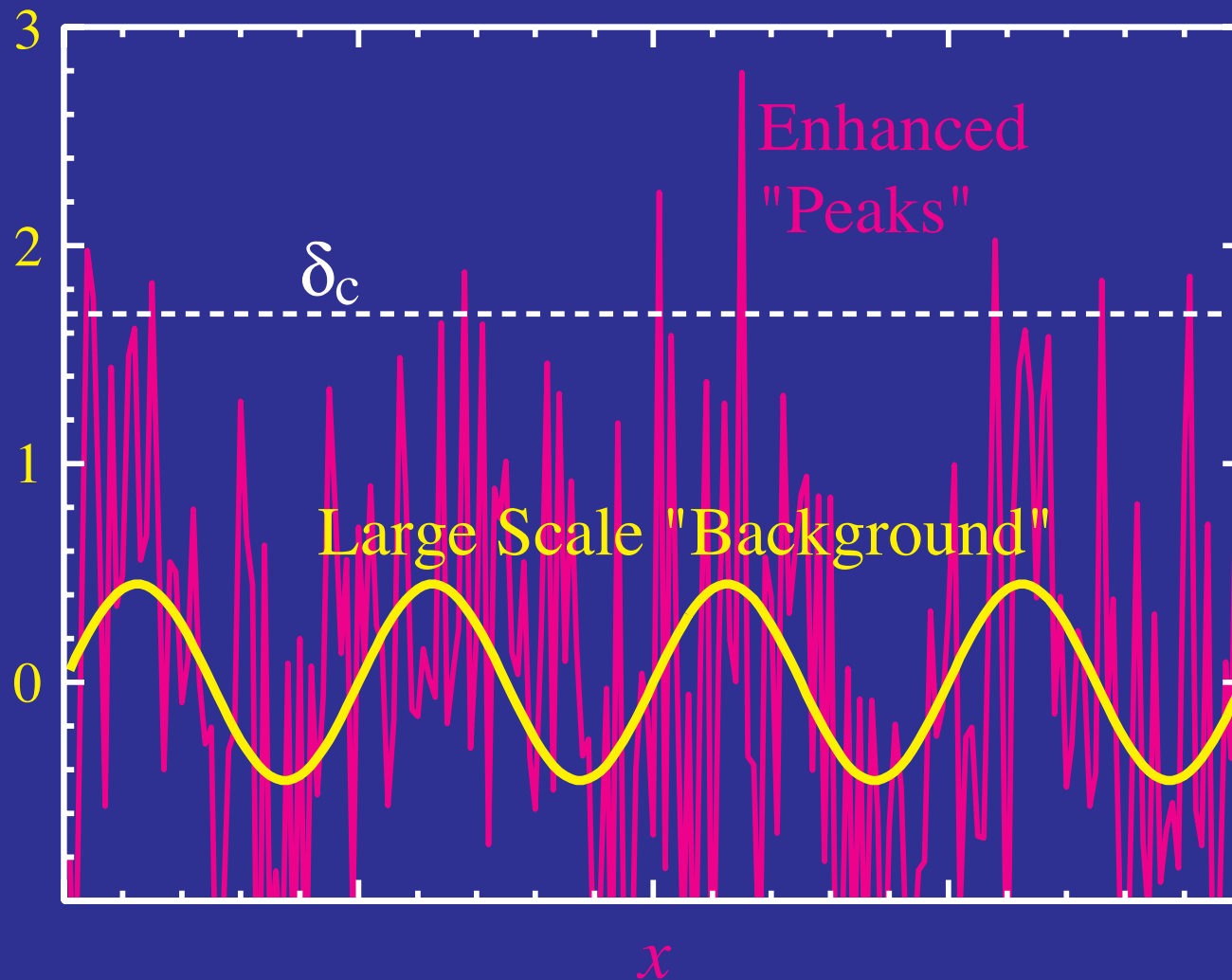
Perturbed Mass Function

- Density fluctuation split

$$\delta = \delta_b + \delta_p$$

- Lowers the threshold for collapse

$$\delta_{cp} = \delta_c - \delta_b$$

so that $\nu = \delta_{cp}/\sigma$

- Taylor expand number density $n_M \equiv dn/d\ln M$

$$n_M + \frac{dn_M}{d\nu}\frac{d\nu}{d\delta_b}\delta_b \ldots = n_M\left[1 + \frac{(\nu^2 - 1)}{\sigma\nu}\right]$$

if mass function is given by Press-Schechter

$$n_M \propto \nu\exp(-\nu^2/2)$$

Halo Bias

- Halos are biased tracers of the "background" dark matter field with a bias $b(M)$ that is given by spherical collapse and the form of the mass function

$$\frac{\delta n_M}{n_M} = [1 + b(M)]\,\delta$$

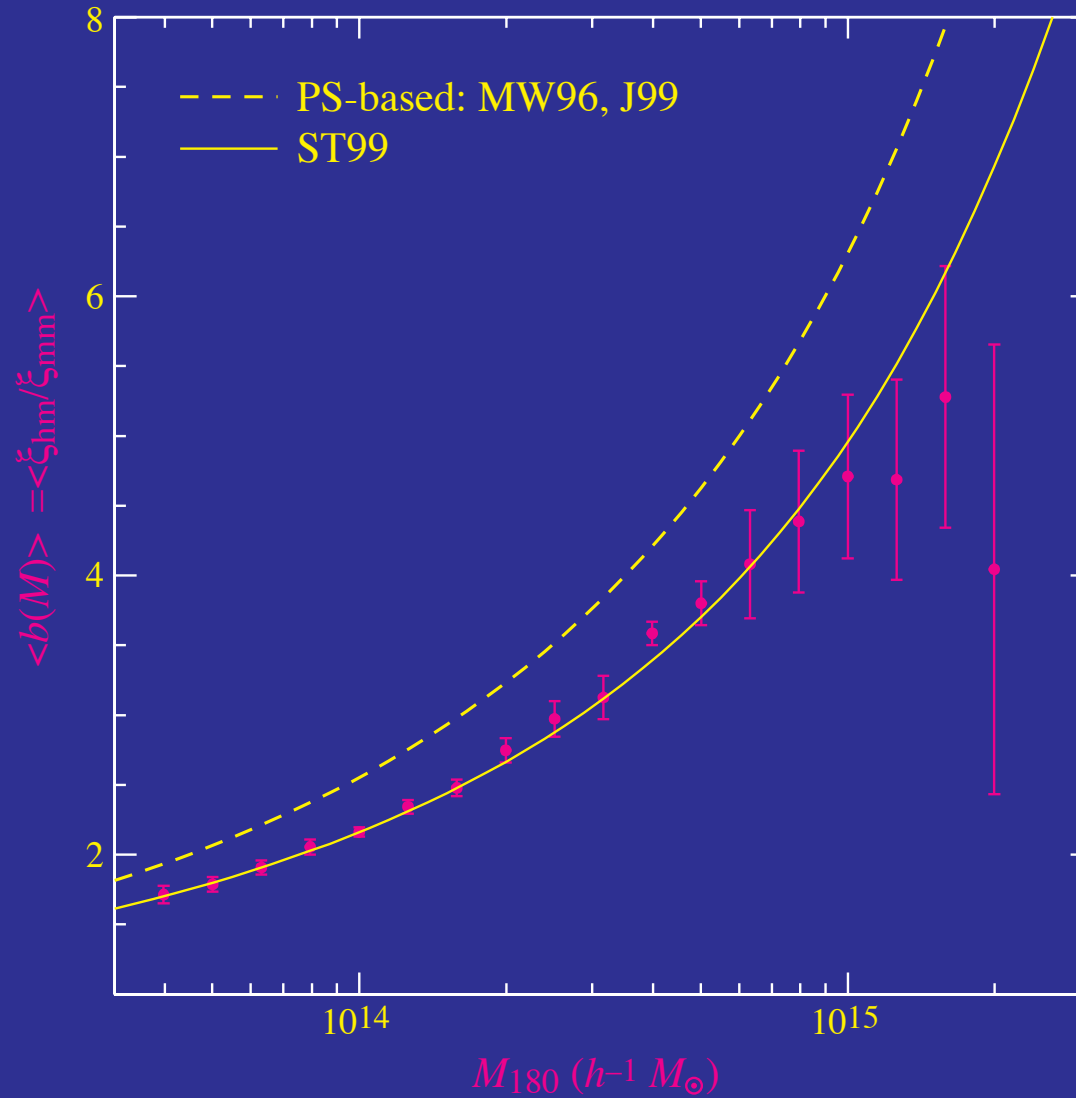- For Press-Schechter

$$b(M) = 1 + \frac{\nu^2 - 1}{\delta_c}$$

- Improved by the Sheth-Torman mass function

$$b(M) = 1 + \frac{a\nu^2 - 1}{\delta_c} + \frac{2p}{\delta_c[1 + (a\nu^2)^p]}$$

with $a = 0.75$ and $p = 0.3$ to match simulations.

Numerical Bias

- Example of halo bias from a simulation (from Hu & Kravstov 2002)

What is a Halo?

- Mass function and halo bias depend on the definition of mass of a halo

- Agreement with simulations depend on how halos are identified

- Other observables (associated galaxies, X-ray, SZ) depend on the details of the density profile

- Fortunately, simulations have shown that halos take on a near universal form in their density profile at least on large scales.
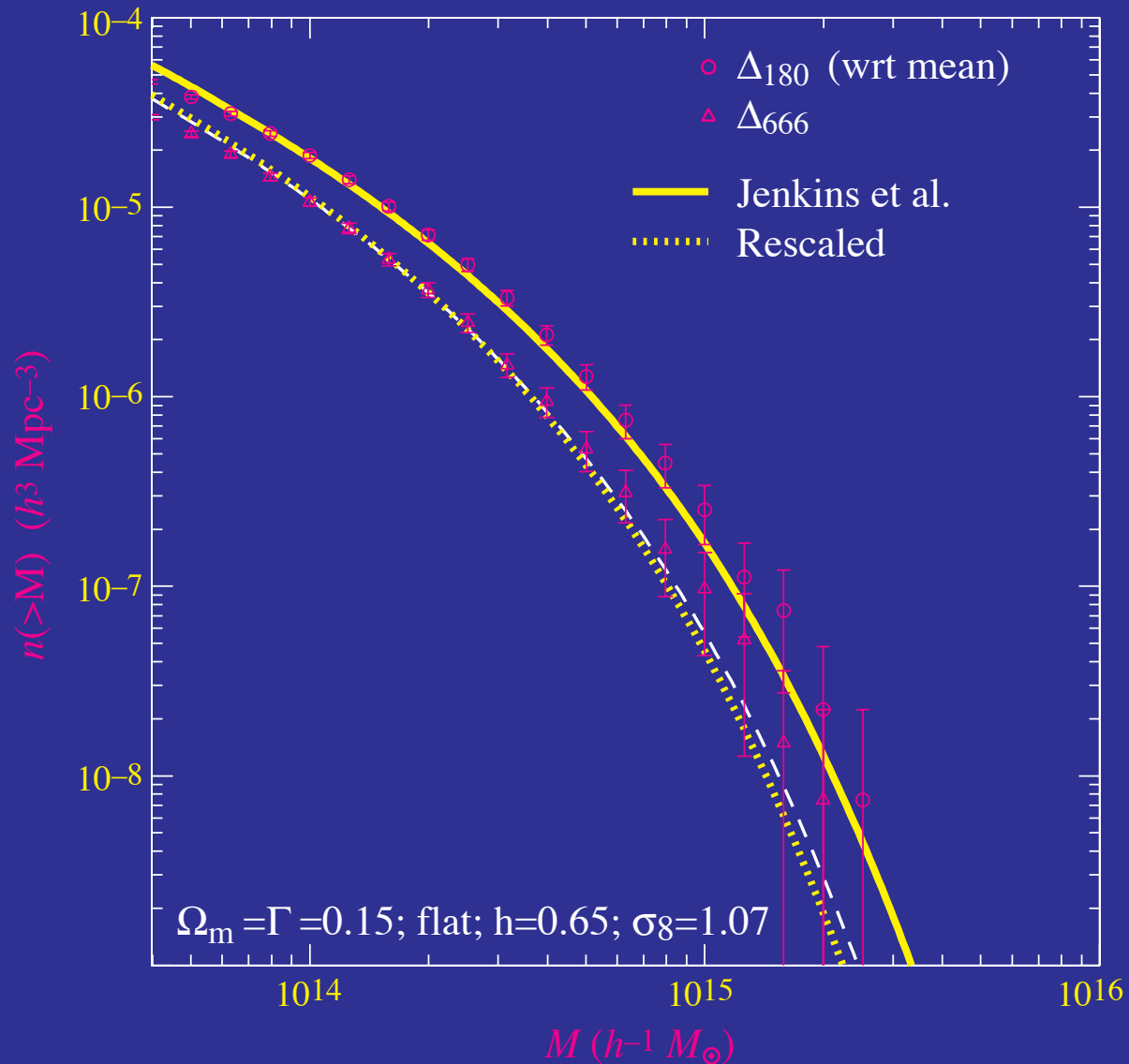
NFW Halo

- Density profile well-described by (Navarro, Frenk & White 1997)

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}$$
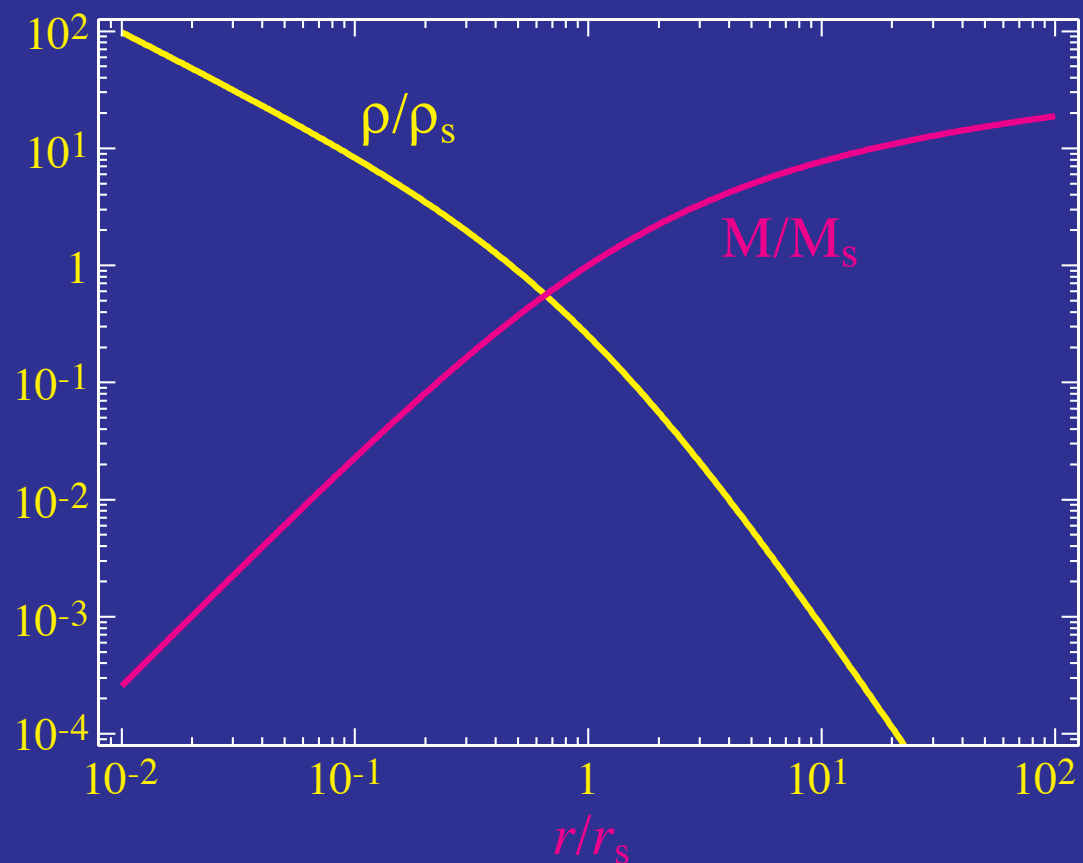
Transforming the Masses

- NFW profile gives a way of transforming different mass definitions

Lack of Concentration?

- NFW parameters may be recast into M_v, the mass of a halo out to the virial radius r_v where the overdensity wrt mean reaches $\Delta_v = 180$.

- Concentration parameter

$$c \equiv \frac{r_v}{r_s}$$

- CDM predicts $c \sim 10$ for M_* halos. Too centrally concentrated for galactic rotation curves?

- Possible discrepancy has lead to the exploration of dark matter alternatives: warm ($m \sim$keV) dark matter, self-interacting dark-matter, annihillating dark matter, ultra-light "fuzzy" dark matter, ...

Incredible, Extensible Halo Model

- An industry developed to build semi-analytic models for wide variety of cosmological observables based on the halo model

- Idea: associate an observable (galaxies, gas, ...) with dark matter halos

- Let the halo model describe the statistics of the observable

- The overextended halo model?

The Halo Model

- NFW halos, of abundance n_M given by mass function, clustered according to the halo bias $b(M)$ and the linear theory $P(k)$
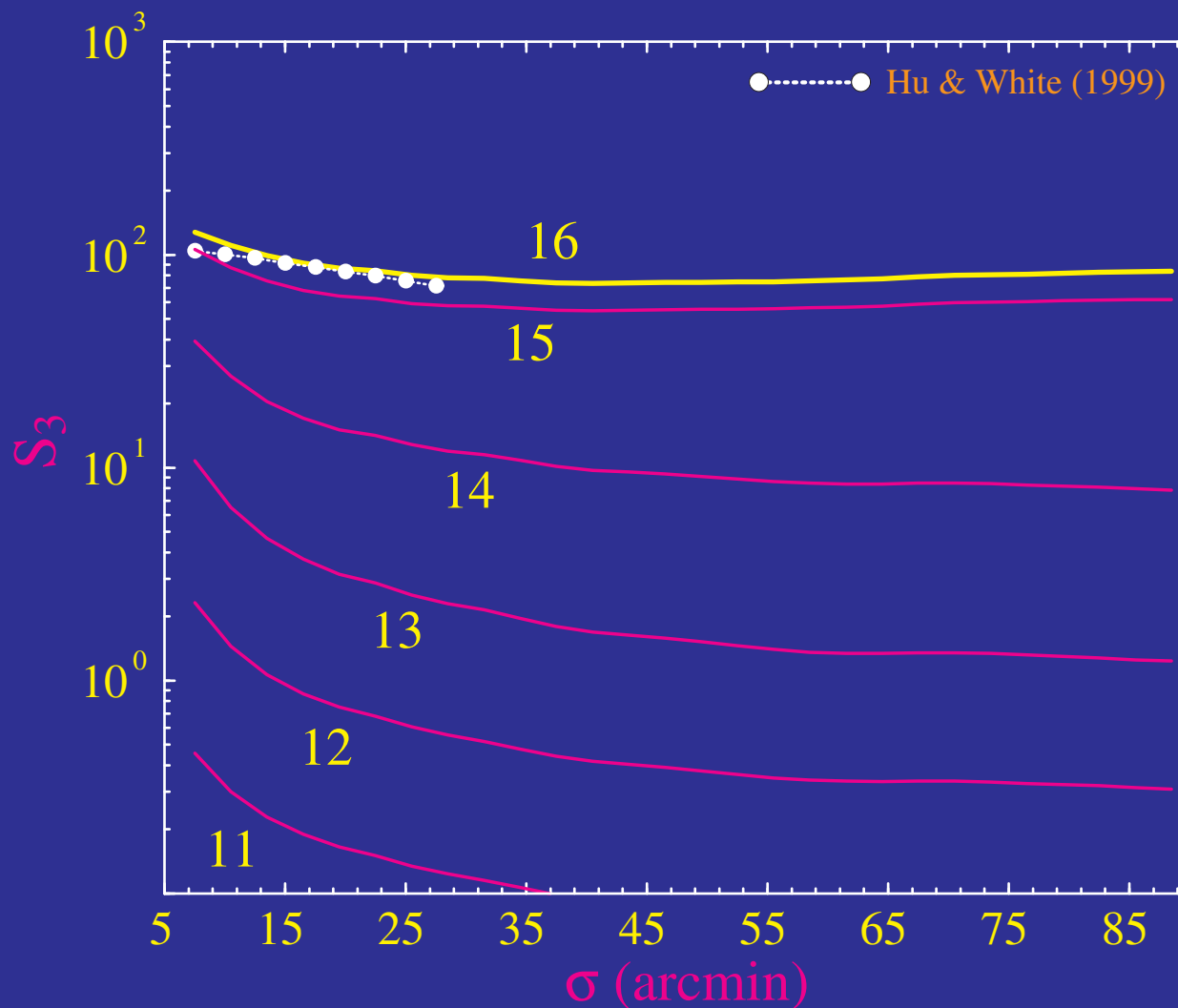
- Power spectrum example:
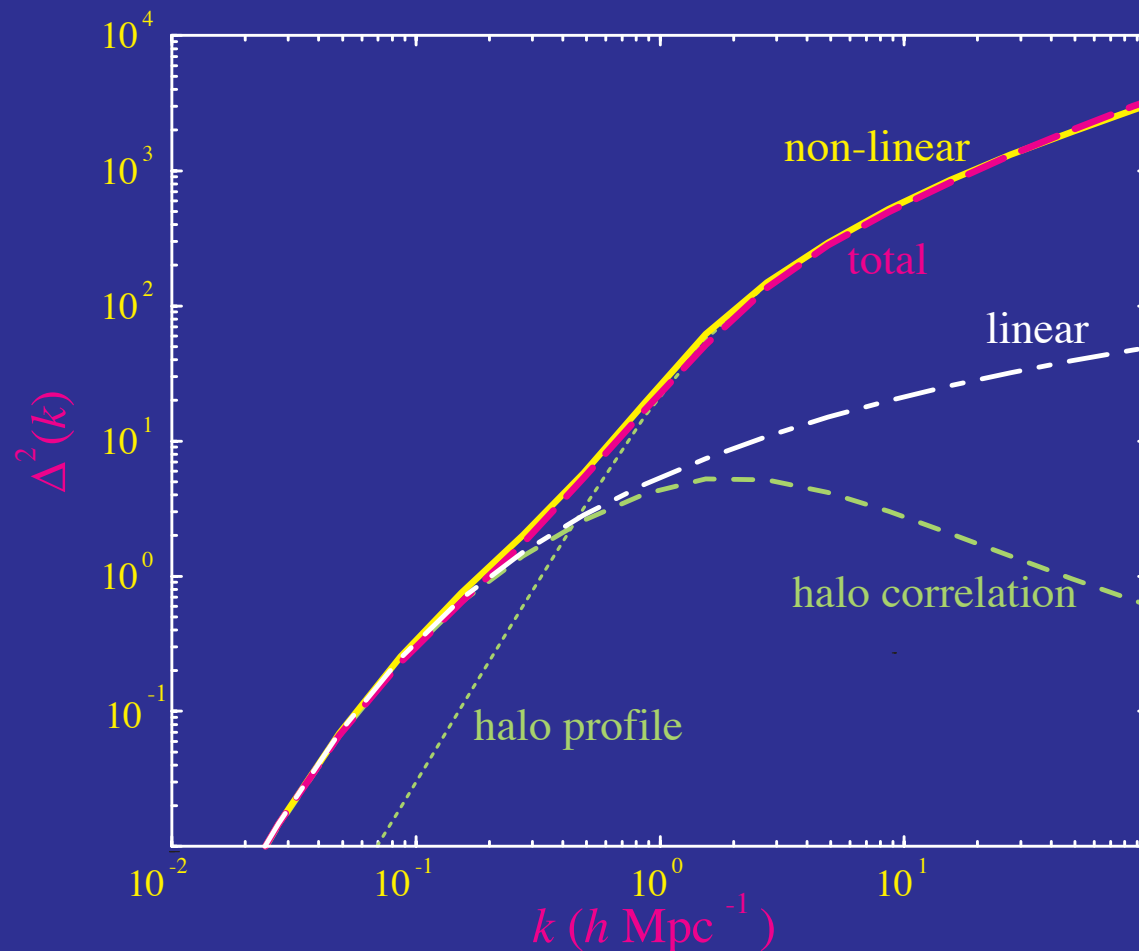
Weak Lensing and the Halo Model

- Power spectrum of shear divided into the halo masses that contribute
- Non-linear regime dominated by halo profile / individual halos increased power spectrum variance and covariance
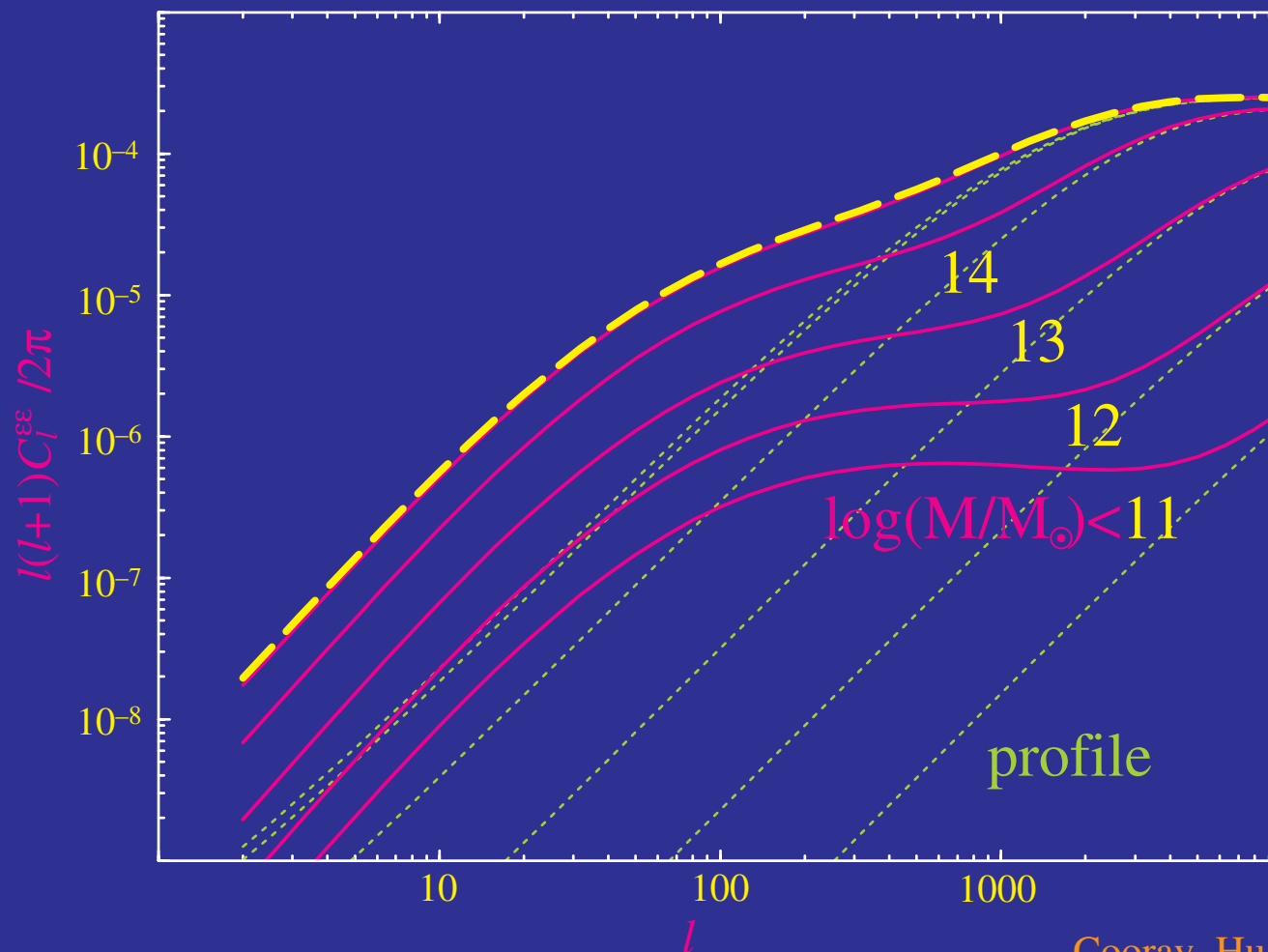
Higher Order Statistics

- Halo model for the bispectrum: S_3 dominated by massive halos

Cooray & Hu (2001)

Halo Temperature

- Motivate with isothermal distribution, correct from simulations

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$

- Express in terms of virial mass M_v enclosed at virial radius r_v

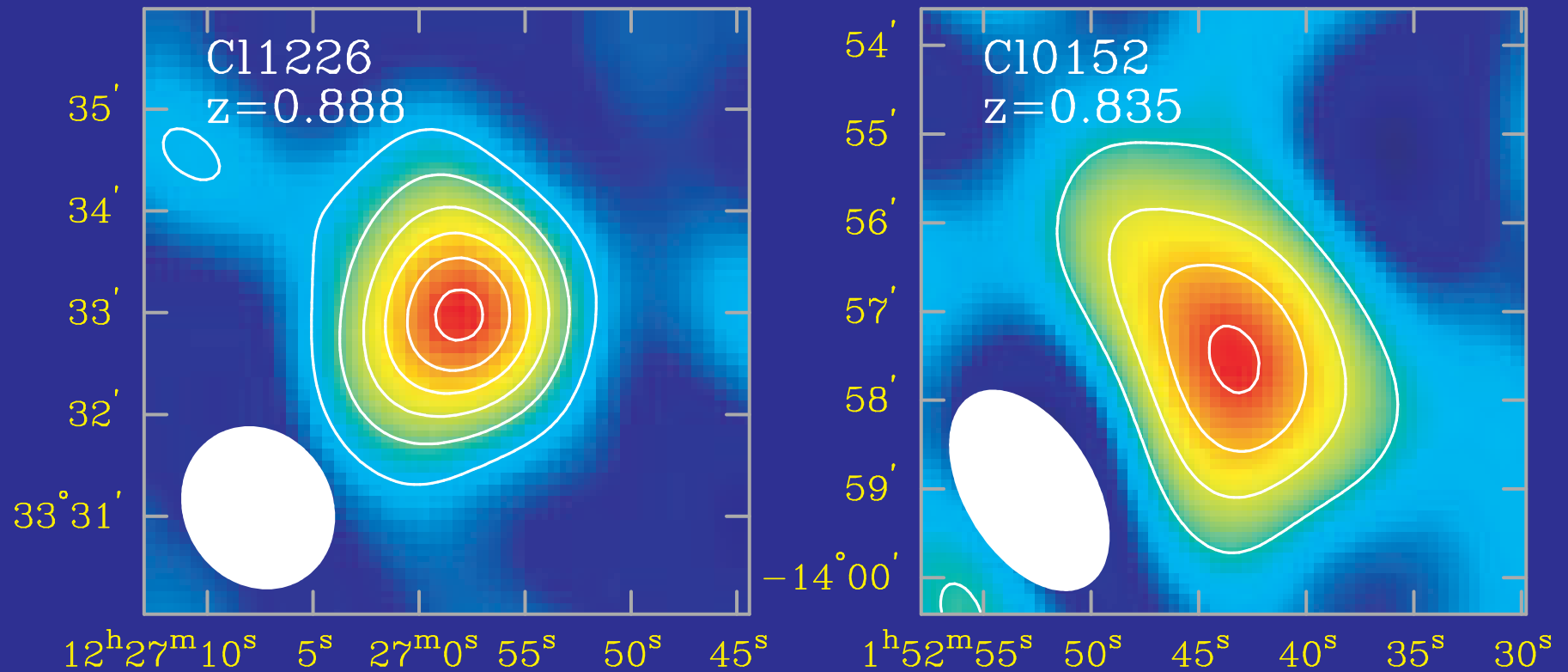$$M_v = \frac{4\pi}{3} r_v^3 \rho_m \Delta_v = \frac{2}{G} r_v \sigma^2$$

- Eliminate r_v, temperature $T \propto \sigma^2$ velocity dispersion^2
- Then $T \propto M_v^{2/3}(\rho_m \Delta_v)^{1/3}$ or

$$\left(\frac{M_v}{10^{15} h^{-1} M_\odot}\right) = \left[\frac{f}{(1+z)(\Omega_m \Delta_v)^{1/3}} \frac{T}{1\text{keV}}\right]^{3/2}$$

- Theory (X-ray weighted): $f \sim 0.75$; observations $f \sim 0.54$. Difference is crucial in determining cosmology from cluster counts!
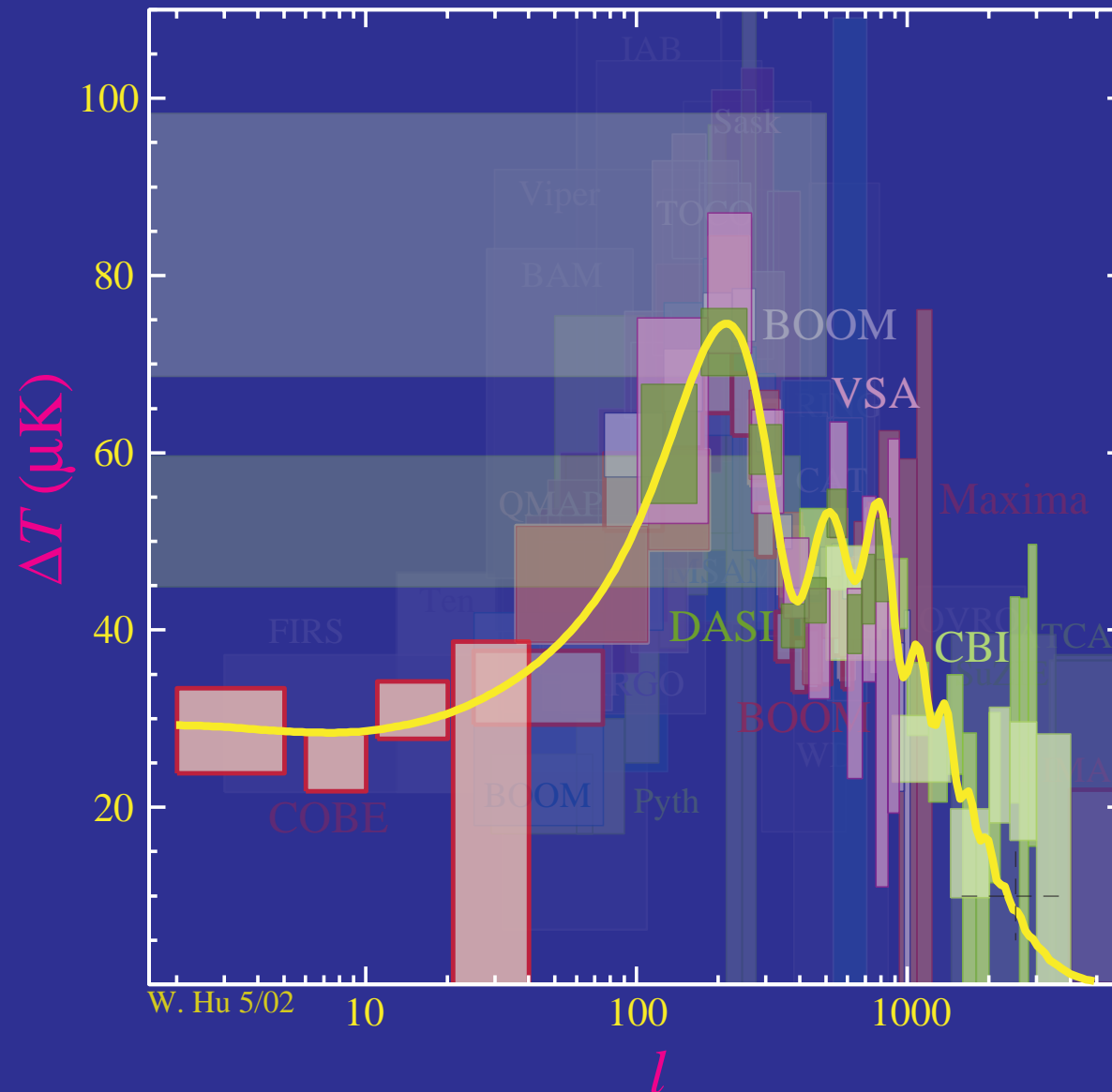
Clusters in the SZE

- Inverse Compton scattering of CMB off hot electrons

Carlstrom et al. (2001)

Clusters in Power Spectrum?

- Excess in arcminute scale CMB anisotropy from CBI

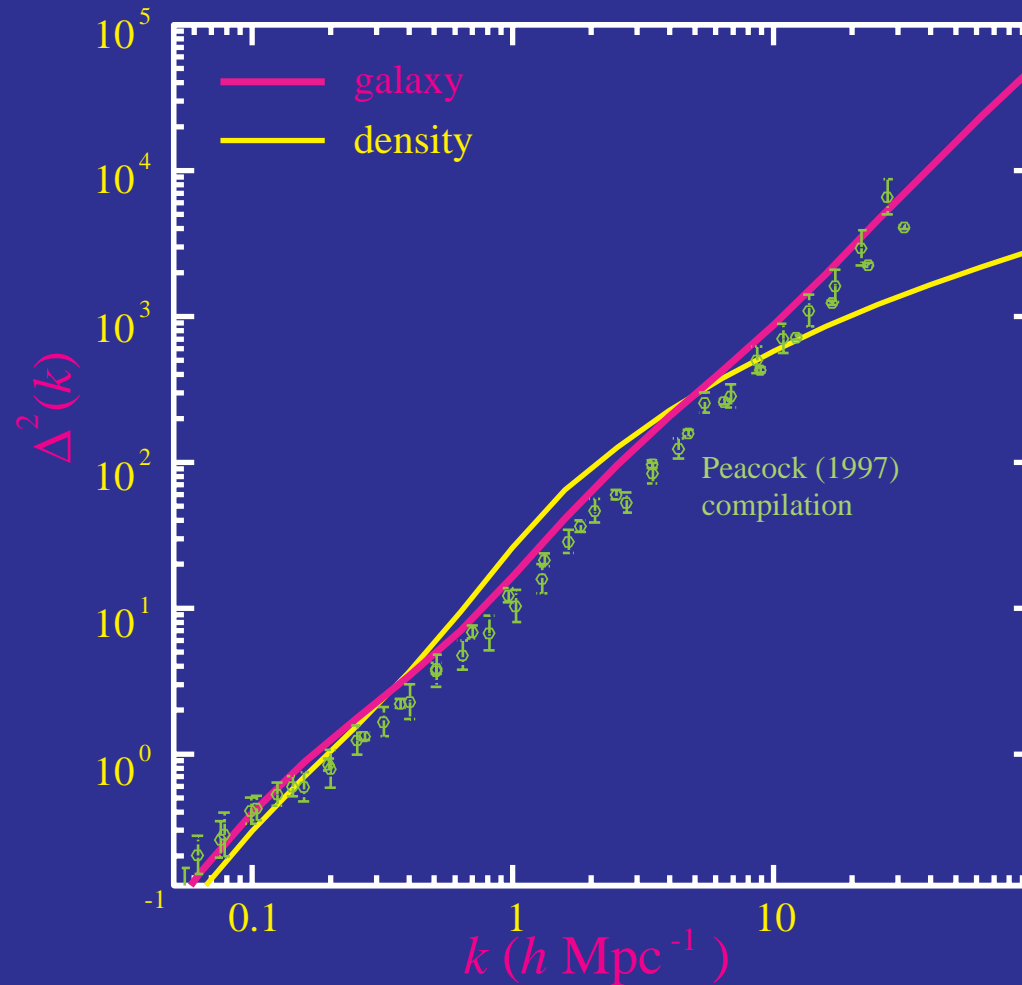W. Hu 5/02

Sensitivity of SZE Power

- Amplitude of fluctuations

Sensitivity of SZE Power

- Dark energy

Galaxy Clustering

- Associate galaxies with halo of mass M: $N(M)$ (Seljak 2001)

- An explanation of the pure power law galaxy spectrum

Summary

- Dark matter simulations well-understood and can be modelled with dark matter halos

- Halo formation modelled by spherical collapse, two magic numbers $\delta_c = 1.686$ and $\Delta_v = 178$

- Halo abundance described by a mass function with exponential high mass cutoff – rare clusters extremely sensitive to power spectrum amplitude and growth rate $\rightarrow$ dark energy

  Possibly too many small halos or sub-structure?

- Halo clustering modelled with peak-background split leading to halo bias

- Halo profile described by NFW halos

  Possibly too high central concentration

- Associate an observable with a halo $\rightarrow$ a halo model