

Outline

- Spherical Collapse

- Mass Function

 - Press-Schechter Formalism

 - Extended Press-Schechter Formalism

 - Halo Abundance

- Halo Bias

- Halo Profile

- Halo Model

 - Density Field

 - Baryonic Gas

 - Galaxies

Closed Universe

- A spherical perturbation of radius r behaves as a **closed universe**
- Radius $r \propto a \rightarrow 0$, collapse in finite time
- **Friedman equation** in a closed universe

$$\frac{1}{a} \frac{da}{dt} = H_0 \left(\Omega_m a^{-3} + (1 - \Omega_m) a^{-2} \right)^{1/2}$$

- Parametric solution in terms of a **development angle**
 $\theta = H_0 \eta (\Omega_m - 1)^{1/2}$, scaled conformal time η

$$r(\theta) = A(1 - \cos \theta)$$

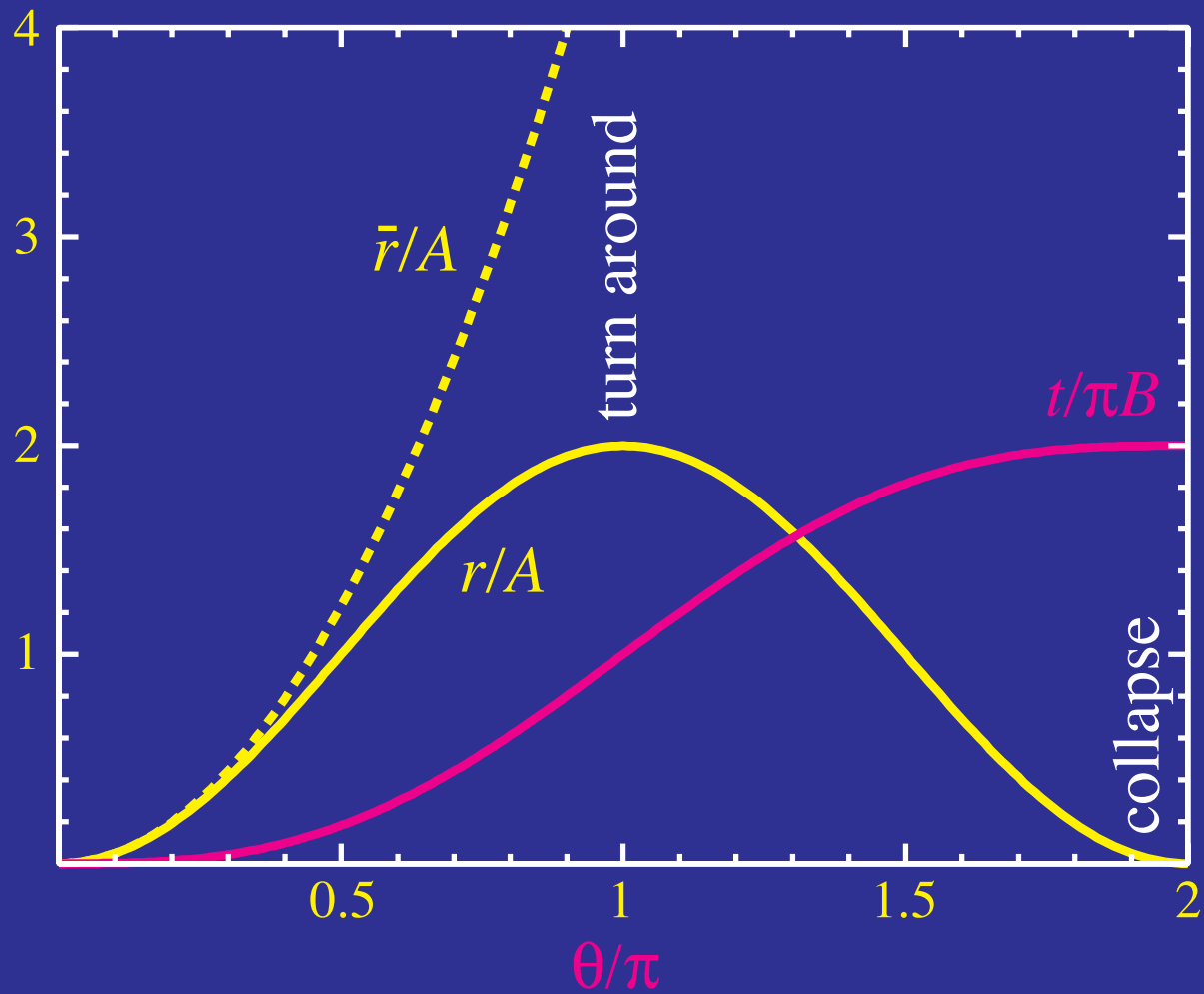
$$t(\theta) = B(\theta - \sin \theta)$$

where $A = r_0 \Omega_m / 2(\Omega_m - 1)$, $B = H_0^{-1} \Omega_m / 2(\Omega_m - 1)^{3/2}$.

- Turn around at $\theta = \pi$, $r = 2A$, $t = B\pi$.
- Collapse at $\theta = 2\pi$, $r \rightarrow 0$, $t = 2\pi B$

Spherical Collapse

- Parametric Solution:



Correspondence

- Eliminate cosmological correspondence in A and B in terms of enclosed mass M

$$M = \frac{4\pi}{3} r_0^3 \Omega_m \rho_c = \frac{4\pi}{3} r_0^3 \Omega_m \frac{3H_0^2}{8\pi G}$$

- Related as $A^3 = GM B^2$, and to initial perturbation

$$\lim_{\theta \rightarrow 0} r(\theta) = A \left(\frac{1}{2} \theta^2 - \frac{1}{4} \theta^4 \right)$$

$$\lim_{\theta \rightarrow 0} t(\theta) = B \left(\frac{1}{6} \theta^3 - \frac{1}{120} \theta^5 \right)$$

- Leading Order: $r = A\theta^2/2$, $t = B\theta^3/6$

$$r = \frac{A}{2} \left(\frac{6t}{B} \right)^{2/3}$$

- Unperturbed matter dominated expansion $r \propto a \propto t^{2/3}$

Next Order

- Iterate r and t solutions

$$\lim_{\theta \rightarrow 0} t(\theta) = \frac{\theta^3}{6} B \left[1 - \frac{1}{20} \left(\frac{6t}{B} \right)^{2/3} \right]$$

$$\theta \approx \left(\frac{6t}{B} \right)^{1/3} \left[1 + \frac{1}{60} \left(\frac{6t}{B} \right)^{2/3} \right]$$

- Substitute back into $r(\theta)$

$$\begin{aligned} r(\theta) &= A \frac{\theta^2}{2} \left(1 - \frac{\theta^2}{12} \right) \\ &= A \left(\frac{6t}{B} \right)^{2/3} \left[1 - \frac{1}{20} \left(\frac{6t}{B} \right)^{2/3} \right] \\ &= (6t)^{2/3} (GM)^{1/3} \left[1 - \frac{1}{20} \left(\frac{6t}{B} \right)^{2/3} \right] \end{aligned}$$

Density Correspondence

- Density

$$\begin{aligned}\rho_m &= \frac{M}{\frac{4}{3}\pi r^3} \\ &= \frac{1}{6\pi t^2 G} \left[1 + \frac{3}{20} \left(\frac{6t}{B} \right)^{2/3} \right]\end{aligned}$$

- Density perturbation

$$\delta \equiv \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m} \approx \frac{3}{20} \left(\frac{6t}{B} \right)^{2/3}$$

- Time \rightarrow scale factor

$$\begin{aligned}t &= \frac{2}{3H_0\Omega_m^{1/2}} a^{3/2} \\ \delta &= \frac{3}{20} a \left(4BH_0\Omega_m^{1/2} \right)^{2/3}\end{aligned}$$

Spherical Collapse Relations

- A and B constants \rightarrow initial cond.

$$B = \frac{1}{2H_0\Omega_m^{1/2}} \left(\frac{3a_i}{5\delta_i} \right)^{3/2}$$

$$A = \frac{3r_i}{10\delta_i}$$

- Scale factor $a \propto t^{2/3}$

$$a = \left(\frac{3}{4} \right)^{2/3} \left(\frac{3a_i}{5\delta_i} \right) (\theta - \sin \theta)^{2/3}$$

- At collapse $\theta = 2\pi$

$$a_{\text{col}} = \left(\frac{3}{4} \right)^{2/3} \left(\frac{3a_i}{5\delta_i} \right) (2\pi)^{2/3} \approx 1.686 \frac{a_i}{\delta_i}$$

- Perturbation collapses when **linear theory** predicts $\delta_c \equiv 1.686$

Virialization

- A real density perturbation is neither spherical nor homogeneous
- **Shell crossing** if δ_i doesn't monotonically decrease
- Collapse does not proceed to a point but reaches **virial equilibrium**

$$U = -2K$$

$$E = U + K = U(r_{\max}) = \frac{1}{2}U(r_{\text{vir}})$$

$$r_{\text{vir}} = \frac{1}{2}r_{\max}$$

since $U \propto r^{-1}$. Thus $\theta_{\text{vir}} = \frac{3}{2}\pi$

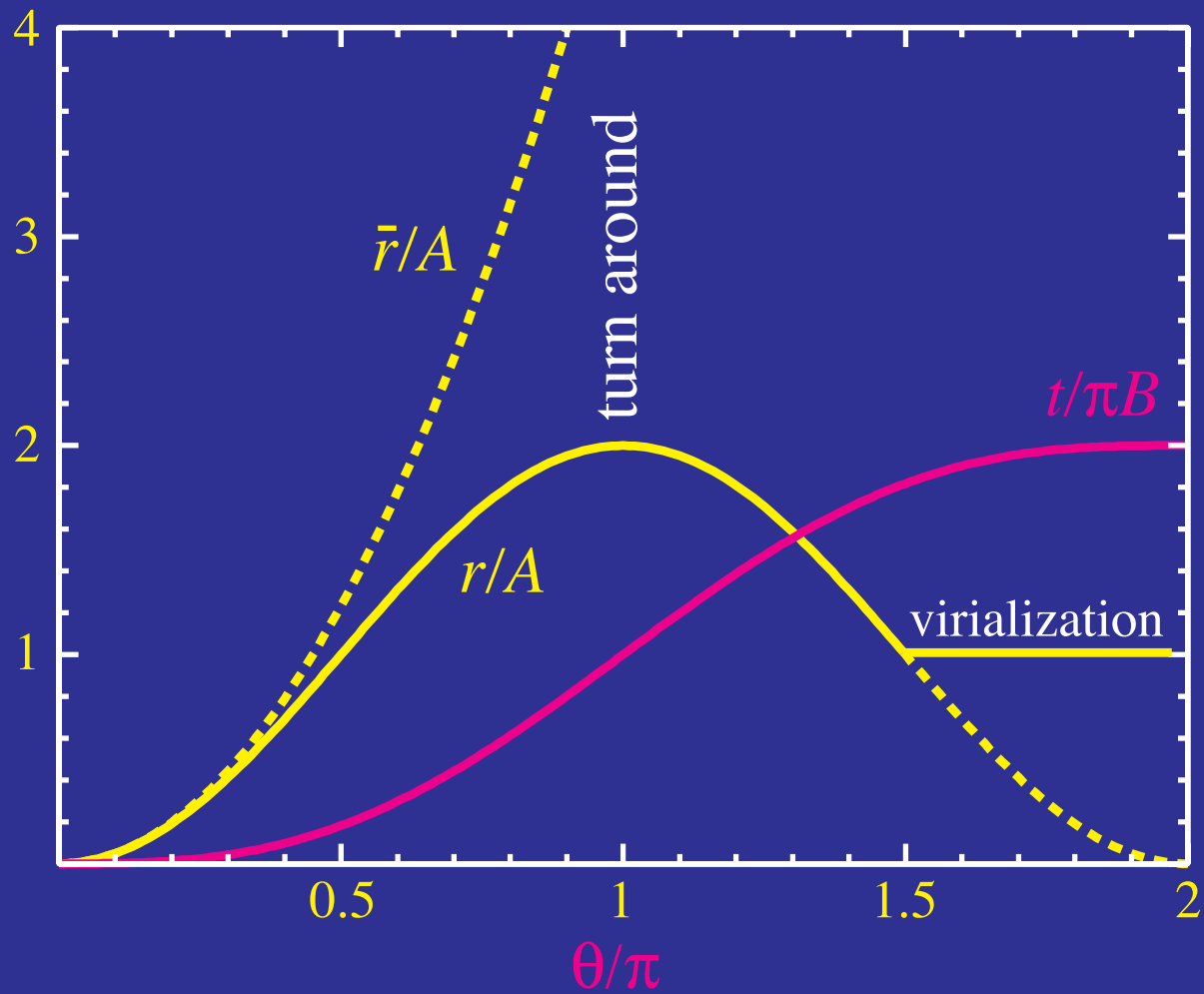
- **Overdensity** at virialization

$$\frac{\rho_m(\theta = 3\pi/2)}{\bar{\rho}_m(\theta = 2\pi)} = 18\pi^2 \approx 178$$

- Threshold $\Delta_v = 178$ often used to define a **collapsed object**

Virialization

- Schematic Picture:



The Mass Function

- **Spherical collapse** predicts the end state as virialized **halos** given an initial density perturbation
- Initial density perturbation is a **Gaussian random field**
- Compare the variance in the linear density field to **threshold** $\delta_c = 1.686$ to determine collapse fraction
- Combine to form the **mass function**, the number density of halos in a range dM around M .
- Halo density defined entirely by linear theory
- Fudge the result to get the right answer compared with simulations (a la Press-Schechter)!

Press-Schechter Formalism

- **Smooth** linear density field on mass scale M with tophat

$$R = \left(\frac{3M}{4\pi} \right)^{1/3}$$

- Result is a Gaussian random field with **variance** $\sigma^2(M)$
- Fluctuations above the threshold δ_c correspond to **collapsed regions**. The fraction in halos $> M$ becomes

$$\frac{1}{\sqrt{2\pi}\sigma(M)} \int_{\delta_c}^{\infty} d\delta \exp\left(-\frac{\delta^2}{2\sigma^2(M)}\right) = \frac{1}{2} \operatorname{erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$

where $\nu \equiv \delta_c/\sigma(M)$

- **Problem:** even as $\sigma(M) \rightarrow \infty$, $\nu \rightarrow 0$, collapse fraction $\rightarrow 1/2$ – only **overdense regions** participate in spherical collapse.
- **Multiply by an ad hoc factor of 2!**

Press-Schechter Mass Function

- **Differentiate** in M to find fraction in range dM and multiply by ρ_m/M the number density of halos if all of the mass were composed of such halos \rightarrow **differential number density** of halos

$$\begin{aligned}\frac{dn}{d \ln M} &= \frac{\rho_m}{M} \frac{d}{d \ln M} \operatorname{erfc} \left(\frac{\nu}{\sqrt{2}} \right) \\ &= \sqrt{\frac{2}{\pi}} \frac{\rho_m}{M} \frac{d \ln \sigma^{-1}}{d \ln M} \nu \exp(-\nu^2/2)\end{aligned}$$

- High mass: **exponential cut off** above M_* where $\sigma(M_*) = \delta_c$

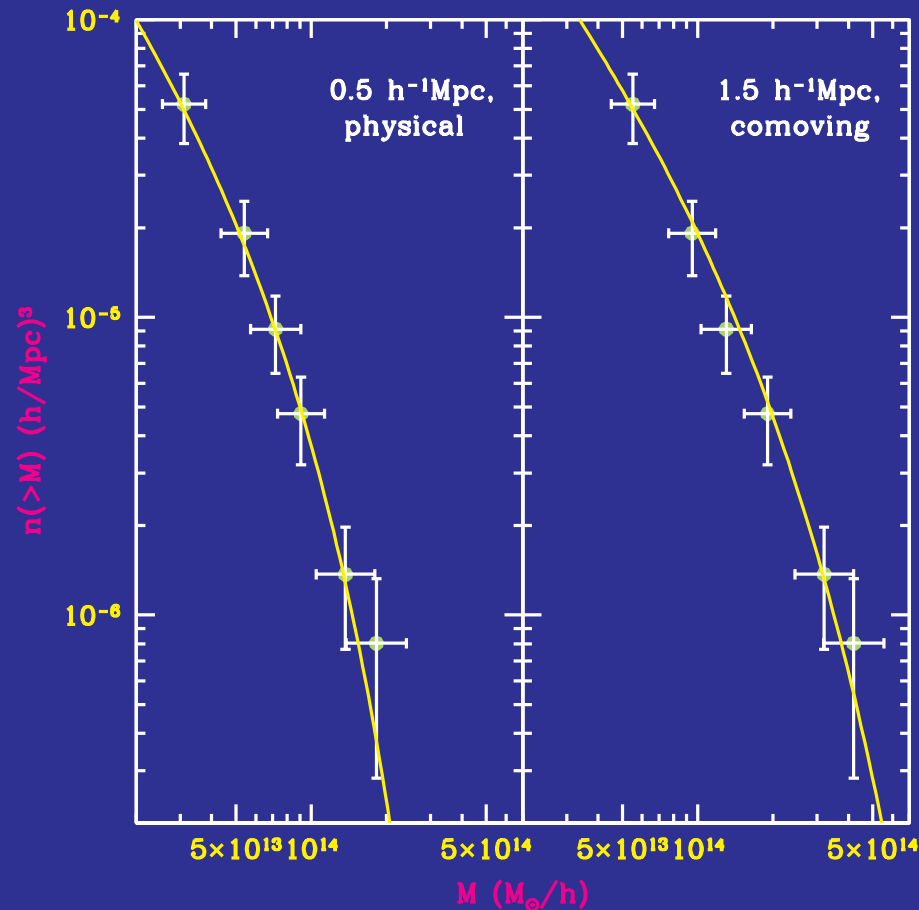
$$M_* \sim 10^{13} h^{-1} M_\odot \quad \text{today}$$

- Low mass **divergence**: (too many for the observations?)

$$\frac{dn}{d \ln M} \propto M^{-1}$$

Observational Mass Functions

- SDSS optically identified clusters (assuming M/L ; Bahcall et al 2002)



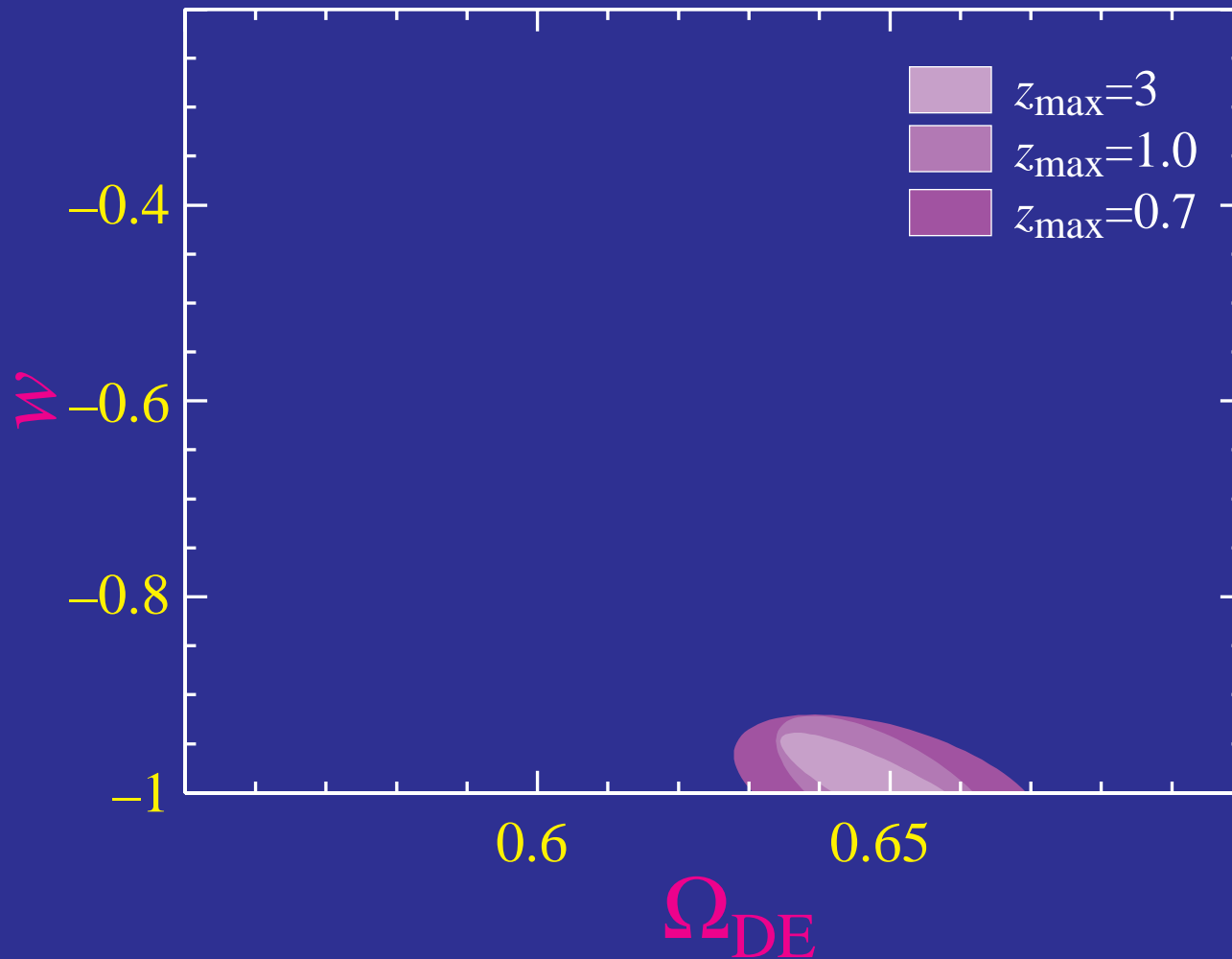
with cluster X -ray temp. function, sensitive to power amplitude σ_8^2

Counting Halos \rightarrow Dark Energy

- Halo abundance exponentially sensitive to growth rate

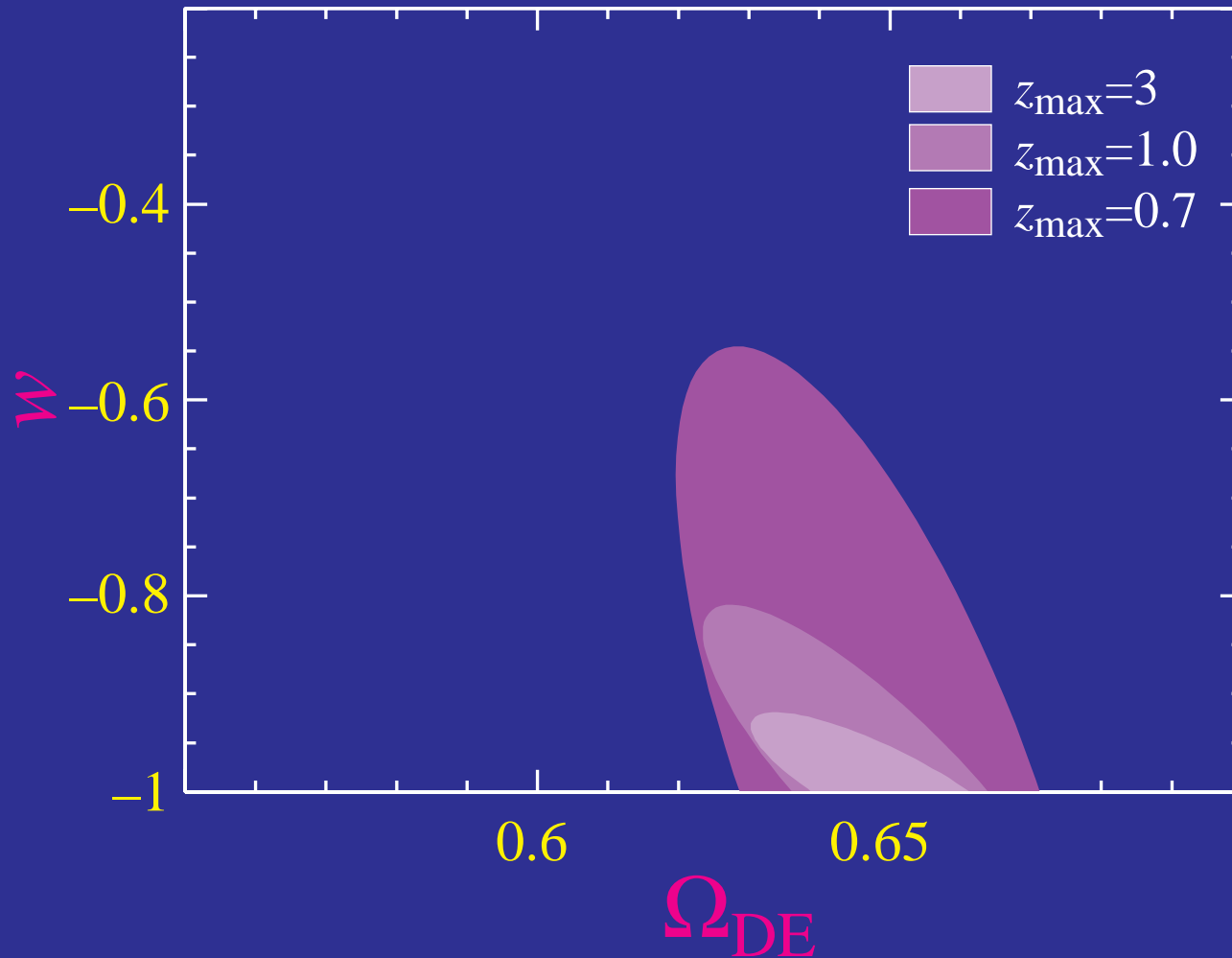
Projected Constraints

- Studies of $M > 2.5 \times 10^{14} M_{\odot}$ (Haiman et al. 2000; Hu & Kravstov)
- All other parameters known



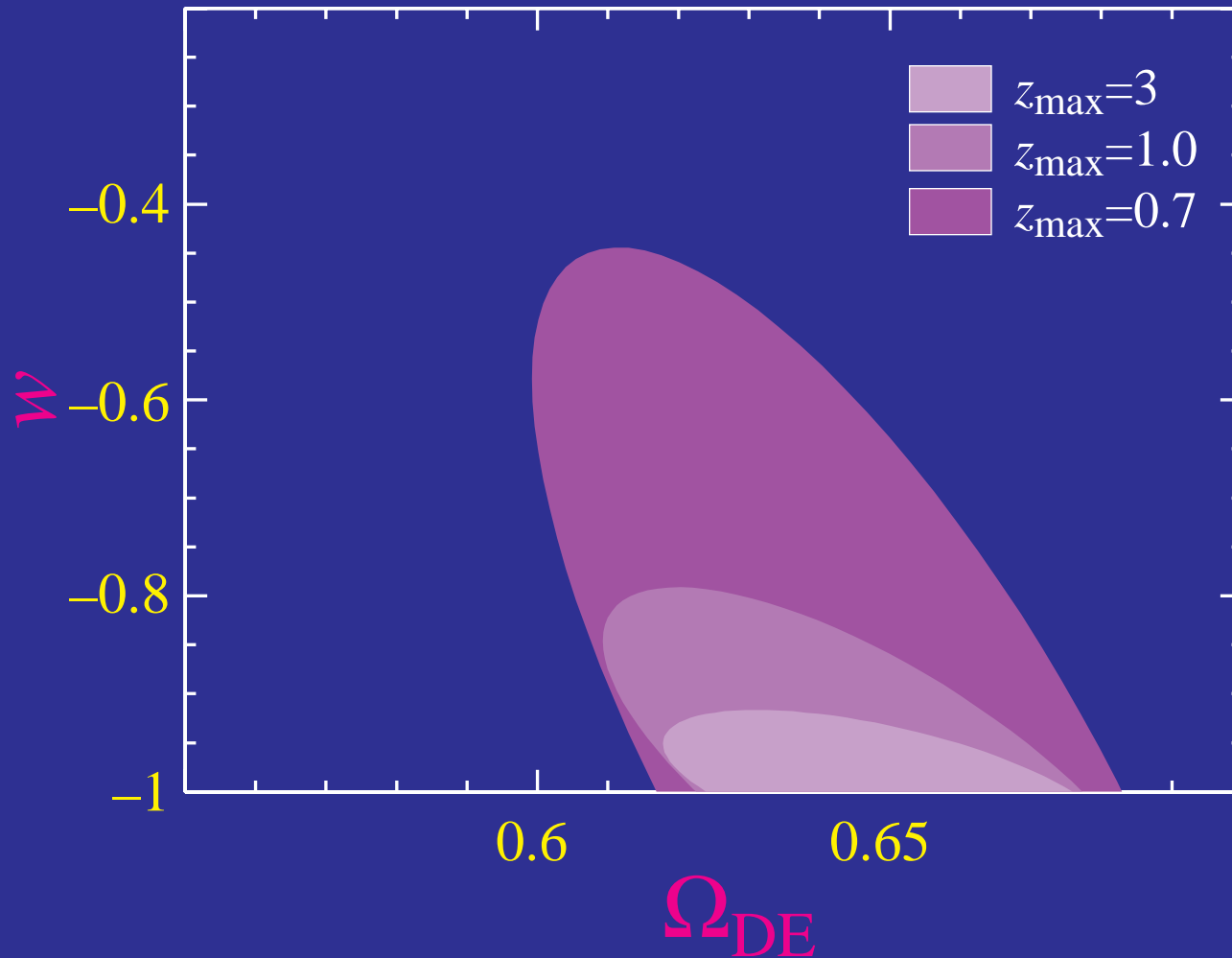
Projected Constraints

- Studies of $M > 2.5 \times 10^{14} M_{\odot}$ (Haiman et al. 2000; Hu & Kravstov)
- Local halo abundance known + present day cosmological params



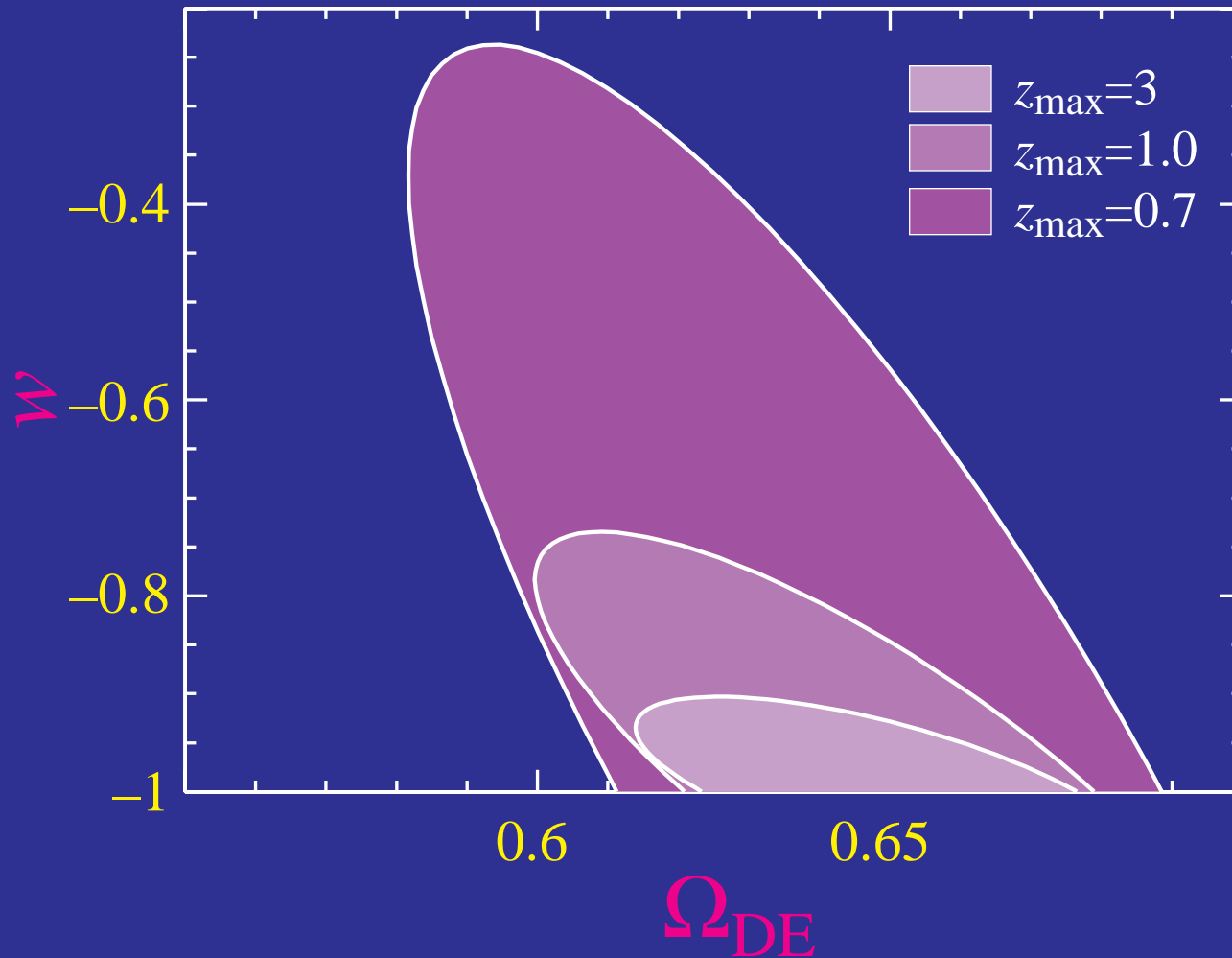
Projected Constraints

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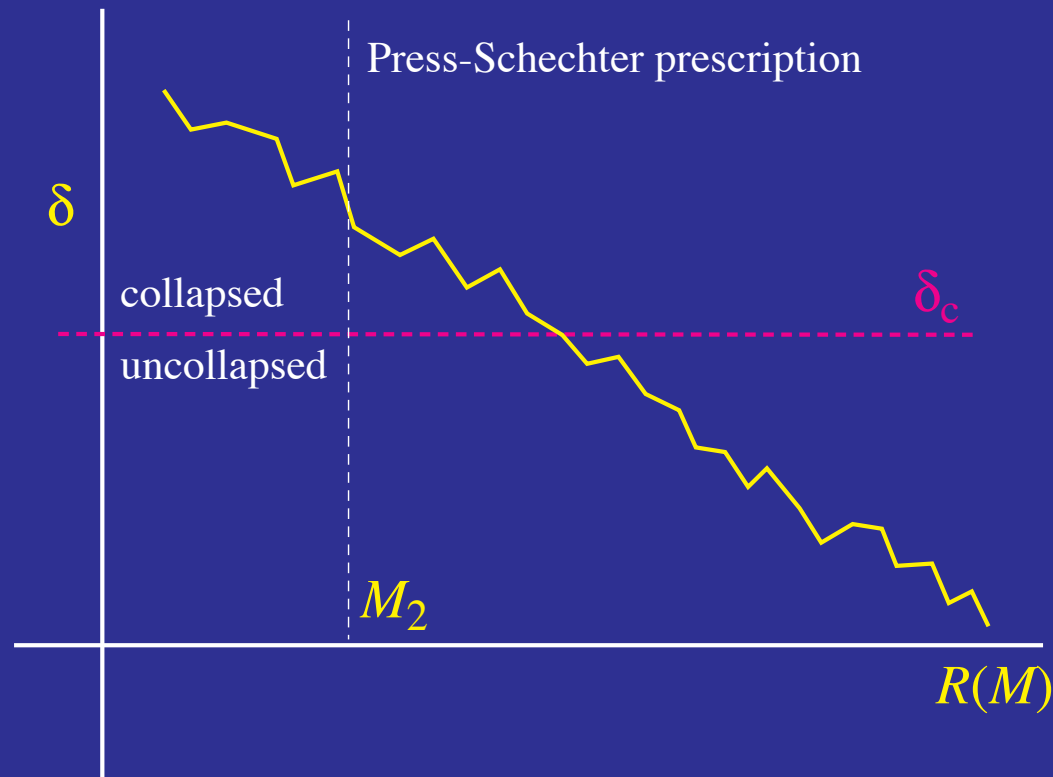
Projected Constraints

- Studies of $M > 2.5 \times 10^{14} M_{\odot}$ (Haiman et al. 2000; Hu & Kravstov)
- Present day cosmological parameters + **sample variance**



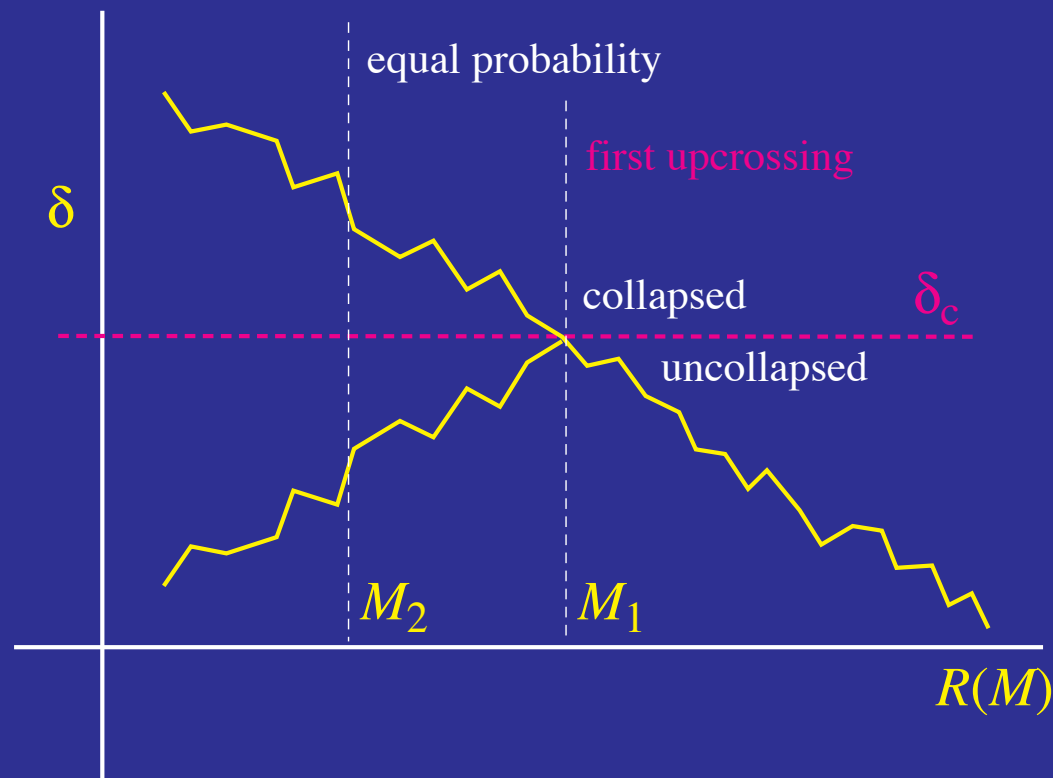
Extended Press-Schechter Formalism

- A region that is **underdense** when smoothed on the scale M may be **overdense** on a scale of a **larger M**
- If smoothing is a tophat in k -space, independence of k -modes implies fluctuation executes a **random walk**



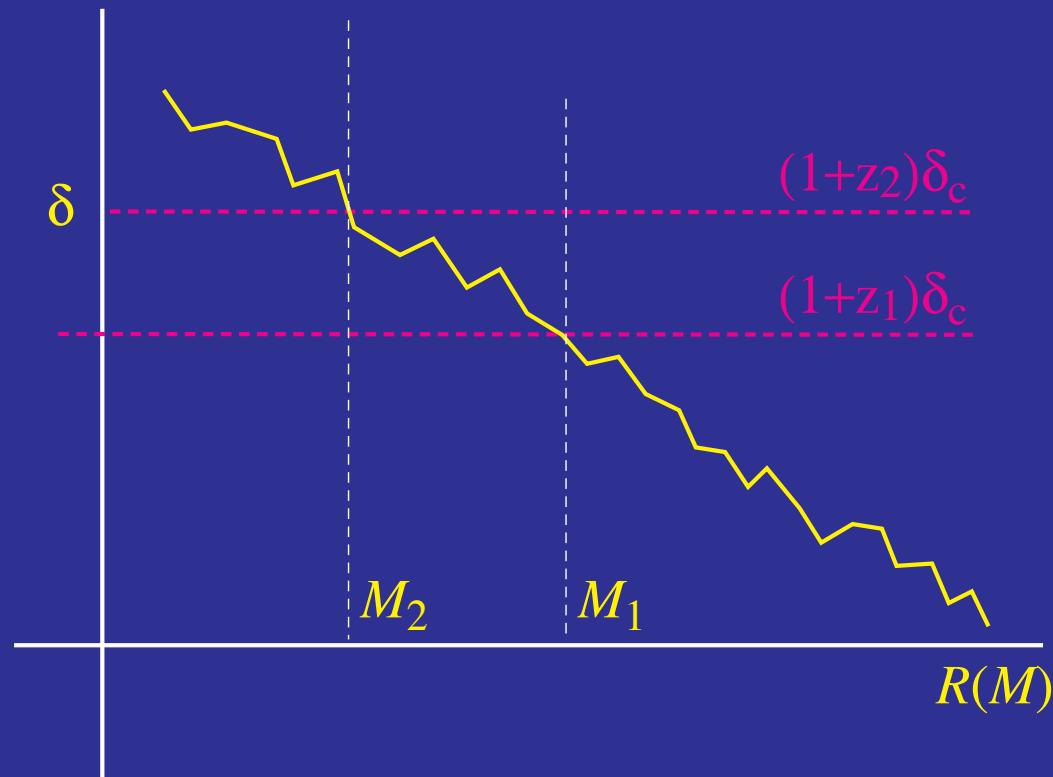
Extended Press-Schechter Formalism

- For each trajectory that lies above threshold at M_2 , there is an **equivalent trajectory** that is its mirror image reflected around δ_c
- Press-Schechter ignored this branch. It supplies the **missing factor of 2**



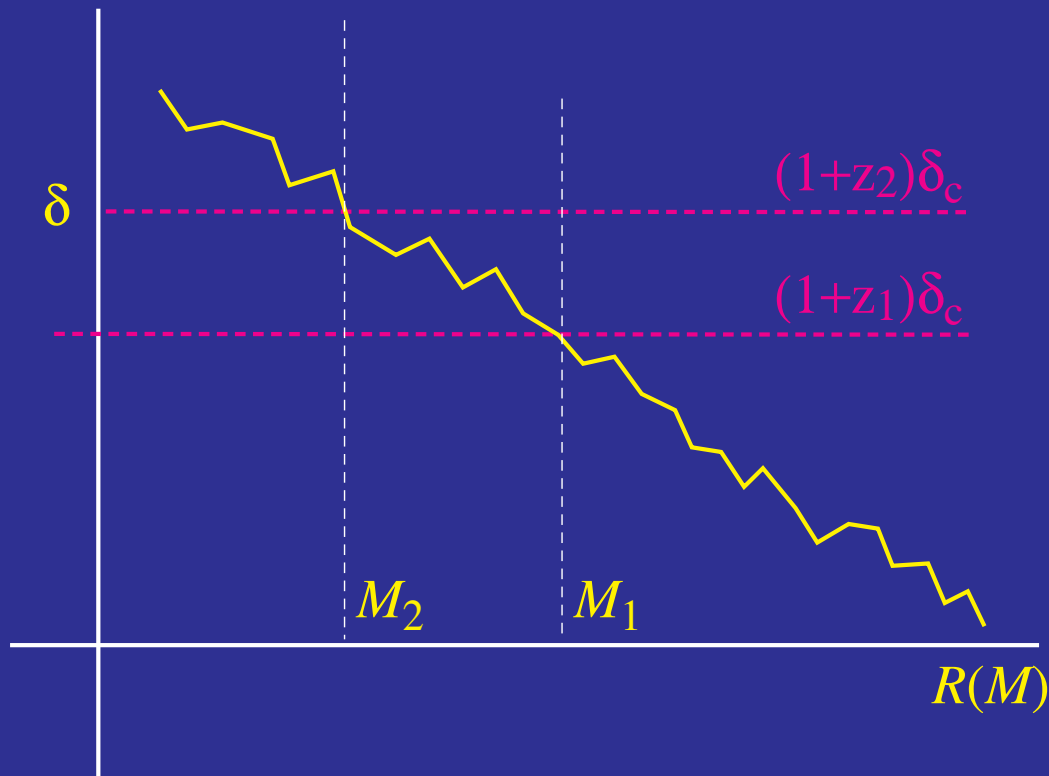
Conditional Mass Function

- Extended Press-Schechter also gives the **conditional mass function**, useful for **merger histories**.
- Given a halo of mass M_1 exists at z_1 , what is the probability that it was part of a halo of mass M_2 at z_2



Conditional Mass Function

- Same as before but with the **origin translated**.
- Conditional mass function is mass function with δ_c and $\sigma^2(M)$ **shifted**



Merger Simulation

- Simulation by Andrey Kravstov

Magic “2” resolved?

- Spherical collapse is defined for a **real-space** not k -space smoothing. Random walk is only a **qualitative explanation**.
- Modern approach: think of spherical collapse as motivating a **fitting form** for the mass function

$$\nu \exp(-\nu^2/2) \rightarrow A[1 + (a\nu^2)^{-p}] \sqrt{a\nu^2} \exp(-a\nu^2/2)$$

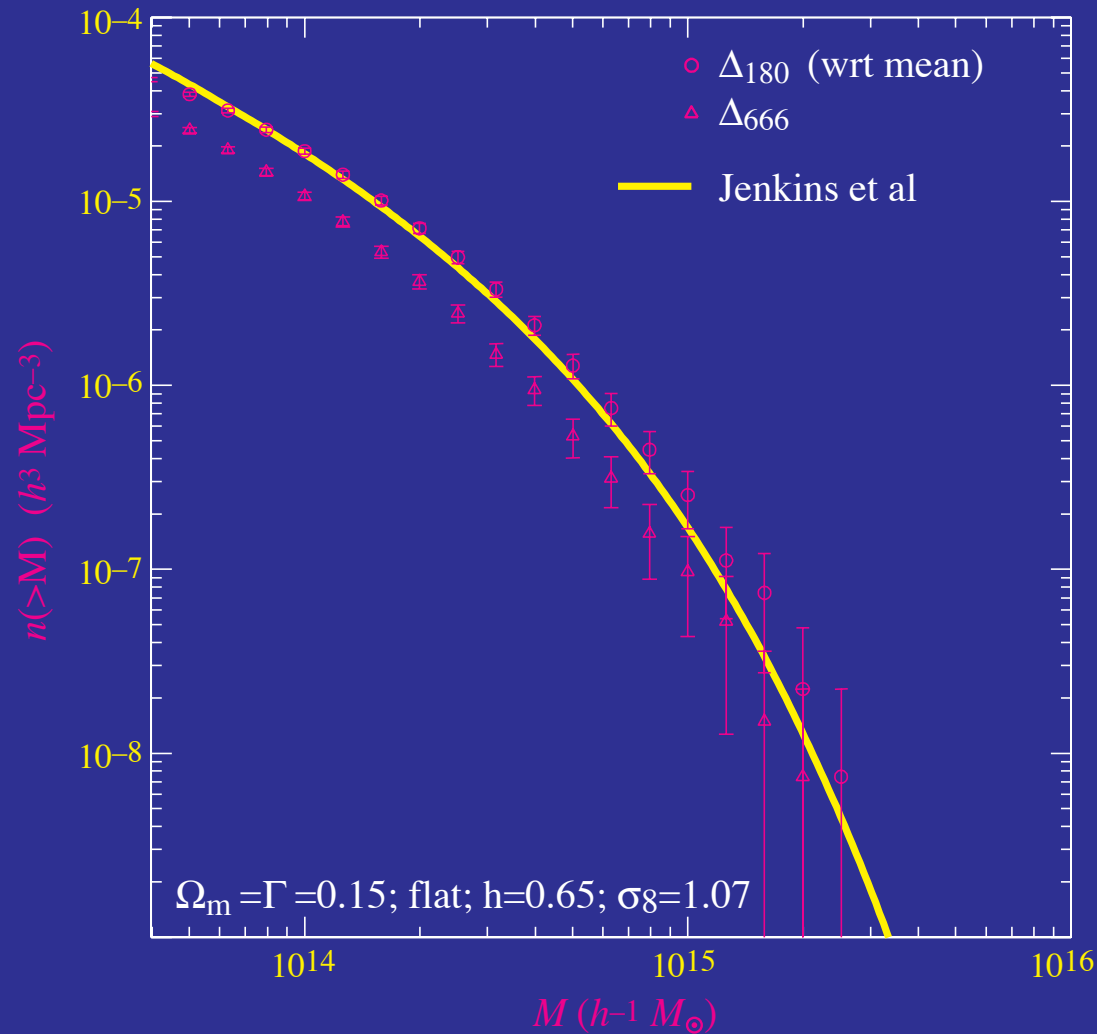
Sheth-Torman 1999, $a = 0.75$, $p = 0.3$. or a completely empirical fitting

$$\frac{dn}{d \ln M} = 0.301 \frac{\rho_m}{M} \frac{d \ln \sigma^{-1}}{d \ln M} \exp[-|\ln \sigma^{-1} + 0.64|^{3.82}]$$

Jenkins et al 2001. Choice is tied up with the question: **what is the mass of a halo?**

Numerical Mass Function

- Example of difference in mass definition (from Hu & Kravstov 2002)



Halo Bias

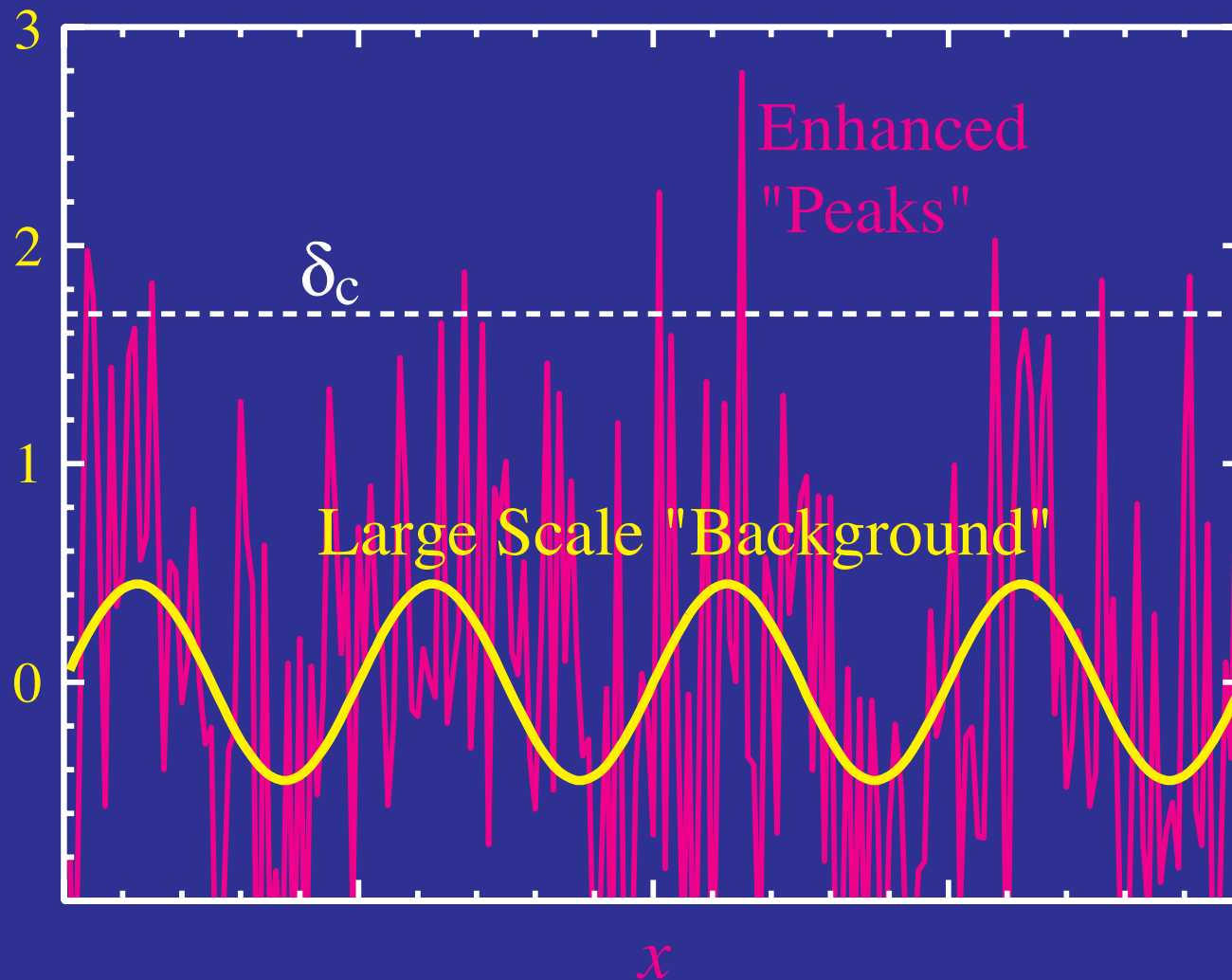
- If halos are formed without regard to the underlying density fluctuation and move under the **gravitational field** then their number density is an **unbiased tracer** of the dark matter density fluctuation

$$\left(\frac{\delta n}{n}\right)_{\text{halo}} = \left(\frac{\delta \rho}{\rho}\right)$$

- However **spherical collapse** says the probability of forming a halo depends on the **initial density field**
- **Large scale density** field acts as “background” enhancement of probability of forming a halo or “peak”
- **Peak-Background Split** (Mo & White 1997)

Peak-Background Split

- Schematic Picture:



Perturbed Mass Function

- Density fluctuation split

$$\delta = \delta_b + \delta_p$$

- Lowers the threshold for collapse

$$\delta_{cp} = \delta_c - \delta_b$$

so that $\nu = \delta_{cp}/\sigma$

- Taylor expand number density $n_M \equiv dn/d \ln M$

$$n_M + \frac{dn_M}{d\nu} \frac{d\nu}{d\delta_b} \delta_b \dots = n_M \left[1 + \frac{(\nu^2 - 1)}{\sigma\nu} \right]$$

if mass function is given by **Press-Schechter**

$$n_M \propto \nu \exp(-\nu^2/2)$$

Halo Bias

- Halos are **biased tracers** of the “background” dark matter field with a bias $b(M)$ that is given by spherical collapse and the form of the mass function

$$\frac{\delta n_M}{n_M} = [1 + b(M)] \delta$$

- For Press-Schechter

$$b(M) = 1 + \frac{\nu^2 - 1}{\delta_c}$$

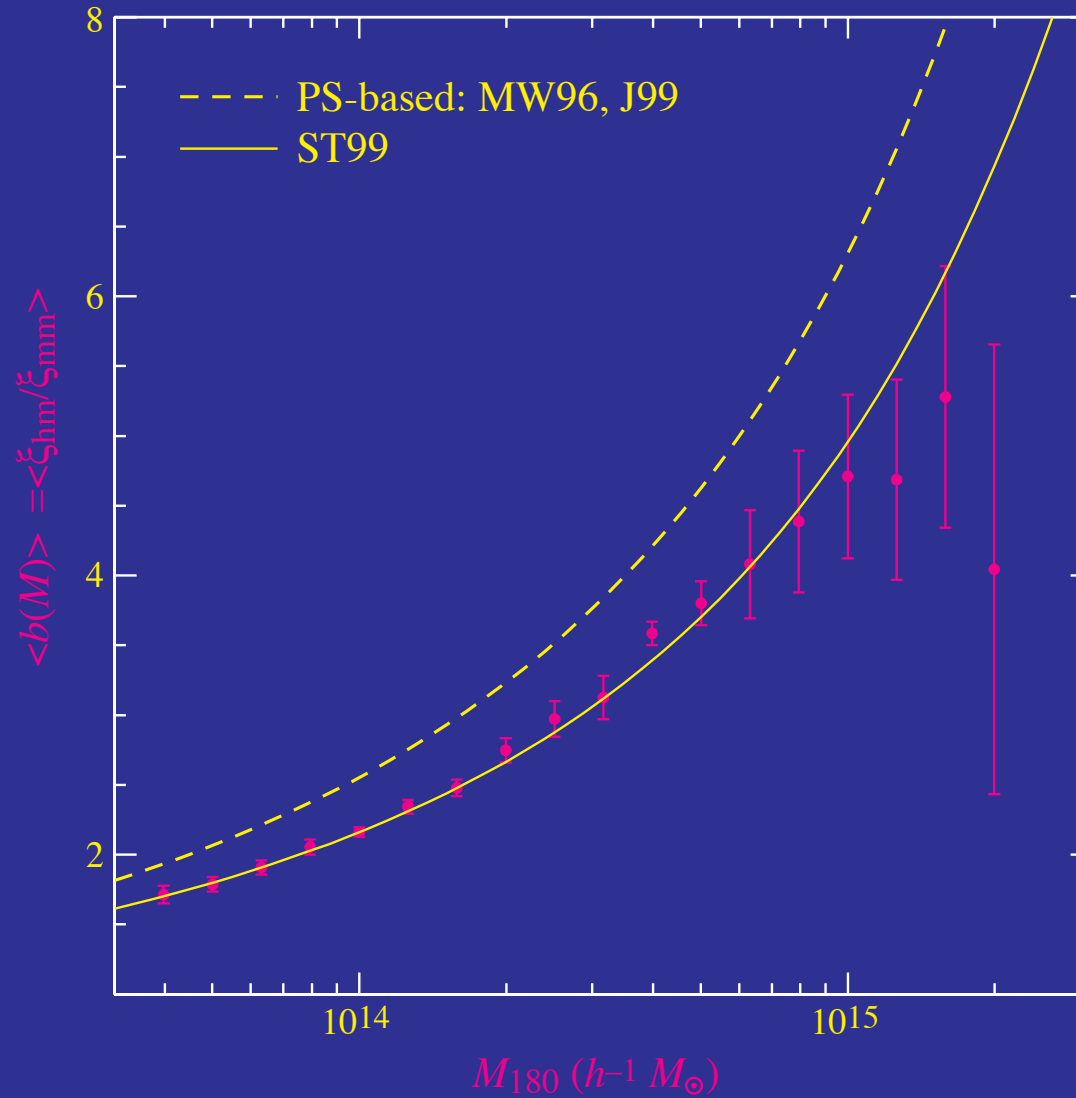
- Improved by the Sheth-Tormen mass function

$$b(M) = 1 + \frac{a\nu^2 - 1}{\delta_c} + \frac{2p}{\delta_c [1 + (a\nu^2)^p]}$$

with $a = 0.75$ and $p = 0.3$ to match simulations.

Numerical Bias

- Example of **halo bias** from a simulation (from [Hu & Kravstov 2002](#))



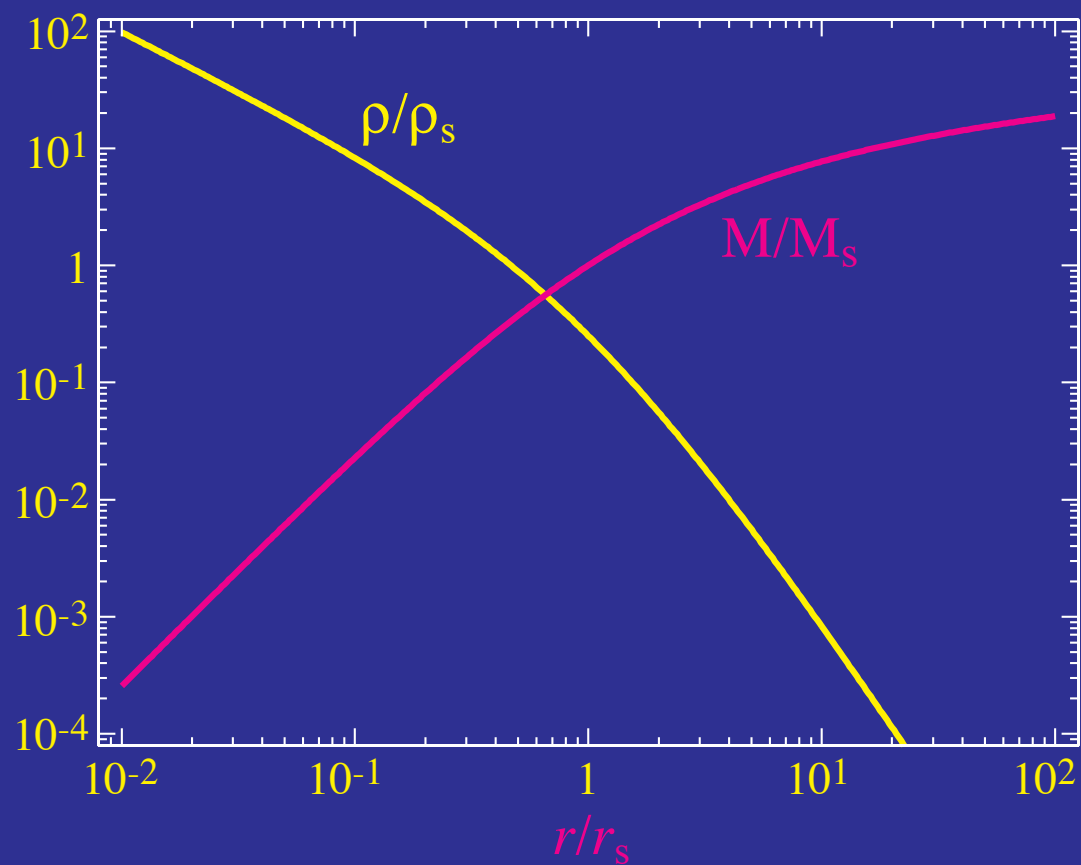
What is a Halo?

- Mass function and halo bias depend on the definition of **mass of a halo**
- Agreement with simulations depend on how **halos are identified**
- Other **observables** (associated galaxies, *X*-ray, SZ) depend on the details of the density profile
- Fortunately, simulations have shown that halos take on a near **universal form** in their **density profile** at least on large scales.

NFW Halo

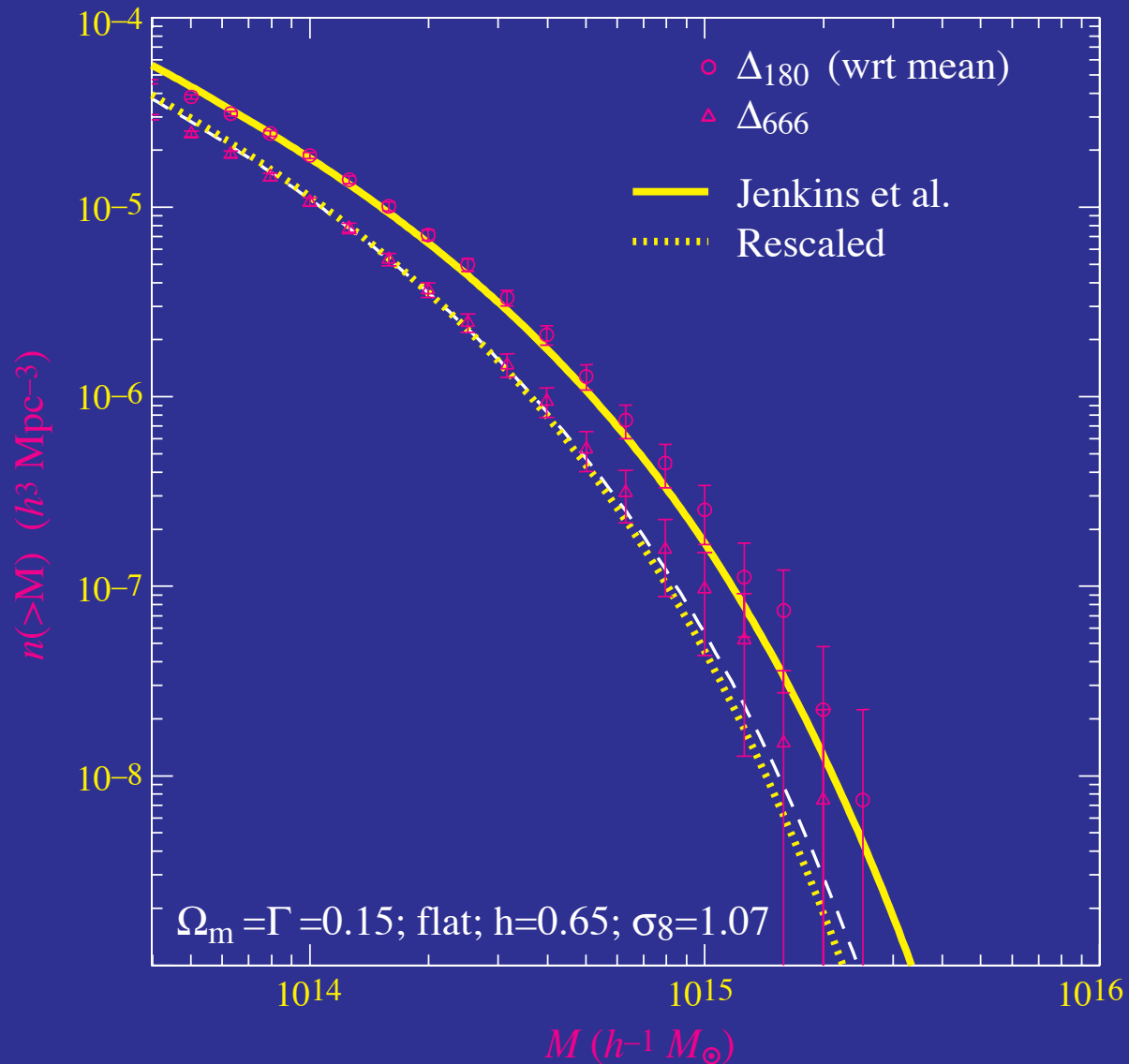
- Density profile well-described by (Navarro, Frenk & White 1997)

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}$$



Transforming the Masses

- NFW profile gives a way of transforming different mass definitions



Lack of Concentration?

- NFW parameters may be recast into M_v , the mass of a halo out to the **virial radius** r_v where the overdensity wrt mean reaches $\Delta_v = 180$.

- **Concentration** parameter

$$c \equiv \frac{r_v}{r_s}$$

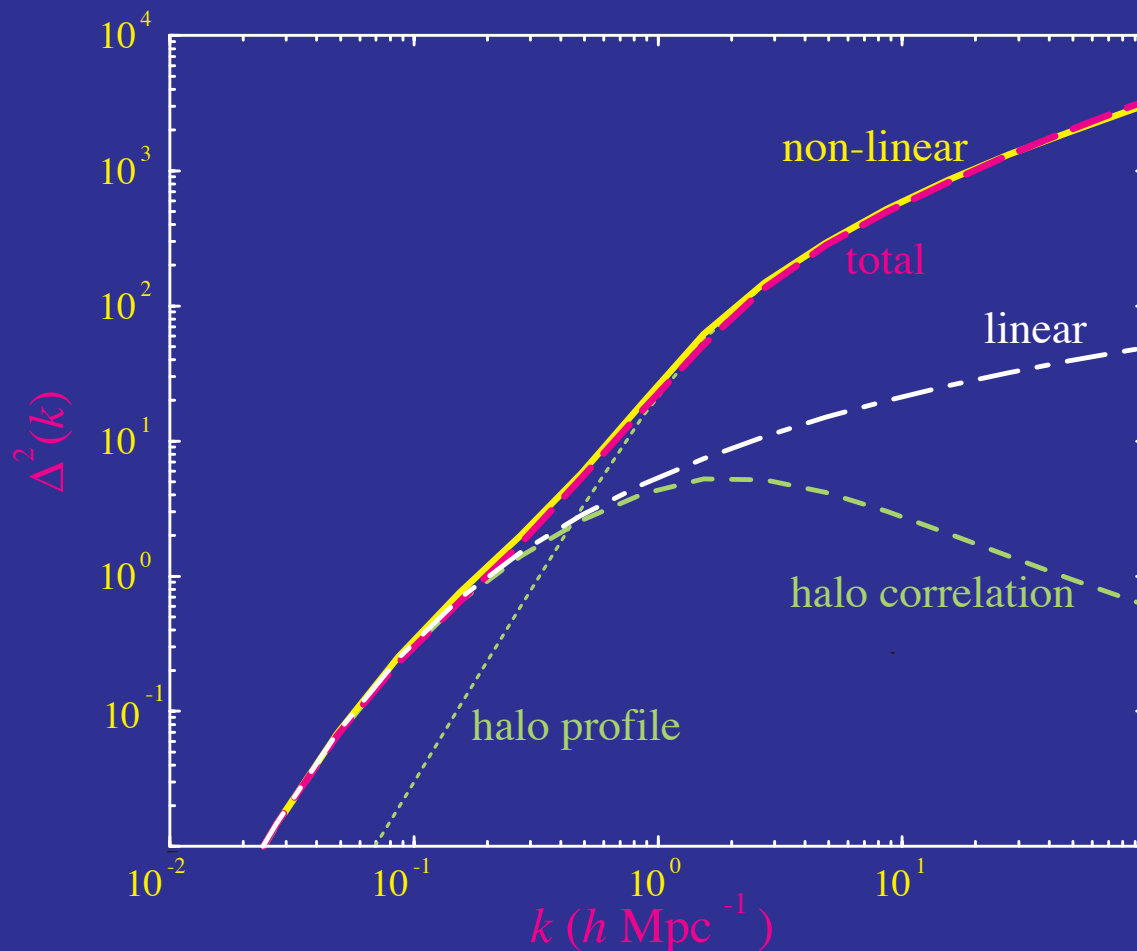
- CDM predicts $c \sim 10$ for M_* halos. **Too centrally concentrated** for galactic rotation curves?
- Possible discrepancy has lead to the exploration of **dark matter alternatives**: warm ($m \sim \text{keV}$) dark matter, self-interacting dark-matter, annihilating dark matter, ultra-light “fuzzy” dark matter, ...

Incredible, Extensible Halo Model

- An industry developed to build **semi-analytic models** for wide variety of **cosmological observables** based on the halo model
- Idea: associate an **observable** (galaxies, gas, ...) with **dark matter halos**
- Let the **halo model** describe the statistics of the observable
- The **overextended** halo model?

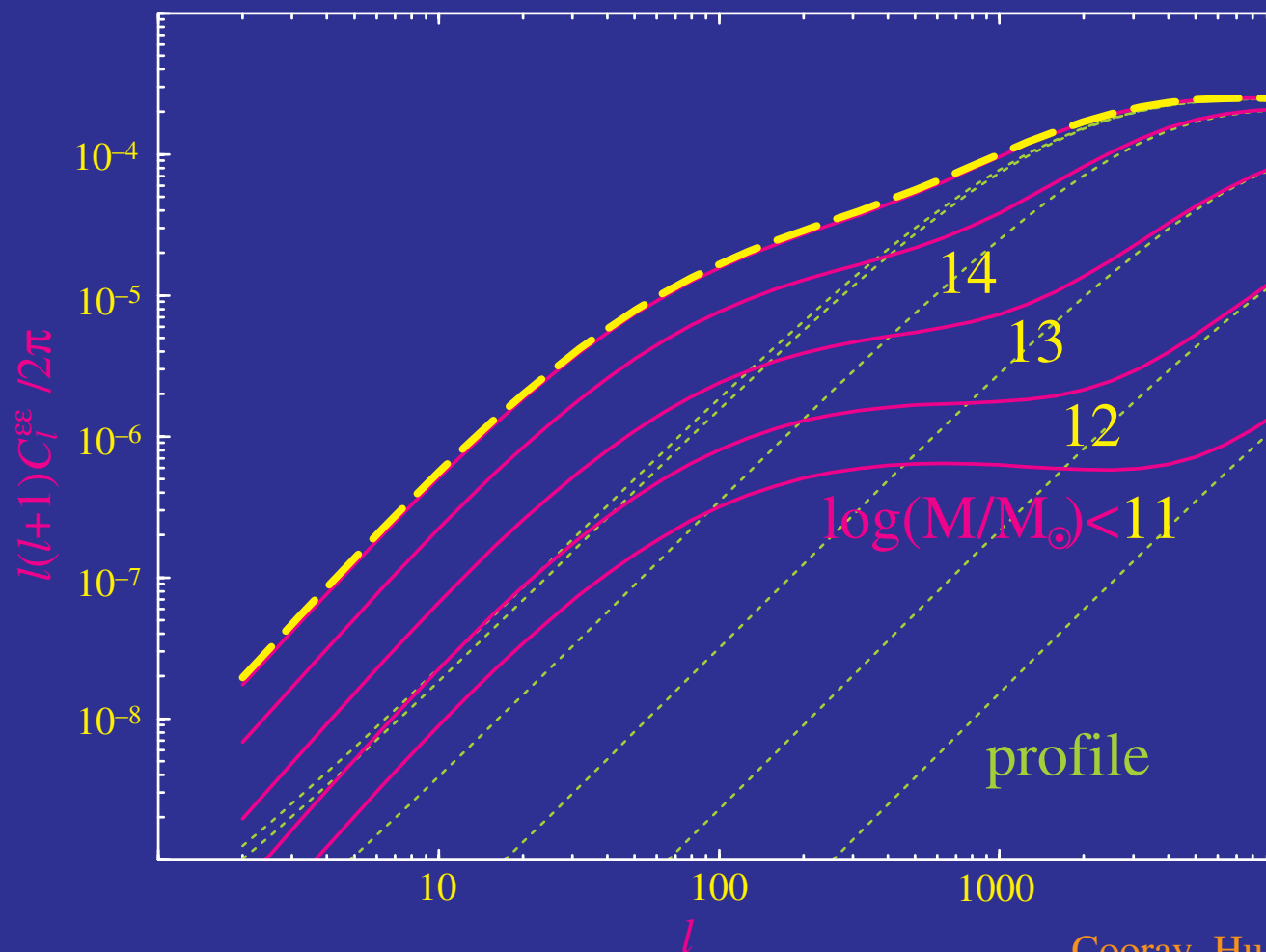
The Halo Model

- **NFW halos**, of abundance n_M given by **mass function**, clustered according to the **halo bias** $b(M)$ and the **linear theory** $P(k)$
- **Power spectrum example:**



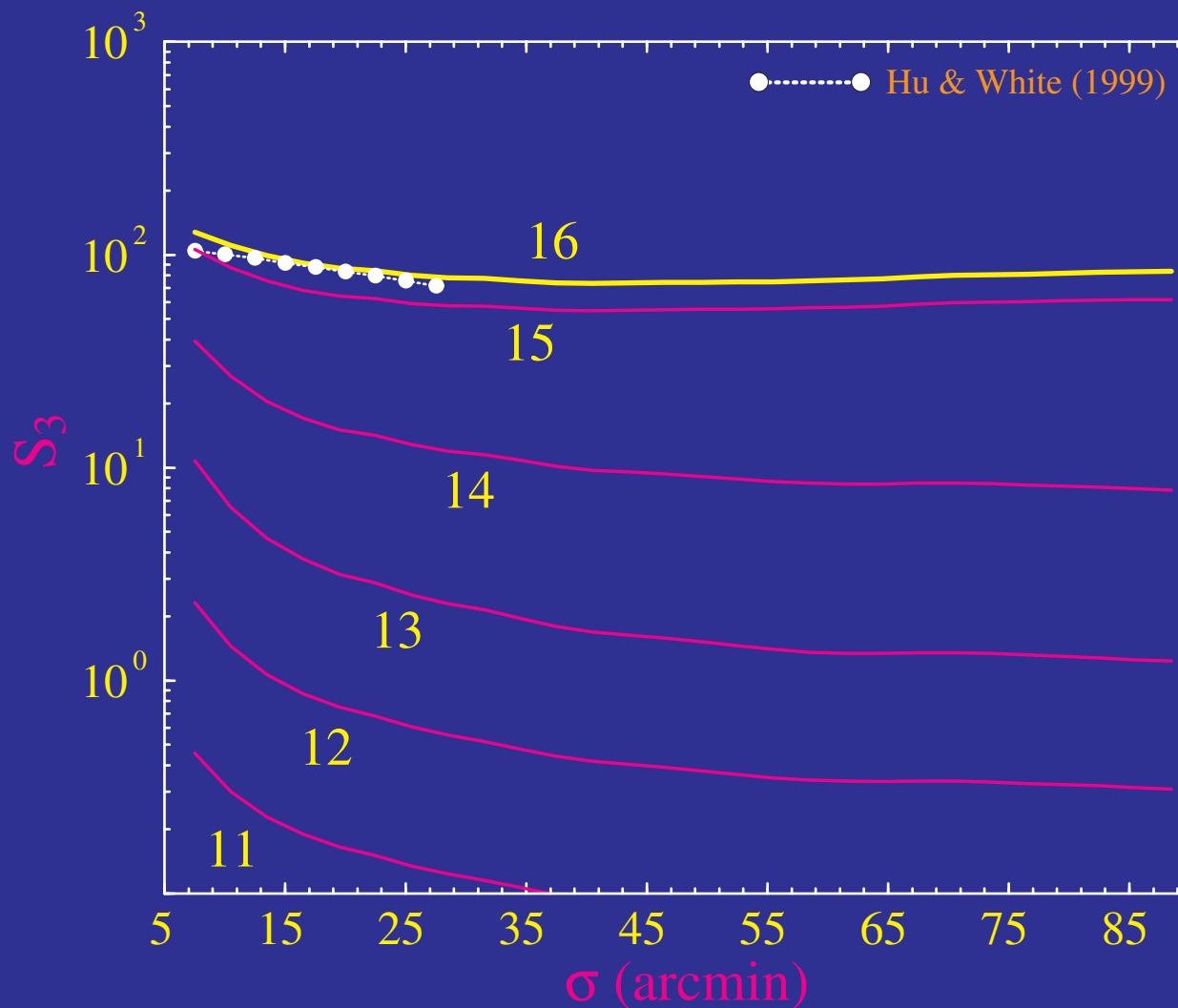
Weak Lensing and the Halo Model

- Power spectrum of shear divided into the halo masses that contribute
- Non-linear regime dominated by halo profile / **individual halos**
increased power spectrum **variance and covariance**



Higher Order Statistics

- Halo model for the bispectrum: S_3 dominated by massive halos



Halo Temperature

- Motivate with **isothermal distribution**, correct from simulations

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$

- Express in terms of **virial mass** M_v enclosed at **virial radius** r_v

$$M_v = \frac{4\pi}{3} r_v^3 \rho_m \Delta_v = \frac{2}{G} r_v \sigma^2$$

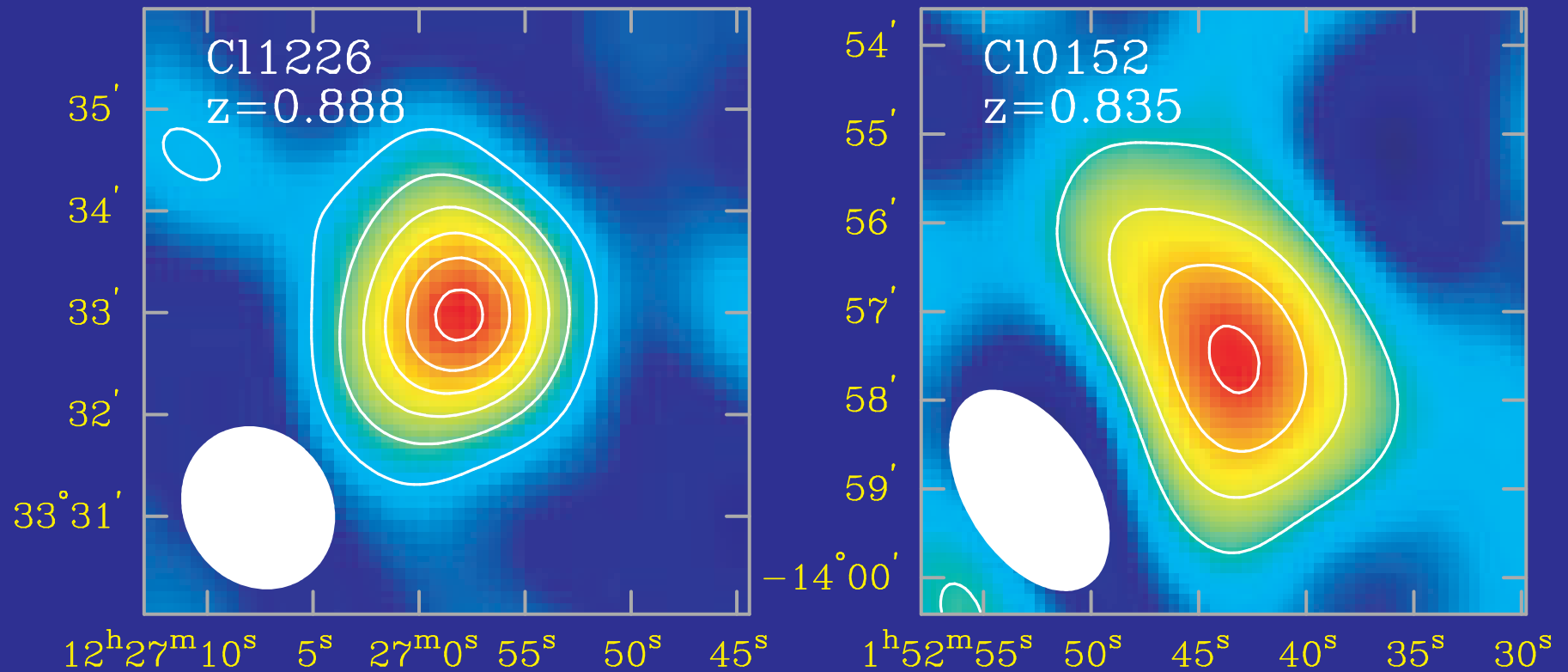
- Eliminate r_v , temperature $T \propto \sigma^2$ velocity dispersion²
- Then $T \propto M_v^{2/3} (\rho_m \Delta_v)^{1/3}$ or

$$\left(\frac{M_v}{10^{15} h^{-1} M_\odot} \right) = \left[\frac{f}{(1+z)(\Omega_m \Delta_v)^{1/3}} \frac{T}{1 \text{keV}} \right]^{3/2}$$

- Theory (X -ray weighted): $f \sim 0.75$; observations $f \sim 0.54$.
Difference is **crucial** in determining cosmology from **cluster counts!**

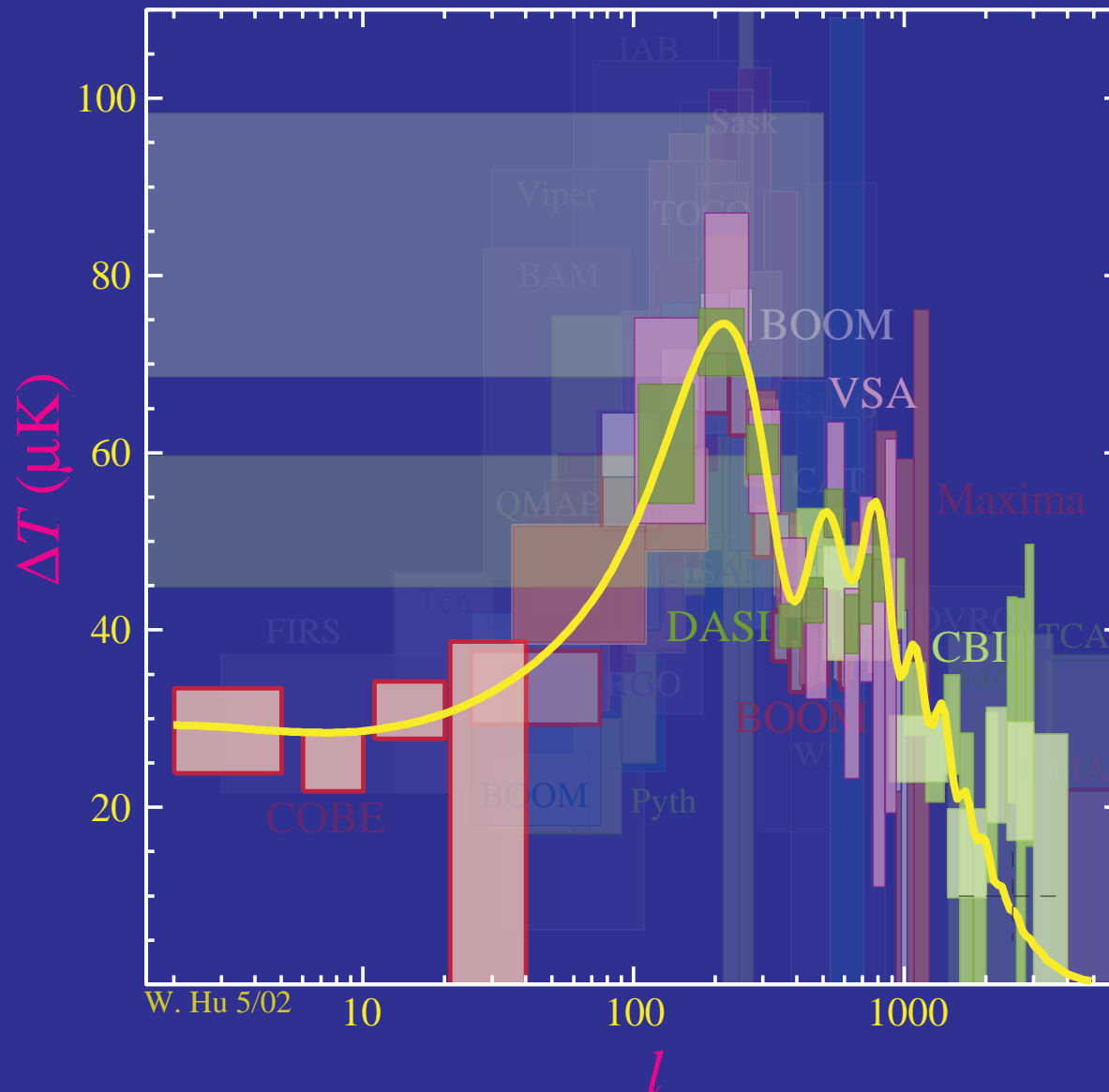
Clusters in the SZE

- Inverse Compton scattering of CMB off hot electrons



Clusters in Power Spectrum?

- Excess in arcminute scale CMB anisotropy from CBI



Sensitivity of SZE Power

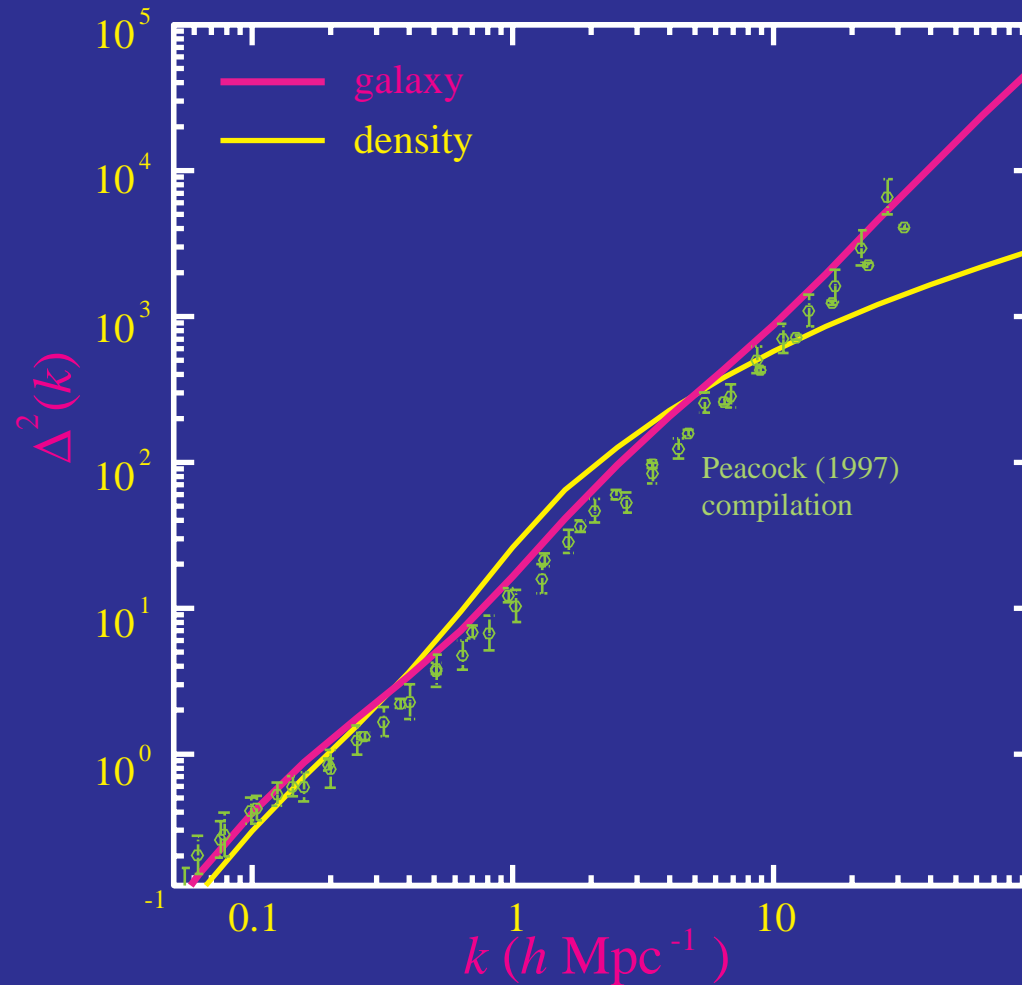
- Amplitude of fluctuations

Sensitivity of SZE Power

- Dark energy

Galaxy Clustering

- Associate **galaxies** with halo of mass M : $N(M)$ (Seljak 2001)



- An explanation of the pure **power law** galaxy spectrum

Summary

- **Dark matter simulations** well-understood and can be modelled with dark matter **halos**
- Halo formation modelled by **spherical collapse**, two magic numbers $\delta_c = 1.686$ and $\Delta_v = 178$
- Halo abundance described by a **mass** function with **exponential** high mass cutoff – **rare clusters** extremely sensitive to power spectrum amplitude and **growth rate** → **dark energy**
Possibly too many small halos or **sub-structure**?
- Halo clustering modelled with peak-background split leading to **halo bias**
- **Halo profile** described by NFW halos
Possibly too high central **concentration**
- Associate an **observable** with a halo → **a halo model**