# Astro 321

# Set 4: Inflation

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# Horizon Problem

- The horizon in a decelerating universe scales as $\eta \propto a^{(1+3w)/2}$, $w > -1/3$. For example in a matter dominated universe

$$\eta \propto a^{1/2}$$

- CMB decoupled at $a_* = 10^{-3}$ so subtends an angle on the sky

$$\frac{\eta_*}{\eta_0} = a_*^{1/2} \approx 0.03 \approx 2^\circ$$

- So why is the CMB sky isotropic to $10^{-5}$ in temperature if it is composed of $\sim 10^4$ causally disconnected regions
- If smooth by fiat, why are there $10^{-5}$ fluctuations correlated on superhorizon scales

# Flatness & Relic Problems

- Flatness problem: why is the radius of curvature larger than the observable universe. (Before the CMB determinations, why is it at least comparable to observable universe $|\Omega_K| \lesssim \Omega_m$)
- Also phrased as a coincidence problem: since $\rho_K \propto a^{-2}$ and $\rho_m \propto a^{-3}$, why would they be comparable today – modern version is dark energy coincidence $\rho_\Lambda = \text{const.}$
- Relic problem – why don't relics like monopoles dominate the energy density
- Inflation is a theory that solves all three problems at once and also supplies a source for density perturbations

# Accelerating Expansion

- In a matter or radiation dominated universe, the horizon grows as a power law in $a$ so that there is no way to establish causal contact on a scale longer than the inverse Hubble length ($1/aH$, comoving coordinates) at any given time: general for a decelerating universe

$$\eta = \int d\ln a \frac{1}{aH(a)}$$

- $H^2 \propto \rho \propto a^{-3(1+w)}$, $aH \propto a^{-(1+3w)/2}$, critical value of $w = -1/3$, the division between acceleration and deceleration
- In an accelerating universe, the Hubble length shrinks in comoving coordinates and so the horizon gets its contribution at the earliest times, e.g. in a cosmological constant universe, the horizon saturates to a constant value

# Causal Contact

- Note confusion in nomenclature: the true horizon always grows meaning that one always sees more and more of the universe. The Hubble length decreases: the difference in conformal time, the distance a photon can travel between two epochs denoted by the scale factor decreases. Regions that were in causal contact, leave causal contact.
- Horizon problem solved if the universe was in an acceleration phase up to $\eta_i$ and the conformal time since then is shorter than the total conformal age

$$\eta_0 \gg \eta_0 - \eta_i$$

$$\text{total distance} \gg \text{distance traveled since inflation}$$

$$\text{apparent horizon}$$

# Flatness & Relic

- Comoving radius of curvature is constant and can even be small compared to the full horizon $R \ll \eta_0$ yet still $\eta_0 \gg R \gg \eta_0 - \eta_i$
- In physical coordinates, the rapid expansion of the universe makes the current observable universe much smaller than the curvature scale
- Likewise, the number density of relics formed before the accelerating (or inflationary) epoch is diluted to make them rare in the current observable volume
- Common to reference time to the end of inflation $\tilde{\eta} \equiv \eta - \eta_i$. Here conformal time is negative during inflation and its value (as a difference in conformal time) reflects the comoving Hubble length - defines leaving the horizon as $k|\tilde{\eta}| = 1$

# Exponential Expansion

- If the accelerating component has equation of state $w = -1$, $\rho =$ const., $H = H_i$ const. so that $a \propto \exp(Ht)$

$$\tilde{\eta} = -\int_a^{a_i} d\ln a \frac{1}{aH} = \frac{1}{aH_i}\Big|_a^{a_i}$$
$$\approx -\frac{1}{aH_i} \qquad (a_i \gg a)$$

- In particular, the current horizon scale $H_0\tilde{\eta}_0 \approx 1$ exited the horizon during inflation at

$$\tilde{\eta}_0 \approx H_0^{-1} = \frac{1}{a_H H_i}$$
$$a_H = \frac{H_0}{H_i}$$

# Sufficient Inflation

- Current horizon scale must have exited the horizon during inflation so that the start of inflation could not be after $a_H$. How long before the end of inflation must it have began?

$$\frac{a_H}{a_i} = \frac{H_0}{H_i a_i}$$

$$\frac{H_0}{H_i} = \sqrt{\frac{\rho_c}{\rho_i}}, \qquad a_i = \frac{T_{\rm CMB}}{T_i}$$

- $\rho_c^{1/4} = 3 \times 10^{-12}$ GeV, $T_{\rm CMB} = 3 \times 10^{-13}$ GeV

$$\frac{a_H}{a_i} = 3 \times 10^{-29} \left( \frac{\rho_i^{1/4}}{10^{14}\text{GeV}} \right)^{-2} \left( \frac{T_i}{10^{10}\text{GeV}} \right)$$

$$\ln \frac{a_i}{a_H} = 65 + 2 \ln \left( \frac{\rho_i^{1/4}}{10^{14}\text{GeV}} \right) - \ln \left( \frac{T_i}{10^{10}\text{GeV}} \right)$$

# Perturbation Generation

- Horizon scale $\tilde{\eta}$ during inflation acts like an even horizon - things leaving causal contact
- Particle creation similar to Hawking radiation from a black hole with hubble length replacing the BH horizon

$$T_{\rm H} \approx H_i$$

- Because $H_i$ remains roughly constant during inflation the result is a scale invariant spectrum of fluctuations due to zero-point fluctuations becoming classical
- Fluctuations in the field driving inflation (inflaton) carry the energy density of the universe and so their zero point fluctuations are net energy density or curvature fluctuations
- Any other light field (gravitational waves, etc...) will also carry scale invariant perturbations but are iso-curvature fluctuations

# Scalar Fields

- Stress-energy tensor of a scalar field

$$T^{\mu}{}_{\nu} = \nabla^{\mu}\varphi\,\nabla_{\nu}\varphi - \frac{1}{2}(\nabla^{\alpha}\varphi\,\nabla_{\alpha}\varphi + 2V)\delta^{\mu}{}_{\nu}\,.$$

- For the background $\langle\phi\rangle \equiv \phi_0$ ($a^{-2}$ from conformal time)

$$\rho_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + V, \quad p_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 - V$$

- So for kinetic dominated $w_\phi = p_\phi/\rho_\phi \to 1$
- And potential dominated $w_\phi = p_\phi/\rho_\phi \to -1$
- A slowly rolling (potential dominated) scalar field can accelerate the expansion and so solve the horizon problem or act as a dark energy candidate

# Equation of Motion

- Can use general equations of motion of dictated by stress energy conservation

$$\dot{\rho}_\phi = -3(\rho_\phi + p_\phi)\frac{\dot{a}}{a}\,,$$

to obtain the equation of motion of the background field $\phi$

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2 V' = 0\,,$$

- In terms of time instead of conformal time

$$\frac{d^2\phi_0}{dt^2} + 3H\frac{d\phi_0}{dt} + V' = 0$$

- Field rolls down potential hill but experiences "Hubble friction" to create slow roll. In slow roll $3H d\phi_0/dt \approx -V'$ and so kinetic energy is determined by field position $\rightarrow$ adiabatic – both kinetic and potential energy determined by single degree of freedom $\phi_0$

# Slow Roll Inflation

- Alternately can derive directly from the Klein-Gordon equation for scalar field
- Scalar field equation of motion $V' \equiv dV/d\phi$

$$\nabla_\mu \nabla^\mu \phi + V'(\phi) = 0$$

so that in the background $\phi = \phi_0$ and

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2 V' = 0$$

$$\frac{d^2\phi_0}{dt^2} + 3H\frac{d\phi_0}{dt} + V' = 0$$

- Simply the continuity equation with the associations

$$\rho_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + V \qquad p_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 - V$$

# Slow Roll Parameters

- Net energy is dominated by potential energy and so acts like a cosmological constant $w \to -1$
- First slow roll parameter

$$\epsilon = \frac{3}{2}(1+w) = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2$$

- Second slow roll parameter $d^2\phi_0/dt^2 \approx 0$, or $\ddot{\phi}_0 \approx (\dot{a}/a)\dot{\phi}_0$

$$\delta = \frac{\ddot{\phi}_0}{\dot{\phi}_0}\left(\frac{\dot{a}}{a}\right)^{-1} - 1 = \epsilon - \frac{1}{8\pi G}\frac{V''}{V}$$

- Slow roll condition $\epsilon, \delta \ll 1$ corresponds to a very flat potential

# Perturbations

- Linearize perturbation $\phi = \phi_0 + \phi_1$ then

$$\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + k^2\phi_1 + a^2 V''\phi_1 = 0$$

  in slow roll inflation $V''$ term negligible

- Implicitly assume that the spatial metric fluctuations (curvature $\mathcal{R}$) vanishes, otherwise covariant derivatives pick these up
- GR: work in the spatially flat slicing and transform back to comoving slicing once done.
- Curvature is local scale factor $a \to (1+\mathcal{R})a$ or $\delta a/a = \mathcal{R}$

$$\mathcal{R} = \frac{\delta a}{a} = \frac{\dot{a}}{a}\delta\eta = \frac{\dot{a}}{a}\frac{\phi_1}{\dot{\phi}_0}$$

  a change in the field value $\phi_1$ defines a change in the epoch that inflation ends, imprinting a curvature fluctuation

# Slow-Roll Evolution

- Rewrite in $u \equiv a\phi$ to remove expansion damping

$$\ddot{u} + [k^2 - 2\left(\frac{\dot{a}}{a}\right)^2]u = 0$$

- or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\rm end}$$
$$\tilde{\eta} = \int_{a_{\rm end}}^{a} \frac{da}{Ha^2} \approx -\frac{1}{aH}$$

- Compact, slow-roll equation:

$$\ddot{u} + [k^2 - \frac{2}{\tilde{\eta}^2}]u = 0$$

# Slow Roll Limit

- Slow roll equation has the exact solution:

$$u = A(k \pm \frac{i}{\tilde{\eta}})e^{\mp ik\tilde{\eta}}$$

- For $|k\tilde{\eta}| \gg 1$ (early times, inside Hubble length) behaves as free oscillator

$$\lim_{|k\tilde{\eta}|\to\infty} u = Ake^{\mp ik\tilde{\eta}}$$

- Normalization $A$ will be set by origin in quantum fluctuations of free field

# Slow Roll Limit

- For $|k\tilde{\eta}| \ll 1$ (late times, $\gg$ Hubble length) fluctuation freezes in

$$\lim_{|k\tilde{\eta}|\to 0} u = \pm\frac{i}{\tilde{\eta}}A = \pm iHaA$$

$$\phi_1 = \pm iHA$$

$$\mathcal{R} = \mp iHA\left(\frac{\dot{a}}{a}\right)\frac{1}{\dot{\phi}_0}$$

- Slow roll replacement

$$\left(\frac{\dot{a}}{a}\right)^2\frac{1}{\dot{\phi}_0^2} = \frac{8\pi G a^2 V}{3}\frac{3}{2a^2 V\epsilon} = \frac{4\pi G}{\epsilon} = \frac{1}{2\epsilon M_{\rm pl}^2}$$

- Comoving curvature power spectrum

$$\Delta_{\mathcal{R}}^2 \equiv \frac{k^3|\mathcal{R}|^2}{2\pi^2} = \frac{k^3}{4\pi^2}\frac{H^2}{\epsilon M_{\rm pl}^2}A^2$$

# Quantum Fluctuations

- Simple harmonic oscillator $\ll$ Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k,\eta)\hat{a} + u^*(k,\eta)\hat{a}^\dagger$$

where $u(k,\eta)$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \qquad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\eta] = i$

$$u(k,\eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

# Quantum Fluctuations

- Zero point fluctuations of ground state

$$
\begin{aligned}
\langle u^2 \rangle &= \langle 0 | u^\dagger u | 0 \rangle \\
&= \langle 0 | (u^* \hat{a}^\dagger + u \hat{a})(u \hat{a} + u^* \hat{a}^\dagger) | 0 \rangle \\
&= \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle |u(k, \tilde{\eta})|^2 \\
&= \langle 0 | [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} | 0 \rangle |u(k, \tilde{\eta})|^2 \\
&= |u(k, \tilde{\eta})|^2 = \frac{1}{2k}
\end{aligned}
$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in. Slow roll equation
- So $A = (2k^3)^{-1/2}$ and curvature power spectrum

$$
\Delta^2_{\mathcal{R}} \equiv \frac{1}{8\pi^2} \frac{H^2}{\epsilon M^2_{\rm pl}}
$$

# Tilt

- Curvature power spectrum is scale invariant to the extent that $H$ is constant
- Scalar spectral index

$$\frac{d\ln \Delta^2_{\mathcal{R}}}{d\ln k} \equiv n_S - 1$$
$$= 2\frac{d\ln H}{d\ln k} - \frac{d\ln \epsilon}{d\ln k}$$

- Evaluate at horizon crossing where fluctuation freezes

$$\frac{d\ln H}{d\ln k}\Big|_{-k\tilde\eta=1} = \frac{k}{H}\frac{dH}{d\tilde\eta}\Big|_{-k\tilde\eta=1}\frac{d\tilde\eta}{dk}\Big|_{-k\tilde\eta=1}$$
$$= \frac{k}{H}(-aH^2\epsilon)\Big|_{-k\tilde\eta=1}\frac{1}{k^2} = -\epsilon$$

where $aH = -1/\tilde\eta = k$

# Tilt

- Evolution of $\epsilon$

$$\frac{d\ln\epsilon}{d\ln k} = -\frac{d\ln\epsilon}{d\ln\tilde{\eta}} = -2(\delta+\epsilon)\frac{\dot{a}}{a}\tilde{\eta} = 2(\delta+\epsilon)$$

- Tilt in the slow-roll approximation

$$n_S = 1 - 4\epsilon - 2\delta$$

# Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation $(K = 0)$, same as scalar field

$$\ddot{h}_{+,\times} + 2\frac{\dot{a}}{a}\dot{h}_{+,\times} + k^2 h_{+,\times} = 0\,.$$

- Acquires quantum fluctuations in same manner as $\phi$. Lagrangian sets the normalization
- Scale-invariant gravitational wave amplitude

$$\Delta^2_{+,\times} = 16\pi G \Delta^2_{\phi_1} = 16\pi G \frac{H^2}{(2\pi)^2} = \frac{H^2}{2\pi^2 M_{\rm pl}^2}$$

# Gravitational Waves

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where $E_i$ is the energy scale of inflation
- Tensor-scalar ratio - various definitions - WMAP standard is

$$r \equiv 4\frac{\Delta_+^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon$$

- Tensor tilt:

$$\frac{d\ln\Delta_+^2}{d\ln k} \equiv n_T = 2\frac{d\ln H}{d\ln k} = -2\epsilon$$

# Gravitational Waves

- Consistency relation between tensor-scalar ratio and tensor tilt

$$r = 16\epsilon = -8n_T$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparision of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself

# Gravitational Wave Phenomenology

- A gravitational wave makes a quadrupolar (transverse-traceless) distortion to metric
- Just like the scale factor or spatial curvature, a temporal variation in its amplitude leaves a residual temperature variation in CMB photons – here anisotropic
- Before recombination, anisotropic variation is eliminated by scattering
- Gravitational wave temperature effect drops sharply at the horizon scale at recombination

# Large Field Models

- For detectable gravitational waves, scalar field must roll by order $M_{\rm pl} = (8\pi G)^{-1/2}$

$$\frac{d\phi_0}{dN} = \frac{d\phi_0}{d\ln a} = \frac{d\phi_0}{dt}\frac{1}{H}$$

- The larger $\epsilon$ is the more the field rolls in an e-fold

$$\epsilon = \frac{r}{16} = \frac{3}{2V}\left(H\frac{d\phi_0}{dN}\right)^2 = \frac{8\pi G}{2}\left(\frac{d\phi_0}{dN}\right)^2$$

- Observable scales span $\Delta N \sim 5$ so

$$\Delta\phi_0 \approx 5\frac{d\phi}{dN} = 5(r/8)^{1/2} M_{\rm pl} \approx 0.6(r/0.1)^{1/2} M_{\rm pl}$$

- Does this make sense as an effective field theory? Lyth (1997)

# Large Field Models

- Large field models include monomial potentials $V(\phi) = A\phi^n$

$$\epsilon \approx \frac{n^2}{16\pi G\phi^2}$$

$$\delta \approx \epsilon - \frac{n(n-1)}{8\pi G\phi^2}$$

- Slow roll requires large field values of $\phi > M_{\rm pl}$
- Thus $\epsilon \sim |\delta|$ and a finite tilt indicates finite $\epsilon$
- Given WMAP tilt, potentially observable gravitational waves

# Small Field Models

- If the field is near an maximum of the potential

$$V(\phi) = V_0 - \frac{1}{2}\mu^2\phi^2$$

- Inflation occurs if the $V_0$ term dominates

$$\epsilon \approx \frac{1}{16\pi G}\frac{\mu^4\phi^2}{V_0^2}$$

$$\delta \approx \epsilon + \frac{1}{8\pi G}\frac{\mu^2}{V_0} \rightarrow \frac{\delta}{\epsilon} = \frac{V_0}{\mu^2\phi^2} \gg 1$$

- Tilt reflects $\delta$: $n_S \approx 1 - 2\delta$ and $\epsilon$ is much smaller
- The field does not roll significantly during inflation and gravitational waves are negligible

# Hybrid Models

- If the field is rolling toward a minimum of the potential

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2$$

- Slow roll parameters similar to small field but a real $m^2$

$$\epsilon \approx \frac{1}{16\pi G}\frac{m^4\phi^2}{V_0^2}$$
$$\delta \approx \epsilon - \frac{1}{8\pi G}\frac{m^2}{V_0}$$

- Then $V_0$ domination $\epsilon, \delta < 0$ and $n_S > 1$ - blue tilt
- For $m^2$ domination, monomial-like.
- Intermediate cases with intermediate predictions - can have observable gravity waves but does not require it.

# Hybrid Models

- But how does inflation end? $V_0$ remains as field settles to minimum
- Implemented as multiple field model with $V_0$ supplied by second field
- Inflation ends when rolling triggers motion in the second field to the true joint minimum