Chapter 4

Multifluid Perturbation Theory

It is the nature of things that they are *ties to each other.*
–Chuang-tzu, 20

In the standard scenario, small perturbations in the early universe grow by gravitational instability to form the wealth of structure observable today. At the early stages of this process, relevant for CMB work, fluctuations are still small and can be described in linear perturbation theory. What makes the problem non-trivial is the fact that different components such as the photons, baryons, neutrinos, and collisionless dark matter, have different equations of state and interactions. It is therefore necessary to employ a fully relativistic multifluid treatment to describe the coupled evolution of the individual particle species.

In this chapter, we discuss the framework for the evolution of fluctuations. Since in linear theory, each normal mode evolves independently we undertake a mode by mode analysis. In open universes, this decomposition implies a lack of structure above the curvature scale for random-field perturbations. We show why this arises and how it might be avoided by generalizing the random field condition [111]. The evolution itself is governed by the energy momentum conservation equations in the perturbed space-time and feeds back into the metric fluctuations through the Einstein equations. In Newtonian gauge, they generalize the Poisson equation familiar from the non-relativistic theory.

It is often useful to express the evolution in other gauges, *e.g.* the popular synchronous gauge and the total matter gauge. We discuss the general issue of gauge transformations and their effect on the interpretation of perturbations. Various aspects of the evolution appear simplest for different choices of gauge. Those that involve the photons are most straightforward to analyze in Newtonian form where redshift and infall correspond to classical intuition. On the other hand, the evolution of the matter and consequently the metric perturbations themselves becomes simpler on its own rest frame. We therefore advocate a hybrid representation for perturbations based on the so-called "gauge invariant" formalism.

4.1 Normal Mode Decomposition

4.1.1 Laplacian Eigenfunctions

Any scalar fluctuation may be decomposed in eigenmodes of the Laplacian

$$\nabla^2 Q \equiv \gamma^{ij} Q_{|ij} = -k^2 Q, \tag{4.1}$$

where '|' represents a covariant derivative with respect to the three metric γ_{ij} of constant curvature $K = -H_0^2(1-\Omega_0-\Omega_\Lambda)$. In flat space $\gamma_{ij} = \delta_{ij}$, and Q is a plane wave $\exp(i\mathbf{k}\cdot\mathbf{x})$. As we shall discuss further in §4.1.3, the eigenfunctions are complete for $k \geq \sqrt{-K}$. Therefore we define the transform of an arbitrary square integrable function $F(\mathbf{x})$ as [110, 111]

$$F(\mathbf{x}) = \sum_{|\mathbf{k}|\geq\sqrt{-K}} F(\mathbf{k})Q(\mathbf{x},\mathbf{k}) = \frac{V}{(2\pi)^3}\int_{|\mathbf{k}|\geq\sqrt{-K}}^{\infty} d^3k F(\mathbf{k})Q(\mathbf{x},\mathbf{k}). \tag{4.2}$$

In the literature, an alternate convention is often employed in order to make the form appear more like the flat space convention [175, 83],

$$F(\mathbf{x}) = \sum_{\tilde{k}} \tilde{F}(\tilde{\mathbf{k}})Q(\mathbf{x},\tilde{\mathbf{k}}) = \frac{V}{(2\pi)^3}\int_0^{\infty} d^3\tilde{k}\tilde{F}(\mathbf{k})Q(\mathbf{x},\tilde{\mathbf{k}}), \tag{4.3}$$

where the auxiliary variable $\tilde{k}^2 = k^2 + K$. The relation between the two conventions is

$$\begin{aligned}\tilde{k}|\tilde{F}(\tilde{k})|^2 &= k|F(k)|^2 \\ &= (\tilde{k}^2 - K)^{1/2}|F([\tilde{k}^2 - K]^{1/2})|^2\end{aligned} \tag{4.4}$$

and should be kept in mind when comparing predictions. In particular, note that power law conditions in $\tilde{k}$ for $\tilde{F}$ are not the same as in k for F.

Vectors and tensors needed in the description of the velocity and stress perturbation can be constructed from the covariant derivatives of Q and the metric tensor,

$$\begin{aligned}Q_i &\equiv -k^{-1}Q_{|i}, \\ Q_{ij} &\equiv k^{-2}Q_{|ij} + \frac{1}{3}\gamma_{ij}Q,\end{aligned} \tag{4.5}$$

where the indices are to be raised and lowered by the three metric γ_{ij} and γ^{ij}. The following identities can be derived from these definitions and the communtation relation for covariant derivatives (see *e.g.* [173] eqn. 8.5.1) [99]

$$\begin{aligned} Q_i^{|i} &= kQ, \\ \nabla^2 Q_i &= -(k^2 - 3K)Q_i, \\ Q_{i|j} &= -k(Q_{ij} - \frac{1}{3}\gamma_{ij}Q), \\ Q^i{}_i &= 0, \\ Q_{ij}{}^{|j} &= \frac{2}{3}k^{-1}(k^2 - 3K)Q_i, \end{aligned} \tag{4.6}$$

and will be useful in simplifying the evolution equations.

4.1.2 Radial Representation

To gain intuition about these functions, let us examine an explicit representation. In radial coordinates, the 3-metric becomes

$$\gamma_{ij}dx^i dx^j = -K^{-1}[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)], \tag{4.7}$$

where the distance is scaled to the curvature radius $\chi = \sqrt{-K}\eta$. Notice that the (comoving) angular diameter distance is $\sinh\chi$, leading to an exponential increase in the surface area of a shell with radial distance $\chi \gg 1$. The Laplacian can now be written as

$$\gamma^{ij}Q_{|ij} = -K\sinh^{-2}\chi\left[\frac{\partial}{\partial\chi}\left(\sinh^2\chi\frac{\partial Q}{\partial\chi}\right) + \sin^{-1}\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Q}{\partial\theta}\right) + \sin^{-2}\theta\frac{\partial^2 Q}{\partial\phi^2}\right]. \tag{4.8}$$

Since the angular part is independent of curvature, we may separate variables such that $Q = X_\nu^\ell(\chi)Y_\ell^m(\theta,\phi)$, where $\nu^2 = \tilde{k}^2/(-K) = -(k^2/K + 1)$. From equation (4.8), it is obvious that the spherically symmetric $\ell = 0$ function is

$$X_\nu^0(\chi) = \frac{\sin(\nu\chi)}{\nu\sinh\chi}. \tag{4.9}$$

As expected, the change in the area element from a flat to curved geometry causes $\chi \to \sinh\chi$ in the denominator. The higher modes are explicitly given by [106, 71]

$$X_\nu^\ell(\chi) = (-1)^{\ell+1}M_\ell^{-1/2}\nu^{-2}(\nu^2+1)^{-\ell/2}\sinh^\ell\chi\frac{d^{\ell+1}(\cos\nu\chi)}{d(\cosh\chi)^{\ell+1}}, \tag{4.10}$$

and become $j_\ell(k\eta)$ in the flat space limit, where

$$\begin{aligned} M_\ell &\equiv \prod_{\ell'=0}^{\ell} K_{\ell'}, \\ K_0 &= 1, \\ K_\ell &= 1-(\ell^2-1)K/k^2, \qquad \ell \geq 1, \end{aligned} \tag{4.11}$$

which all reduce to unity as $K \to 0$. This factor represents our convention for the normalization of the open universe functions,

$$\int X_\nu^\ell(\chi) X_{\nu'}^{\ell'}(\chi) \sinh^2\chi d\chi = \frac{\pi}{2\nu^2}\delta(\nu-\nu')\delta(\ell-\ell'), \tag{4.12}$$

and is chosen to be similar to the flat space case. In the literature, the normalization is often chosen such that $\tilde{X}_\nu^\ell = X_\nu^\ell M_\ell^{-1/2}$ is employed as the radial eigenfunction [175, 83].

It is often more convenient to generate these functions from their recursion relations. One particularly useful relation is [3]

$$\frac{d}{d\eta} X_\nu^\ell = \frac{\ell}{2\ell+1} k K_\ell^{1/2} X_\nu^{\ell-1} + \frac{\ell+1}{2\ell+1} k K_{\ell+1}^{1/2} X_\nu^{\ell+1}. \tag{4.13}$$

Since radiation free streams on radial null geodesics, we shall see that the collisionless Boltzmann equation takes on the same form as equation (4.13).

4.1.3 Completeness and Super Curvature Modes

Open universe eigenfunctions possess the curious property that they are complete for $k \geq \sqrt{-K}$. Mathematically, this is easier to see with a choice of three metric such that $\gamma_{ij} = \delta_{ij}/(-Kz^2)$, the so-called flat-surface representation [175, 111]. In this system $-\infty < x < \infty$, $-\infty < y < \infty$, $0 \leq z < \infty$ and surfaces of constant z are flat. The Laplacian

$$\nabla^2 Q = -Kz^2 \left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2} \right) + Kz\frac{\partial Q}{\partial z}, \tag{4.14}$$

has eigenfunctions

$$Q = z \exp(ik_1 x + ik_2 y) K_{i\nu}(k_\perp z), \tag{4.15}$$

where $K_{i\nu}$ is the modified Bessel function and $k_\perp^2 = k_1^2 + k_2^2$. Since the x and y dependences are just those of plane waves, which we know are complete, we need only concern ourselves with the z coordinate. As pointed out by Wilson [175], it reduces to a Kontorovich-Lebedev transform,

$$g(y) = \int_0^\infty f(x) K_{ix}(y) dx,$$
$$f(x) = 2\pi^{-2} x \sinh(x\pi) \int_0^\infty g(y) K_{ix}(y) y^{-1} dy, \tag{4.16}$$

i.e. there exists a completeness relation,

$$\int_0^\infty d\nu \nu \sinh(\pi\nu) K_{i\nu}(k_\perp z) K_{i\nu}(k_\perp z') = \frac{\pi^2}{2} z \delta(z - z'). \tag{4.17}$$

Therefore an arbitrary square integrable function $F(\mathbf{x})$ can be decomposed into a sum of eigenmodes of $\nu \geq 0$,

$$F(\mathbf{x}) = \int_0^\infty \nu \sinh(\pi\nu) d\nu \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_2 F(\mathbf{k}) Q(\mathbf{x}, \mathbf{k}),$$
$$F(\mathbf{k}) = \frac{1}{2\pi^4} \int_0^\infty \frac{dz}{z^3} \int_0^\infty dx \int_0^\infty dy F(\mathbf{x}) Q(\mathbf{x}, \mathbf{k}), \tag{4.18}$$

where Q is given by equation (4.15) and $\mathbf{x} = (x, y, z)$ and $\mathbf{k} = (k_1, k_2, \nu)$. Since $\nu \geq 0$ implies $k \geq \sqrt{-K}$, this establishes the claimed completeness.

This completeness property leads to a seemingly bizarre consequence if we consider random fields, *i.e.* randomly phased superpositions of these eigenfunctions. To see this, return to the radial representation. In Fig. 4.1, we plot the spherically symmetric $\ell = 0$ mode given by equation (4.9). Notice that its first zero is at $\chi = \pi/\nu$. This is related to the completeness property: as $\nu \to 0$, we can obtain arbitrarily large structures. For this reason, ν or more specifically $\tilde{k} = \nu\sqrt{-K}$ is often thought of as the wavenumber [175, 95]. However, the *amplitude* of the structure above the curvature scale is suppressed as $e^{-\chi}$. Prominent structure lies only at the curvature scale as $\nu \to 0$. In this sense, k should be regarded as the effective wavelength. This is important to bear in mind when considering the meaning of "scale invariant" fluctuations. In fact, the $e^{-\chi}$ behavior is *independent* of the wavenumber and ℓ, if $\chi \gg 1$.

This peculiarity in the eigenmodes has significant consequences. Any random phase superposition of the eigenmodes X_ν^ℓ will have exponentially suppressed structure larger than the curvature radius. Even though completeness tells us that arbitrarily large structure can be built out of the X_ν^ℓ functions, it *cannot* be done without correlating the modes. This is true even if the function is square integrable, *i.e.* has support only to a finite radius possibly above the curvature scale.

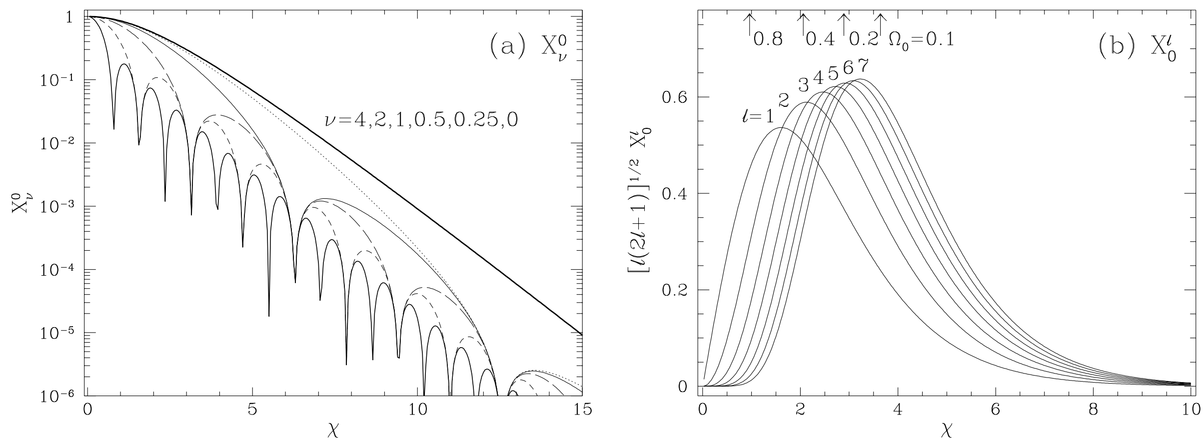

Figure 4.1: Open Radial Eigenfunctions

(a) The isotropic $\ell = 0$ function for several values of the wavenumber ν. The zero crossing moves out to arbitrarily large scales as $\nu \to 0$, reflecting completeness. However, the function retains prominent structure only near the curvature scale $\chi \simeq 1$. A random superposition of these low ν modes cannot produce more than exponentially decaying structure larger than the curvature scale. (b) Low order multipoles in the asymptotic limit $\nu \to 0$. If most power lies on the curvature scale, the ℓ-mode corresponding to the angle that the curvature radius subtends will dominate the anisotropy. The normalization is appropriate for comparing contributions to the anisotropy $\ell(2\ell+1)C_\ell/4\pi$. Also shown is the location of the horizon $\chi = \eta_0\sqrt{-K}$ for several values of Ω_0. If contributions to the anisotropy come from a sufficiently early epoch, the dominant ℓ-mode for the curvature scale will peak at this value (see *e.g.* Fig. 6.10).

Is the lack of structure above the curvature scale reasonable? The fundamental difference between open and flat universes is that the volume increases exponentially with the radial coordinate above the curvature scale $V(\chi_c) \sim [\sinh(2\chi_c) - 2\chi_c]$. Structure above the curvature scale implies correlations over vast volumes [95]. It is in fact difficult to conceive of a model where correlations do not die exponentially above the curvature radius. The random phase hypothesis has been proven to be valid for inflationary perturbations in a pre-existing open geometry [110] and only mildly violated for bubble nucleated open universes [180].

Lyth and Woszcynza [111] show that the simplest way to generalize random fields to include supercurvature scale structure is to employ an overcomplete set of eigenfunctions extended by analytic continuation of the modes to $k \to 0$. Of course, random phase conditions in the overcomplete set can alternatively be expressed as initially phase correlated modes of the complete set. In linear theory, the evolution of each mode is independent and thus there is no distinction between the two. Including supercurvature perturbations merely amounts to extending the treatment to the full range of k: $0 \leq k < \infty$. All of the equations presented here may be extended in this manner with the understanding that $\nu \to |\nu|$.

4.1.4 Higher Angular Functions

We will often need to represent a general function of position $\mathbf{x}$ *and* angular direction $\boldsymbol{\gamma}$, *e.g.* for the radiation distribution. As we have seen, vector and tensors constructed from Q and its covariant derivatives can be used to represent dipoles and quadrupoles, $G_1 = \gamma^i Q_i$ and $G_2 = \frac{3}{2}\gamma^i\gamma^j Q_{ij}$. We can generalize these considerations and form the full multipole decomposition [175]

$$F(\mathbf{x},\boldsymbol{\gamma}) = \sum_{\tilde{\mathbf{k}}}\sum_{\ell=0}^{\infty}\tilde{F}_\ell(\mathbf{k})G_\ell(\mathbf{x},\boldsymbol{\gamma},\mathbf{k}), \tag{4.19}$$

where

$$G_\ell(\mathbf{x},\boldsymbol{\gamma},\mathbf{k}) = (-k)^{-\ell}Q_{|i_1\ldots i_\ell}(\mathbf{x},\mathbf{k})P_\ell^{i_1\ldots i_\ell}(\mathbf{x},\boldsymbol{\gamma}), \tag{4.20}$$

and

$$\begin{aligned} P_0 &= 1, \qquad P_1^i = \gamma^i, \\ P_2^{ij} &= \frac{1}{2}(3\gamma^i\gamma^j - \gamma^{ij}), \\ P_{\ell+1}^{i_1\ldots i_{\ell+1}} &= \frac{2\ell+1}{\ell+1}\gamma^{(i_1}P_\ell^{i_2\ldots i_{\ell+1})} - \frac{\ell}{\ell+1}\gamma^{(i_1 i_2}P_{\ell-1}^{i_3..i_{\ell+1})}, \end{aligned} \tag{4.21}$$

with parentheses denoting symmetrization about the indices. For flat space, this becomes $G_\ell = (-i)^\ell \exp(i\mathbf{k}\cdot\mathbf{x})P_\ell(\mathbf{k}\cdot\boldsymbol{\gamma})$, where P_ℓ is an ordinary Legendre polynomial. Notice that along a path defined by fixed $\boldsymbol{\gamma}$, the flat G_ℓ becomes $j_\ell(k\eta)$ after averaging over k-directions. Traveling on a fixed direction away from a point is the same as following a radial path outwards. Thus fluctuations *along* this path can be decomposed in the radial eigenfunction. It is therefore no surprise that G_ℓ obeys a recursion relation similar to X_ν^ℓ,

$$\begin{aligned} \gamma^i G_{\ell|i} &= \frac{d}{d\eta}G[\mathbf{x}(\eta),\boldsymbol{\gamma}(\eta)] = \dot{x}^i\frac{\partial}{\partial x^i}G_\ell + \dot{\gamma}^i\frac{\partial}{\partial\gamma^i}G_\ell \\ &= k\left\{\frac{\ell}{2\ell+1}K_\ell G_{\ell-1} - \frac{\ell+1}{2\ell+1}G_{\ell+1}\right\}, \end{aligned} \tag{4.22}$$

which follows from equation (4.20) and (4.21) via an exercise in combinatorics involving commutations of covariant derivatives [64]. Here we take $\mathbf{x}(\eta)$ to be the integral path along $\boldsymbol{\gamma}$. By comparing equations (4.13) and (4.22), the open universe generalization of the relation between G_ℓ and the radial eigenfunction is now apparent:

$$G_\ell[\mathbf{x}(\eta), \boldsymbol{\gamma}(\eta)] = M_\ell^{1/2} X_\nu^\ell(\eta). \tag{4.23}$$

The only conceptual difference is that for the radial path that we decompose fluctuations on, $\boldsymbol{\gamma}$ is not constant. The normalization also suggests that to maintain close similarity to the flat space case, the multipole moments should be redefined as

$$F(\mathbf{x}, \boldsymbol{\gamma}) = \sum_{|\mathbf{k}| \ge \sqrt{-K}} \sum_{\ell=0}^{\infty} F_\ell(\mathbf{k}) M_\ell^{-1/2} G_\ell(\mathbf{x}, \boldsymbol{\gamma}, \mathbf{k}), \tag{4.24}$$

which again differ from the conventions of [175, 83] by a factor $M_\ell^{1/2}$.

4.2 Newtonian Gauge Evolution

4.2.1 Metric Fluctuations

In linear theory, the evolution of each k mode is independent. We can therefore assume without loss of generality that the equation of motion for the kth mode can be obtained by taking a metric of the form,

$$\begin{aligned} g_{00} &= -(a/a_0)^2(1 + 2\Psi Q), \\ g_{0i} &= 0, \\ g_{ij} &= (a/a_0)^2(1 + 2\Phi Q)\gamma_{ij}, \end{aligned} \tag{4.25}$$

assuming the Newtonian representation, and correspondingly

$$\begin{aligned} g^{00} &= -(a_0/a)^2(1 - 2\Psi Q), \\ g^{0i} &= 0, \\ g^{ij} &= (a_0/a)^2(1 - 2\Phi Q)\gamma_{ij}, \end{aligned} \tag{4.26}$$

where employ the notation $\Psi(\eta, \mathbf{x}) = \Psi(\eta) Q(\mathbf{x})$, *etc.* and drop the k index where no confusion will arise. Note that we have switched from time to conformal time as the zero component. The Christoffel symbols can now be written as

$$\Gamma^0_{00} = \frac{\dot{a}}{a} + \dot{\Psi} Q,$$

$$
\begin{aligned}
\Gamma^0{}_{0i} &= -k\Psi Q_i, \\
\Gamma^i{}_{00} &= -k\Psi Q^i, \\
\Gamma^i{}_{0j} &= (\frac{\dot{a}}{a} + \dot{\Phi} Q)\delta^i{}_j, \\
\Gamma^0{}_{ij} &= [\frac{\dot{a}}{a} + (-2\frac{\dot{a}}{a}\Psi + 2\frac{\dot{a}}{a}\Phi + \dot{\Phi})Q]\gamma_{ij}, \\
\Gamma^i{}_{jk} &= {}^{(s)}\Gamma^i{}_{jk} - k\Phi(\delta^i{}_j Q_k + \delta^i{}_k Q_j - \gamma_{jk}Q^i),
\end{aligned} \tag{4.27}
$$

where ${}^{(s)}\Gamma^i{}_{jk}$ is the Christoffel symbol on the unperturbed 3-surface γ_{ij}.

Finally we can write the Einstein tensor as $G_{\mu\nu} = \bar{G}_{\mu\nu} + \delta G_{\mu\nu}$, where

$$
\begin{aligned}
\bar{G}^0{}_0 &= -3\left(\frac{a_0}{a}\right)^2 \left[\left(\frac{\dot{a}}{a}\right)^2 + K\right], \\
\bar{G}^i{}_j &= -\left(\frac{a_0}{a^2}\right)^2 \left[2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + K\right]\delta^i{}_j, \\
\bar{G}^0{}_i &= \bar{G}^i{}_0 = 0
\end{aligned} \tag{4.28}
$$

are the background contributions and

$$
\begin{aligned}
\delta G^0{}_0 &= 2\left(\frac{a_0}{a}\right)^2 \left[3\left(\frac{\dot{a}}{a}\right)^2 \Psi - 3\frac{\dot{a}}{a}\dot{\Phi} - (k^2 - 3K)\Phi\right] Q, \\
\delta G^0{}_i &= 2\left(\frac{a_0}{a}\right)^2 \left[\frac{\dot{a}}{a}k\Psi - k\dot{\Phi}\right] Q_i, \\
\delta G^i{}_0 &= -2\left(\frac{a_0}{a}\right)^2 \left[\frac{\dot{a}}{a}k\Psi - k\dot{\Phi}\right] Q^i, \\
\delta G^i{}_j &= 2\left(\frac{a_0}{a}\right)^2 \left\{\left[2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right]\Psi + \frac{\dot{a}}{a}[\dot{\Psi} - \dot{\Phi}]\right. \\
&\quad \left. -\frac{k^2}{3}\Psi - \ddot{\Phi} - \frac{\dot{a}}{a}\Phi - \frac{1}{3}(k^2 - 3K)\Phi\right\}\delta^i{}_j Q \\
&\quad -\left(\frac{a_0}{a}\right)^2 k^2(\Psi + \Phi)Q^i{}_j,
\end{aligned} \tag{4.29}
$$

are the first order contributions from the metric fluctuations.

4.2.2 Conservation Equations

The equations of motion under gravitational interactions are most easily obtained by employing the conservation equations. The stress-energy tensor of a non-interacting fluid is covariantly conserved $T^{\mu\nu}{}_{;\mu} = 0$. The $\nu = 0$ equation gives energy density conservation, *i.e.* the continuity equation; the $\nu = i$ equations give momentum conservation, *i.e.* the Euler equation. To first order, the stress energy tensor of a fluid x, possibly itself a composite of different particle species, is

$$\begin{aligned} T^0_{\ 0} &= -(1+\delta_x Q)\rho_x, \\ T^0_{\ i} &= (\rho_x + p_x)V_x Q_i, \\ T^j_{\ 0} &= -(\rho_x + p_x)V_x Q^j, \\ T^i_{\ j} &= p_x(\delta^i_{\ j} + \frac{\delta p_x}{p_x}\delta^i_{\ j}Q + \Pi_x Q^i_{\ j}), \end{aligned} \tag{4.30}$$

where ρ_x is the energy density, p_x is the pressure, $\delta_x = \delta\rho_x/\rho_x$ and Π_x is the anisotropic stress of the fluid.

Continuity Equation

The zeroth component of the conservation equation becomes

$$\begin{aligned} -\partial_0 T^{00} &= \partial_i T^{i0} + \Gamma^0_{\ \alpha\beta}T^{\alpha\beta} + \Gamma^\alpha_{\ \alpha\beta}T^{0\beta} \\ &= T^{i0}_{\ \ |i} + 2\Gamma^0_{\ 00}T^{00} + \Gamma^0_{\ ij}T^{ij} + \Gamma^i_{\ i0}T^{00}, \end{aligned} \tag{4.31}$$

where we have dropped second order terms. For pedagogical reasons, let us evaluate each term explicitly

$$\begin{aligned} T^{00} &= (1+\delta_x Q - 2\Psi Q)\,(a_0/a)^2\rho_x, \\ \partial_0 T^{00} &= [(1+\delta_x Q - 2\Psi Q)(\frac{\dot\rho_x}{\rho_x} - 2\frac{\dot a}{a}) + (\dot\delta_x - 2\dot\Psi Q)]\,(a_0/a)^2\rho_x, \\ T^{i0}_{\ \ |i} &= (1+w_x)kV_x Q\,(a_0/a)^2\rho_x, \\ \Gamma^0_{\ 00}T^{00} &= [\frac{\dot a}{a}(1+\delta_x Q - 2\Psi Q) + \dot\Psi Q]\,(a_0/a)^2\rho_x, \\ \Gamma^0_{\ ij}T^{ij} &= 3w_x[\frac{\dot a}{a}(1+\frac{\delta p_x}{p_x}Q - 2\Psi Q) + \dot\Phi Q]\,(a_0/a)^2\rho_x, \\ \Gamma^i_{\ i0}T^{00} &= 3[\frac{\dot a}{a}(1+\delta_x Q - 2\Psi Q) + \dot\Phi Q]\,(a_0/a)^2\rho_x, \end{aligned} \tag{4.32}$$

where $w_x \equiv p_x/\rho_x$ gives the equation of state of the fluid.

The zeroth order equation becomes

$$\frac{\dot\rho_x}{\rho_x} = -3(1+w_x)\frac{\dot a}{a}. \tag{4.33}$$

For a constant w_x, $\rho_x \propto a^{-3(1+w_x)}$, *i.e.* $w_r = \frac{1}{3}$ and $\rho_r \propto a^{-4}$ for the radiation, $w_m \simeq 0$ and $\rho_m \propto a^{-3}$ for the matter, and $w_v = -1$ and $\rho_v =$ constant for the vacuum or cosmological constant contribution. The first order equation is the continuity equation for perturbations,

$$\dot{\delta}_x = -(1+w_x)(kV_x + 3\dot{\Phi}) - 3\frac{\dot{a}}{a}\delta w_x, \tag{4.34}$$

where the fluctuation in the equation of state

$$\begin{aligned} \delta w_x &= \frac{p_x + \delta p_x}{\rho_x + \delta\rho_x} - w_x \\ &= \left(\frac{\delta p_x}{\delta \rho_x} - w_x\right)\delta_x. \end{aligned} \tag{4.35}$$

This may occur for example if the temperature of a non-relativistic fluid is spatially varying and can be important at late times when astrophysical processes can inject energy in local regions.

We can recast equation (4.34) into the form

$$\frac{d}{d\eta}\left(\frac{\delta_x}{1+w_x}\right) = -(kV_x + 3\dot{\Phi}) - 3\frac{\dot{a}}{a}w_x\Gamma_x, \tag{4.36}$$

where the entropy fluctuation is

$$w_x\Gamma_x = (\delta p_x/\delta\rho_x - c_x^2)\delta_x, \tag{4.37}$$

with the sound speed $c_x^2 \equiv \dot{p}_x/\dot{\rho}_x$. Here we have used the relation

$$\begin{aligned} \dot{w}_x &= \frac{\dot{\rho}_x}{\rho_x}(\frac{\dot{p}_x}{\dot{\rho}_x} - w_x) \\ &= -3(1+w_x)(c_x^2 - w_x)\frac{\dot{a}}{a}, \end{aligned} \tag{4.38}$$

which follows from equation (4.33). Entropy fluctuations are generated if the fluid is composed of species for which both the equation of state and the number density fluctuations differ. For a single particle fluid, this term vanishes.

Let us interpret equation (4.36). In the limit of an ultra-relativistic or non-relativistic single particle fluid, the quantity

$$\frac{\delta_x}{1+w_x} = \frac{\delta n_x}{n_x} \tag{4.39}$$

is the number density fluctuation in the fluid. Equation (4.36) thus reduces to the ordinary continuity equation for the number density of particles in the absence of creation and annihilation processes. Aside from the usual kV_x term, there is a $3\dot{\Phi}$ term. We have shown in §2.1.2 that this term represents the stretching of space due to the presence of space curvature, *i.e.* the spatial metric has a factor $a(1+\Phi)$. Just as the expansion term a causes an a^{-3} dilution of number density, there is a corresponding perturbative effect of 3Φ from the fluctuation. For the radiation energy density, there is also an effect on the wavelength which brings the total to 4Φ as equation (4.34) requires.

Euler Equation

Similarly, the conservation of momentum equation is obtained from the space component of the conservation equation,

$$\begin{aligned} -\partial_0 T^{0i} &= \partial_j T^{ji} + \Gamma^i{}_{\alpha\beta} T^{\alpha\beta} + \Gamma^\alpha{}_{\alpha\beta} T^{i\beta} \\ &= T^{ji}{}_{|j} + \Gamma^i{}_{00} T^{00} + 2\Gamma^i{}_{0j} T^{0j} + \Gamma^0{}_{00} T^{i0} + \Gamma^0{}_{0j} T^{ij} + \Gamma^j{}_{j0} T^{i0}. \end{aligned} \tag{4.40}$$

Explicitly, the contributions are

$$\begin{aligned} \partial_0 T^{0i} &= [(1+w_x)(\frac{\dot\rho_x}{\rho_x} - 2\frac{\dot a}{a}) + \dot w_x] V_x Q^i\,(a_0/a)^2 \rho_x, \\ T^{ij}{}_{|j} &= [-\frac{\delta p_x}{p_x} + \frac{2}{3}(1-3K/k^2)\Pi_x] k w_x Q^i\,(a_0/a)^2\rho_x, \\ \Gamma^i{}_{00} T^{00} &= -k\Psi Q^i\,(a_0/a)^2\rho_x, \\ \Gamma^i{}_{0j} T^{0j} &= \frac{\dot a}{a}(1+w_x) Q^i\,(a_0/a)^2\rho_x, \\ &= \Gamma^0{}_{00} T^{i0} \\ &= \frac{1}{3}\Gamma^j{}_{j0} T^{i0}. \end{aligned} \tag{4.41}$$

These terms are all first order in the perturbation and form the Euler equation

$$\dot V_x = -\frac{\dot a}{a}(1-3w_x)V_x - \frac{\dot w_x}{1+w_x}V_x + \frac{\delta p_x/\delta\rho_x}{1+w_x}k\delta_x - \frac{2}{3}\frac{w_x}{1+w_x}(1-3K/k^2)k\Pi_x + k\Psi. \tag{4.42}$$

Employing equation (4.38) for the time variation of the equation of state and equation (4.37) for the entropy, we can rewrite this as

$$\dot V_x + \frac{\dot a}{a}(1-3c_x^2)V_x = \frac{c_x^2}{1+w_x}k\delta_x + \frac{w_x}{1+w_x}k\Gamma_x - \frac{2}{3}\frac{w_x}{1+w_x}(1-3K/k^2)k\Pi_x + k\Psi. \tag{4.43}$$

The gradient of the gravitational potential provides a source to velocities from infall. The expansion causes a drag term on the matter but not the radiation. This is because the expansion redshifts particle momenta as a^{-1}. For massive particles, the velocity consequently decays as $V_m \propto a^{-1}$. For radiation, the particle energy or equivalently the temperature of the distribution redshifts. The bulk velocity V_r represents a *fractional* temperature fluctuation with a dipole signature. Therefore, the decay scales out. Stress in the fluid, both isotropic (pressure) and anisotropic, prevents gravitational infall. The pressure contribution is separated into an acoustic part proportional to the sound speed c_x^2 and an entropy part which contributes if the fluid is composed of more than one particle species.

4.2.3 Total Matter and Its Components

If the fluid x in the last section is taken to be the total matter T, equations (4.34) and (4.43) describe the evolution of the whole system. However, even considering the metric fluctuations Ψ and Φ as external fields, the system of equations is not closed since the anisotropic stress Π_T and the entropy Γ_T remain to be defined. The fluid must therefore be broken into particle components for which these quantities are known.

We can reconstruct the total matter variables from the components via the relations,

$$\rho_T \delta_T = \sum_i \rho_i \delta_i, \tag{4.44}$$

$$\delta p_T = \sum_i \delta p_i, \tag{4.45}$$

$$(\rho_T + p_T) V_T = \sum_i (\rho_i + p_i) V_i, \tag{4.46}$$

$$p_T \Pi_T = \sum_i p_i \Pi_i, \tag{4.47}$$

$$\dot\rho_T c_T^2 = \sum_i \dot\rho_i c_i^2, \tag{4.48}$$

which follow from the form of the stress-energy tensor. Vacuum contributions are usually not included in the total matter. Similarly, the entropy fluctuation can be written

$$\begin{aligned} p_T \Gamma_T &= \delta p_T - \frac{\dot p_T}{\dot\rho_T} \delta\rho_T \\ &= \sum_i \delta p_i - \frac{\dot p_i}{\dot\rho_i} \delta\rho_i + \left(\frac{\dot p_i}{\dot\rho_i} - \frac{\dot p_T}{\dot\rho_T} \right) \delta\rho_i \\ &= \sum_i p_i \Gamma_i + (c_i^2 - c_T^2) \delta\rho_i. \end{aligned} \tag{4.49}$$

Even supposing the entropy of the individual fluids vanishes, there can be a non-zero Γ_T due to differing density contrasts between the components which have different equations of state w_i. If the universe consists of non-relativistic matter and fully-relativistic radiation only, there are only two relevant equations of state $w_r = 1/3$ for the radiation and $w_m \simeq 0$ for the matter. The relative entropy contribution then becomes,

$$\Gamma_T = -\frac{4}{3}\frac{1-3w_T}{1+w_T}S, \tag{4.50}$$

where the S is the fluctuation in the matter to radiation number density

$$S = \delta(n_m/n_r) = \delta_m - \frac{3}{4}\delta_r, \tag{4.51}$$

and is itself commonly referred to as the entropy fluctuation for obvious reasons.

Although covariant conservation applies equally well to particle constituents as to the total fluid, we have assumed in the last section that the species were non-interacting. To generalize the conservation equations, we must consider momentum transfer between components. Let us see how this is done.

4.2.4 Radiation

In the standard model for particle physics, the universe contains photons and three flavors of massless neutrinos as its radiation components. For the photons, we must consider the momentum transfer with the baryons through Compton scattering. We have in fact already obtained the full evolution equation for the photon component through the derivation of the Boltzmann equation in Chapter 2. In real space, the temperature fluctuation is given by [see equation (2.63)]

$$\begin{aligned}\frac{d}{d\eta}(\Theta+\Psi) &\equiv \dot{\Theta}+\dot{\Psi}+\dot{x}^i\frac{\partial}{\partial x^i}(\Theta+\Psi)+\dot{\gamma}^i\frac{\partial}{\partial\gamma^i}(\Theta+\Psi)\\ &= \dot{\Psi}-\dot{\Phi}+\dot{\tau}(\Theta_0-\Theta+\gamma_i v_b^i+\frac{1}{16}\gamma_i\gamma_j\Pi_\gamma^{ij}),\end{aligned} \tag{4.52}$$

recall that τ is the Compton optical depth, $\Theta_0 = \delta_\gamma/4$ is the isotropic component of Θ, and Π_γ^{ij} the quadrupole moments of the photon energy density are given by equation (2.64).

The angular fluctuations in a given spatial mode Q can be expressed by the multipole decomposition of equation (4.24)

$$\Theta(\eta,\mathbf{x},\boldsymbol{\gamma}) = \sum_{\ell=0}^{\infty}\Theta_\ell(\eta)M_\ell^{-1/2}G_\ell(\mathbf{x},\boldsymbol{\gamma}). \tag{4.53}$$

Be employing the recursion relations (4.22), we can break equation (4.52) into the standard hierarchy of coupled equations for the ℓ-modes:

$$\dot{\Theta}_0 = -\frac{k}{3}\Theta_1 - \dot{\Phi},$$

$$\begin{aligned}\dot{\Theta}_1 &= k\left[\Theta_0 + \Psi - \frac{2}{5}K_2^{1/2}\Theta_2\right] - \dot{\tau}(\Theta_1 - V_b),\\ \dot{\Theta}_2 &= k\left[\frac{2}{3}K_2^{1/2}\Theta_1 - \frac{3}{7}K_3^{1/2}\Theta_3\right] - \frac{9}{10}\dot{\tau}\Theta_2,\\ \dot{\Theta}_\ell &= k\left[\frac{\ell}{2\ell-1}K_\ell^{1/2}\Theta_{\ell-1} - \frac{\ell+1}{2\ell+3}K_{\ell+1}^{1/2}\Theta_{\ell+1}\right] - \dot{\tau}\Theta_\ell, \quad (\ell > 2)\end{aligned} \tag{4.54}$$

where $\gamma_i v_b^i(\mathbf{x}) = V_b G_1(\mathbf{x}, \boldsymbol{\gamma})$ and recall $K_\ell = 1 - (\ell^2 - 1)K/k^2$. Since $V_\gamma = \Theta_1$, comparison with equation (4.43) gives the relation between the anisotropic stress perturbation of the photons and the quadrupole moment

$$\Pi_\gamma = \frac{12}{5}(1 - 3K/k^2)^{-1/2}\Theta_2. \tag{4.55}$$

Thus anisotropic stress is generated by the streaming of radiation from equation (4.54) once the mode enters the horizon $k\eta \gtrsim 1$. The appearance of the curvature term is simply an artifact of our convention for the multipole moment normalization. For supercurvature modes, it is also a convenient rescaling of the anisotropic stress since in the Euler equation (4.43), the term $(1 - 3K/k^2)k\Pi_\gamma = 12(k^2 - 3K)^{1/2}\Theta_2/5$ is manifestly finite as $k \to 0$.

By analogy to equation (4.54), we can immediately write down the corresponding Boltzmann equation for (massless) neutrino temperature perturbations $N(\eta, \mathbf{x}, \boldsymbol{\gamma})$ with the replacements

$$\Theta_\ell \to N_\ell, \dot{\tau} \to 0, \tag{4.56}$$

in equation (4.54). This is sufficient since neutrino decoupling occurs before any scale of interest enters the horizon.

4.2.5 Matter

There are two non-relativistic components of dynamical importance to consider: the baryons and collisionless cold dark matter. The collisionless evolution equations for the baryons are given by (4.34) and (4.43) with $w_b \simeq 0$ if $T_e/m_e \ll 1$. However, before recombination, Compton scattering transfers momentum between the photons and baryons. It is unnecessary to derive the baryon transport equation from first principles since the momentum of the total photon-baryon fluid is still conserved. Conservation of momentum yields

$$(\rho_\gamma + p_\gamma)\delta V_\gamma = \frac{4}{3}\rho_\gamma \delta V_\gamma = \rho_b \delta V_b. \tag{4.57}$$

Thus equations (4.34), (4.43) and (4.54) imply

$$\begin{aligned}\dot{\delta}_b &= -kV_b - 3\dot{\Phi}, \\ \dot{V}_b &= -\frac{\dot{a}}{a}V_b + k\Psi + \dot{\tau}(V_\gamma - V_b)/R,\end{aligned} \tag{4.58}$$

where $R = 3\rho_b/4\rho_\gamma$. The baryon continuity equation can also be combined with the photon continuity equation [$\ell = 0$ in (4.54)] to obtain

$$\dot{\delta}_b = -k(V_b - V_\gamma) + \frac{3}{4}\dot{\delta}_\gamma. \tag{4.59}$$

As we shall see, this is useful since it has a gauge invariant interpretation: it represents the evolution of the number density or entropy fluctuation [see equation (4.51)]. Finally, any collisionless non-relativistic component can described with equation (4.58) by dropping the interaction term $\dot{\tau}$. The equations can also be obtained from (4.34) and (4.43) by noting that for a collisionless massive particle, the pressure, sound speed and entropy fluctuation may be ignored.

4.2.6 Einstein Equations

The Einstein equations close the system by expressing the time evolution of the metric in terms of the matter sources,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{4.60}$$

where $T_{\mu\nu}$ is now the total stress-energy tensor (including any vacuum contributions). The background equations give matter conservation for the space-space equation. This is already contained in equation (4.33). The time-space component vanishes leaving only the time-time component

$$\left(\frac{\dot{a}}{a}\right)^2 + K = \frac{8\pi G}{3}\left(\frac{a}{a_0}\right)^2(\rho_T + \rho_v), \tag{4.61}$$

where ρ_v is the vacuum contribution and we have used equation (4.28). This evolution equation for the scale factor is often written in terms of the Hubble parameter,

$$\begin{aligned}H^2 &\equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \left(\frac{\dot{a}}{a}\frac{a_0}{a}\right)^2 \\ &= \left(\frac{a_0}{a}\right)^4 \frac{a_{eq}+a}{a_{eq}+a_0}\Omega_0 H_0^2 - \left(\frac{a_0}{a}\right)^2 K + \Omega_\Lambda H_0^2,\end{aligned} \tag{4.62}$$

where recall $\Omega_0 = \rho_T/\rho_{crit}$ and $\Omega_\Lambda = \rho_v/\rho_{crit}$ with $\rho_{crit} = 3H_0^2/8\pi G$. Here a_{eq} is the epoch of matter-radiation equality. Notice that as a function of a, the expansion will be dominated successively by radiation, matter, curvature, and Λ. Of course, either or both of the latter terms may be absent in the real universe.

The first order equations govern the evolution of Ψ and Φ. They are the time-time term,

$$3\left(\frac{\dot{a}}{a}\right)^2\Psi - 3\frac{\dot{a}}{a}\dot{\Phi} - (k^2 - 3K)\Phi = -4\pi G\left(\frac{a}{a_0}\right)^2 \rho_T\delta_T, \tag{4.63}$$

the time-space term,

$$\frac{\dot{a}}{a}\Psi - \dot{\Phi} = 4\pi G\left(\frac{a}{a_0}\right)^2 (1 + w_T)\rho_T V_T/k, \tag{4.64}$$

and the traceless space-space term

$$k^2(\Psi + \Phi) = -8\pi G\left(\frac{a}{a_0}\right)^2 p_T\Pi_T. \tag{4.65}$$

The other equations express the conservation laws which we have already found. Equations (4.63) and (4.64) can be combined to form the generalized Poisson equation

$$(k^2 - 3K)\Phi = 4\pi G\left(\frac{a}{a_0}\right)^2 \rho_T[\delta_T + 3\frac{\dot{a}}{a}(1 + w_T)V_T/k]. \tag{4.66}$$

Equations (4.65) and (4.66) form the two fundamental evolution equations for metric perturbations in Newtonian gauge.

Notice that the form of (4.66) reduces to the ordinary Poisson equation of Newtonian mechanics if the last term in the brackets is negligible. Employing the matter continuity equation (4.34), this occurs when $k\eta \gg 1$, *i.e.* when the fluctuation is well inside the horizon as one would expect. This extra piece represents a relativistic effect and depends on the frame of reference in which the perturbation is defined. This suggests that we can simplify the form and interpretation of the evolution equations by a clever choice of gauge.

4.3 Gauge

Sayings from a perspective work nine times out of ten, wise sayings work seven times out of ten. Adaptive sayings are new every day, smooth them out on the whetstone of Heaven.

–Chuang-tzu, 27

Fluctuations are defined on hypersurfaces of constant time. Since in general relativity, we can choose the coordinate system arbitrarily, this leads to an ambiguity in the definition of fluctuations referred to as *gauge freedom*. There is *no* gauge invariant meaning to density fluctuations. For example, even a completely homogeneous and isotropic

Friedmann-Robertson-Walker space can be expressed with an inhomogeneous metric by choosing an alternate time slicing that is warped (see Fig. 4.2). Conversely, a fluctuation can be thought of as existing in a homogeneous and isotropic universe where the initial time slicing is altered (see §5.1.2). Two principles are worthwhile to keep in mind when considering the gauge:

1. Choose a gauge whose coordinates are completely fixed.
2. Choose a gauge where the physical interpretation and/or form of the evolution is simplest.

The first condition is the most important. Historically, much confusion has arisen from the use of a particular gauge choice, the synchronous gauge, which *alone* does not fix the coordinates entirely [133]. An ambiguity in the mapping onto this gauge appears, for example, at the initial conditions. Usually this problem is solved by completely specifying the initial hypersurface. Improper mapping can lead to artificial "gauge modes" in the solution. The second point is that given gauge freedom exists, we may as well exploit it by choosing one which simplifies either the calculation or the interpretation. It turns out that the two often conflict. For this reason, we advocate a hybrid choice of representation for fluctuations.

How is a hybrid choice implemented? This is the realm of the so-called "gauge invariant" formalism. Let us consider for a moment the meaning of the term gauge invariant. If the coordinates are completely specified, the fluctuations are real geometric objects and may be represented in any coordinate system. They are therefore manifestly gauge invariant. However, in the new frame they may take on a different *interpretation*, *e.g.* density fluctuations in general will not remain density fluctuations. The "gauge invariant" program reduces to the task of writing down fluctuations in a given gauge in terms of quantities in an arbitrary gauge. It is therefore a problem in mapping. The only quantities that are not "gauge invariant" in this sense are those that are ill defined, *i.e.* represent fluctuations in a gauge whose coordinates have not been completely fixed. This should be distinguished from objects that actually have a gauge invariant interpretation. As we shall see, quantities such as anisotropies of $\ell \geq 2$ are the same in any frame. This is because the coordinate system is defined by a scalar function in space to describe the "warping" of the time slicing and a vector to define the "boost," leaving higher order quantities invariant.

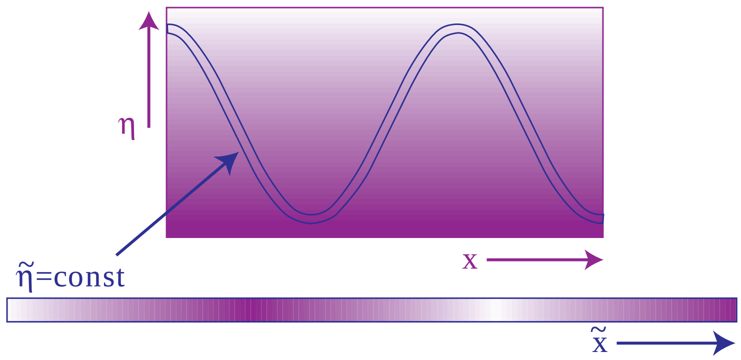

Figure 4.2: Gauge Ambiguity

Gauge ambiguity refers to the freedom to choose the time slicing on which perturbations are defined. In this simple example, a homogeneous FRW universe appears to have density perturbations for a warped choice of time slicing. One usually employs a set of standard "observers" to define the coordinate slicing. The Newtonian gauge boosts observers into a frame where the expansion rate looks isotropic (shear free). The synchronous gauge can be implemented to follow the collisionless non-relativistic particles. The total matter gauge employs the rest frame of the total energy density fluctuations.

4.3.1 Gauge Transformations

The most general form of a metric perturbed by scalar fluctuations is [99]

$$\begin{aligned} g_{00} &= -(a/a_0)^2[1+2A^G Q], \\ g_{0j} &= -(a/a_0)^2 B^G Q_j, \\ g_{ij} &= (a/a_0)^2[\gamma_{ij} + 2H_L^G Q\gamma_{ij} + 2H_T^G Q_{ij}], \end{aligned} \tag{4.67}$$

where the superscript G is employed to remind the reader that the actual values vary from gauge to gauge. A gauge transformation is a change in the correspondence between the perturbation and the background represented by the coordinate shift

$$\begin{aligned} \tilde{\eta} &= \eta + TQ, \\ \tilde{x}^i &= x^i + LQ^i. \end{aligned} \tag{4.68}$$

T corresponds to a choice in time slicing and L the choice of the spatial coordinate grid. They transform the metric as

$$\begin{aligned} \tilde{g}_{\mu\nu}(\eta, x^i) &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\eta - TQ, x^i - LQ^i) \\ &\simeq g_{\mu\nu}(\eta, x^i) + g_{\alpha\nu}\delta x^\alpha{}_{,\mu} + g_{\alpha\mu}\delta x^\alpha{}_{,\nu} - g_{\mu\nu,\lambda}\delta x^\lambda. \end{aligned} \tag{4.69}$$

From this, we obtain the relations for the metric fluctuations

$$\begin{aligned}
A^{\tilde{G}} &= A^G - \dot{T} - \frac{\dot{a}}{a}T, \\
B^{\tilde{G}} &= B^G + \dot{L} + kT, \\
H_L^{\tilde{G}} &= H_L^G - \frac{k}{3}L - \frac{\dot{a}}{a}T, \\
H_T^{\tilde{G}} &= H_T^G + kL.
\end{aligned} \tag{4.70}$$

An analogous treatment of the stress energy tensor shows that

$$\begin{aligned}
v_x^{\tilde{G}} &= v_x^G + \dot{L}, \\
\delta_x^{\tilde{G}} &= \delta_x^G + 3(1+w_x)\frac{\dot{a}}{a}T, \\
\delta p_x^{\tilde{G}} &= \delta p_x^G + 3c_x^2\rho_x(1+w_x)\frac{\dot{a}}{a}T, \\
\Pi_x^{\tilde{G}} &= \Pi_x^G.
\end{aligned} \tag{4.71}$$

Therefore any ambiguity in the time slicing T leads to freedom in defining the density contrast δ_x. Notice that the anisotropic stress Π_x has a truly gauge invariant meaning as does any higher order tensor contribution. Furthermore, relative quantities do as well, *e.g.*

$$\begin{aligned}
\frac{\delta_x^{\tilde{G}}}{1+w_x} - \frac{\delta_y^{\tilde{G}}}{1+w_y} &= \frac{\delta_x^G}{1+w_x} - \frac{\delta_y^G}{1+w_y}, \\
v_x^{\tilde{G}} - v_y^{\tilde{G}} &= v_x^G - v_y^G, \\
\Gamma_x^{\tilde{G}} &= \Gamma_x^G,
\end{aligned} \tag{4.72}$$

the relative number density, velocity, and entropy fluctuation. We hereafter drop the superscript from such quantities.

4.3.2 Newtonian Gauge

In the Newtonian gauge, $B^N = H_T^N = 0$. Physically, it is a time slicing in which the expansion is isotropic. This considerably simplifies the interpretation of effects such as gravitational infall and redshift. From an arbitrary coordinate system G, the Newtonian gauge is reached by employing [see equation (4.70)]

$$\begin{aligned}
T &= -B^G/k + \dot{H}_T^G/k^2, \\
L &= -H_T^G/k.
\end{aligned} \tag{4.73}$$

From equations (4.70) and (4.71), the fundamental perturbations on this choice of hypersurface slicing are

$$
\begin{aligned}
\Psi \equiv A^N &= A^G + \frac{1}{a}\frac{d}{d\eta}[aB^G/k - a\dot{H}_T^G/k^2], \\
\Phi \equiv H_L^N &= H_L^G + \frac{1}{3}H_T^G + \frac{\dot{a}}{a}(B^G/k - \dot{H}_T^G/k^2), \\
\delta_x^N &= \delta_x^G + 3(1+w_x)\frac{\dot{a}}{a}(-B^G/k + \dot{H}_T^G/k^2), \\
\delta p_x^N &= \delta p_x^G + 3c_x^2\rho_x(1+w_x)\frac{\dot{a}}{a}(-B^G/k + \dot{H}_T^G/k^2), \\
V_x \equiv v_x^N &= v_x^G - \dot{H}_T^G/k. \qquad (4.74)
\end{aligned}
$$

This is commonly referred to as the "gauge invariant" definition of Newtonian perturbations. Note that the general form of the Poisson equation becomes

$$
\Phi = 4\pi G\left(\frac{a}{a_0}\right)^2 \rho_T\left(\delta_T^G + 3\frac{\dot{a}}{a}(1+w_T)(v_T^G - B^G)/k\right). \qquad (4.75)
$$

As we have seen, density perturbations in this gauge grow due to infall into the potential Ψ and metric stretching effects from Φ. In the absence of changes in Φ, they will therefore not grow outside the horizon since causality prevents infall growth.

4.3.3 Synchronous Gauge

The synchronous gauge, defined by $A^S = B^S = 0$ is a popular and in many cases computationally useful choice. The condition $A^S = 0$ implies that proper time corresponds with coordinate time, and $B^S = 0$ that constant space coordinates are orthogonal to constant time hypersurfaces. This is the natural coordinate system for freely falling observers.

From an arbitrary coordinate choice, the synchronous condition is satisfied by the transformation

$$
\begin{aligned}
T &= a^{-1}\int d\eta a A^G + c_1 a^{-1}, \\
L &= -\int d\eta (B^G + kT^G) + c_2, \qquad (4.76)
\end{aligned}
$$

where c_1 and c_2 are integration constants. There is therefore residual gauge freedom in synchronous gauge. It manifests itself as a degeneracy in the mapping of fluctuations onto the synchronous gauge and appears, for example as an ambiguity in δ_x^S of $3(1+w_x)c_1\dot{a}/a^2$. This represents an unphysical gauge mode. To eliminate it, one must carefully define the initial conditions.

It is a simple exercise in algebra to transform the evolution equations from Newtonian to synchronous representation. The metric perturbations are commonly written as

$$\begin{aligned} h_L &\equiv 6H_L^S, \\ \eta_T &\equiv -H_L^S - \frac{1}{3}H_T^S. \end{aligned} \tag{4.77}$$

Equation (4.76) tells us that

$$\begin{aligned} T &= -\dot{L}/k = (v_x^N - v_x^S)/k \\ &= \frac{1}{2}(\dot{h}_L + 6\dot{\eta}_T)/k^2, \end{aligned} \tag{4.78}$$

from which it follows

$$\dot{\Phi} = \frac{1}{6}\dot{h}_L - k(v_x^N - v_x^S)/3 + \frac{d}{d\eta}[\frac{\dot{a}}{a}(v_x^N - v_x^S)/k]. \tag{4.79}$$

Furthermore, the density and pressure relations

$$\begin{aligned} \delta_x^N &= \delta_x^S - 3(1+w_x)\frac{\dot{a}}{a}(v_x^N - v_x^S)/k, \\ \delta p_x^N &= \delta p_x^S - 3(1+w_x)c_x^2\rho_x\frac{\dot{a}}{a}(v_x^N - v_x^S)/k, \end{aligned} \tag{4.80}$$

and equation (4.38) yields

$$\dot{\delta}_x^N = \dot{\delta}_x^S - (1+w_x)\left\{3(\dot{\Phi} - \frac{1}{6}\dot{h}_L) + \left[k^2 - 9(c_x^2 - w_x)\left(\frac{\dot{a}}{a}\right)^2\right](v_x^N - v_x^S)/k\right\}, \tag{4.81}$$

and

$$3\frac{\dot{a}}{a}\left(\frac{\delta p_x^N}{\delta \rho_x^N} - w_x\right)\delta_x^N = 3\frac{\dot{a}}{a}\left(\frac{\delta p_x^S}{\delta \rho_x^S} - w_x\right)\delta_x^S + 9(1+w_x)(c_x^2 - w_x)\left(\frac{\dot{a}}{a}\right)^2(v_x^N - v_x^S)/k. \tag{4.82}$$

Thus the continuity equation of (4.34) becomes

$$\dot{\delta}_x^S = -(1+w_x)(kv_x^S + \dot{h}_L/2) - 3\frac{\dot{a}}{a}\left(\frac{\delta p_x^S}{\delta \rho_x^S} - w_x\right)\delta_x^S. \tag{4.83}$$

Likewise with the relation

$$\dot{v}_x^S + \frac{\dot{a}}{a}v_x^S = \dot{v}_x^N + \frac{\dot{a}}{a}v_x^N - k\Psi, \tag{4.84}$$

and equation (4.38), the transformed Euler equation immediately follows:

$$\dot{v}_x^S = -\frac{\dot{a}}{a}(1-3w_x)v_x^S - \frac{\dot{w}_x}{1+w_x}v_x^S + \frac{\delta p_x^S/\delta\rho_x^S}{1+w_x}k\delta_x^S - \frac{2}{3}\frac{w_x}{1+w_x}(1-3K/k^2)k\Pi_x. \tag{4.85}$$

Finally, one can also work in the reverse direction and obtain the Newtonian variables in terms of the synchronous gauge perturbations. Given the residual gauge freedom, this is a many to one mapping. The Newtonian metric perturbation follows from equation (4.75), $B^S = 0$, and the gauge invariance of Π_T:

$$\begin{aligned}(k^2 - 3K)\Phi &= 4\pi G \left(\frac{a}{a_0}\right)^2 \rho_T[\delta_T^S + 3\frac{\dot{a}}{a}(1+w_T)v_T^S/k], \\ k^2(\Psi + \Phi) &= -8\pi G \left(\frac{a}{a_0}\right)^2 p_T \Pi_T. \end{aligned} \tag{4.86}$$

They can also be written in terms of the synchronous gauge metric perturbations as

$$\begin{aligned}\Psi &= \frac{1}{2k^2}\left[\ddot{h}_L + 6\ddot{\eta}_T + \frac{\dot{a}}{a}(\dot{h}_L + 6\dot{\eta}_T)\right], \\ \Phi &= -\eta_T + \frac{1}{2k^2}\frac{\dot{a}}{a}(\dot{h}_L + 6\dot{\eta}_T). \end{aligned} \tag{4.87}$$

In fact, equations (4.86) and (4.87) close the system by expressing the time evolution of the metric variables η_T and h_L in terms of the matter sources.

Now let us return to the gauge mode problem. The time slicing freedom can be fixed by a choice of the initial hypersurface. The natural choice is one in which the velocity vanishes $v_x^S(\eta_i) = 0$ for some set of "observer" particle species x. This condition fixes c_1 and removes the gauge ambiguity in the density perturbations. Notice also that the synchronous gauge has an elegant property. Since it is the coordinate system of freely falling observers, if the velocity of a *non-interacting* pressureless species is set to zero initially it will remain so. In the Euler equation (4.85), the infall term that sources velocities has been transformed away by equation (4.84). Thus in the absence of pressure and entropy terms, there are no sources to the velocity.

The synchronous gauge therefore represents a "Lagrangian" coordinate system as opposed to the more "Eulerian" choice of a Newtonian coordinate system. In this gauge, the coordinate grid follows freely falling particles so that density growth due to infall is transformed into dilation effects from the stretching of the grid. Although the coordinate grid must be redefined when particle trajectories cross, this does not occur in linear perturbation theory if the defining particles are non-relativistic. Thus in synchronous gauge, the dynamics are simpler since we employ the rest frame of the collisionless matter. The only drawback to this gauge choice is that physical intuition is more difficult to obtain since we have swept dynamical effects into the behavior of the coordinate grid.

4.3.4 Total Matter Gauge

As an obvious extension of the ideas which make the synchronous gauge appealing, it is convenient to employ the rest frame of the total rather than collisionless matter. The total matter velocity is thus set to be orthogonal to the constant time hypersurfaces $v_T^T = B^T$. With the additional constraint $H_T^T = 0$, the transformation is obtained by

$$\begin{aligned} T &= (v_T^G - B^G)/k, \\ L &= -H_T^G/k, \end{aligned} \tag{4.88}$$

which fixes the coordinates completely. The matter perturbation quantities become

$$\begin{aligned} \Delta_x \equiv \delta_x^T &= \delta_x^G + 3(1+w_x)\frac{\dot a}{a}(v_T^G - B^G)/k, \\ \delta p_x^T &= \delta p_x^G + 3(1+w_x)c_x^2\rho_x\frac{\dot a}{a}(v_T^G - B^G)/k, \\ V_x \equiv V_x^T &= v_x^G - \dot H_T^G/k. \end{aligned} \tag{4.89}$$

Notice that the Newtonian gauge $B^N = H_N^T = 0$ and $v_x^T = v_x^N = V_x$. In synchronous gauge, $B^S = 0$ as well. If the rest frame of the total matter is the same as the collisionless non-relativistic matter, as is the case for adiabatic conditions, $\delta_x^S \simeq \Delta_x^T$ if $v_x^S(0) = 0$.

The evolution equations are easily obtained from Newtonian gauge with the help of the following relations,

$$\frac{d}{d\eta}\left(\frac{\dot a}{a}\right) = -\frac{1}{2}\left[\left(\frac{\dot a}{a}\right)^2 + K\right](1+3w_T) + \frac{3}{2}(1+w_T)\left(\frac{a}{a_0}\right)^2\Omega_\Lambda H_0^2, \tag{4.90}$$

which follows from equation (4.61) and

$$\frac{\dot a}{a}\Psi + \dot\Phi = \frac{3}{2}\left[\left(\frac{\dot a}{a}\right)^2 + K - \left(\frac{a}{a_0}\right)^2\Omega_\Lambda H_0^2\right](1+w_T)V_T/k \tag{4.91}$$

from equation (4.64). The Newtonian Euler equation can also be rewritten as

$$\begin{aligned} \frac{d}{d\eta}\left(\frac{\dot a}{a}(1+w_T)V_T\right) &= -\left(\frac{\dot a}{a}\right)^2(1-3w_T)(1+w_T)V_T + \frac{\dot a}{a}\frac{\delta p_T^N}{\delta\rho_T^N}k\delta_T^N \\ &\quad -\frac{2}{3}\frac{\dot a}{a}w_T(1-3K/k^2)k\Pi_T + (1+w_T)\frac{\dot a}{a}k\Psi \\ &\quad -\frac{1}{2}(1+3w_T)(1+w_T)\left[\left(\frac{\dot a}{a}\right)^2 + K\right]V_T \\ &\quad +\frac{3}{2}(1+w_T)^2\left(\frac{a}{a_0}\right)^2\Omega_\Lambda H_0^2 V_T. \end{aligned} \tag{4.92}$$

With this relation, the total matter continuity and Euler equations readily follow,

$$\dot{\Delta}_T - 3w_T\frac{\dot{a}}{a}\Delta_T = -(1-3K/k^2)(1+w_T)kV_T - 2(1-3K/k^2)\frac{\dot{a}}{a}w_T\Pi_T, \tag{4.93}$$

$$\dot{V}_T + \frac{\dot{a}}{a}V_T = \frac{c_T^2}{1+w_T}k\Delta_T + k\Psi + \frac{w_T}{1+w_T}k\Gamma_T - \frac{2}{3}(1-3K/k^2)\frac{w_T}{1+w_T}k\Pi_T. \tag{4.94}$$

The virtue of this representation is that the evolution of the total matter is simple. This is reflected by the form of the Poisson equation,

$$(k^2-3K)\Phi = 4\pi G\left(\frac{a}{a_0}\right)^2 \rho_T\Delta_T, \tag{4.95}$$

$$k^2(\Psi+\Phi) = -8\pi G\left(\frac{a}{a_0}\right)^2 p_T\Pi_T. \tag{4.96}$$

In the total matter rest frame, there are no relativistic effects from the velocity and hence the Poisson equation takes its non-relativistic form. Again the drawback is that the interpretation is muddled.

4.3.5 Hybrid Formulation

We have seen that the Newtonian gauge equations correspond closely with classical intuition and thus provide a simple representation for relativistic perturbation theory. However, since density perturbations grow by the causal mechanism of potential infall, we have build a fundamental scale, the particle horizon, into the evolution. Frames that comove with the matter, *i.e.* in which the particle velocity vanishes, have no fundamental scale. This simplifies the perturbation equations and in many cases admit scale invariant, *i.e.* power law solutions (see §5). Two such frames are commonly employed: the rest frame of the collisionless non-relativistic mater and that of the total matter. The former is implemented under a special choice of the synchronous gauge condition and the latter by the total matter gauge. For the case of adiabatic fluctuations, where non-relativistic and relativistic matter behave similarly, they are essentially identical. For entropy fluctuations, the total matter gauge is more ideal.

Since we can express fluctuations on any given frame by combination of variables on any other, we can mix and match quantities to suit the purpose at hand. To be explicit, we will hereafter employ total matter gauge density fluctuations $\Delta_x \equiv \delta_x^T$, but Newtonian temperature $\Theta \equiv \delta_\gamma^N/4$ and metric perturbations Ψ and Φ. The velocity perturbation is the same in both these frames, which we denote $V_x = v_x^N = v_x^T$. To avoid confusion, we will hereafter employ *only* this choice. We now turn to the solution of these equations and their implications for the CMB.