CMBology

- Universe is currently bathed in 2.725K blackbody radiation which composes the majority of the radiation density of the universe
  mm-cm wavelength, 100 GHz photons near peak
  400 photon cm$^{-3}$

- Radiation is extremely isotropic: aside from the $10^{-3}$ temperature variations due to the Doppler shift of our own motion, fluctuations in the temperature are at the $10^{-5}$ level.

- Fluctuations are the imprint of the origin of structure

- Fluctuations are polarized at the 10% level reflecting scattering processes by which they last interacted with matter

- Place CMB in cosmological context
Astro 282
FRW Cosmology
FRW Cosmology

- FRW cosmology = homogeneous and isotropic on large scales
- Universe observed to be nearly isotropic (e.g. CMB, radio point sources, galaxy surveys)
- Copernican principle: must be isotropic to all observers (all locations)
- Implies homogeneity; also galaxy redshift surveys (LCRS, 2dF, SDSS) have seen the “end of greatness”, large scale homogeneity directly
- FRW cosmology (homogeneity, isotropy & Einstein equations) generically implies the expansion of the universe, except for special unstable cases
• Spatial geometry is that of a constant curvature (positive, negative, zero) surface

• Metric tells us how to measure distances on this surface

• Consider the closed geometry of a sphere of radius $R$ and suppress one dimension
Angular Diameter Distance

- Spatial distance: restore 3rd dimension with the usual spherical polar angles

\[ d\Sigma^2 = dD^2 + D_A^2 d\alpha^2 \]

\[ = dD^2 + D_A^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

- \(D_A\) is called the angular diameter distance since \(D_A d\alpha\) corresponds to the transverse separation or size as opposed to the Euclidean \(D d\alpha\), i.e. is the apparent distance to an object through the gravitational lens of the background geometry.

- In a positively curved geometry \(D_A < D\) and objects are further than they appear.

- In a negatively curved universe \(R\) is imaginary and \(R \sin(D/R) = i|R| \sin(D/i|R|) = |R| \sinh(D/|R|)\) – and \(D_A > D\) objects are closer than they appear.
Volume Element

- Metric also defines the volume element

\[ dV = (dD)(D_A d\theta)(D_A \sin \theta d\phi) \]
\[ = D_A^2 dD d\Omega \]

- Most of classical cosmology boils down to these three quantities, (comoving) distance, (comoving) angular diameter distance, and volume element

- For example, distance to a high redshift supernova, angular size of the horizon at last scattering, number density of clusters...
Comoving Coordinates

- Remaining degree of freedom (preserving homogeneity and isotropy) is an overall scale factor that relates the geometry (fixed by the radius of curvature $R$) to physical coordinates – a function of time only

\[
d\sigma^2 = a^2(t)d\Sigma^2
\]

our conventions are that the scale factor today $a(t_0) \equiv 1$

- Similarly physical distances are given by $d(t) = a(t)D$, $d_A(t) = a(t)D_A$.

- Distances in capital case are *comoving* i.e. they comove with the expansion and do not change with time – simplest coordinates to work out geometrical effects
Redshift

- Wavelength of light “stretches” with the scale factor, so that it is convenient to define a shift-to-the-red or redshift as the scale factor increases.

\[ \lambda(a) = a(t)\Lambda \]

\[ \frac{\lambda(1)}{\lambda(a)} = \frac{1}{a} \equiv (1 + z) \]

\[ \frac{\delta\lambda}{\lambda} = -\frac{\delta\nu}{\nu} = z \]

- Given known frequency of emission \( \nu(a) \), redshift can be precisely measured (modulo Doppler shifts from peculiar velocities) – interpreting the redshift as a Doppler shift, objects recede in an expanding universe - \( \nu = zc \)
Time and Conformal Time

- As in special relativity, time comes in with the opposite signature in measuring space-time separation

- Proper time

\[
d\tau^2 = dt^2 - d\sigma^2
\]

\[
= dt^2 - a^2(t) d\Sigma^2
\]

\[
\equiv a^2(t) (d\eta^2 - d\Sigma^2)
\]

- Special relativity: physics invariant under the set of linear coordinate transformations (Lorentz transformation) that preserve lengths \((d\tau^2)\)

- General relativity: physics invariant under a general coordinate transformation that preserves lengths
A GR Aside

- We will generally skirt around General Relativity but knowledge of the language will be useful
- Proper time defines the metric $g_{\mu\nu}$

\[ d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu \]

- Usually we will use comoving coordinates and conformal time as the “x” ’s unless otherwise specified – metric for other choices are related by $a(t)$ – e.g. in spherical coordinates $\mu \in \eta, \theta, \phi, D$

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -D_A^2 & 0 & 0 \\
0 & 0 & -D_A^2 \sin^2 \theta & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}
\]
 Photon Cartography

- Classical cosmology is photon cartography – mapping out the expansion by tracking the distance a photon travels as a function of scale factor or redshift

- Taking out the scale factor in the time coordinate \( d\eta = dt/a \) defines \textit{conformal time} – useful in that photons travelling radially from observer then obey

\[
\Delta D = \Delta \eta = \int \frac{dt}{a}
\]

so that time and distance may be interchanged
Horizon

- Distance travelled by a photon in the whole lifetime of the universe defines the horizon

- Since $d\tau = 0$, the horizon is simply the conformal time elapsed

\[ D_{\text{horizon}}(t) = \int_0^t \frac{dt'}{a} = \eta(t) \]

- Since the horizon always grows with time, there is always a point in time before which two observers separated by a distance $D$ could not have been in causal contact

- Horizon problem: why is the universe homogeneous and isotropic on large scales, near the current horizon – problem deepens for objects seen at early times, e.g. CMB
Hubble Parameter

- Useful to define the expansion rate or Hubble parameter

\[ H(t) \equiv \frac{1}{a} \frac{da}{dt} \]

since dynamics (Einstein equations) will give this directly as \( H(a) \equiv H(t(a)) \)

- Time becomes

\[ t = \int dt = \int \frac{da}{aH(a)} \]

- Conformal time becomes

\[ \eta = \int \frac{dt}{a} = \int \frac{da}{a^2 H(a)} \]
Distance-Redshift Relation

- All distance redshift relations based on comoving distance $D(z)$

$$D(a) = \int dD = \int_a^1 \frac{da'}{a^2 H(a)}$$

$$(da = -(1 + z)^{-2}dz = -a^2dz)$$

$$D(z) = -\int_z^0 \frac{dz'}{H(z')} = \int_0^z \frac{dz'}{H(z')}$$

- Note limiting case is the Hubble law

$$\lim_{z \to 0} D(z) = z/H(z = 0) \equiv z/H_0$$

redshift (recession velocity) increases linearly with distance

- Hubble constant usually quoted as $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$, observationally $h \sim 0.7$; in natural units $H_0 = (2997.9)^{-1} h \text{ Mpc}^{-1}$ defines an inverse length scale
Distance-Redshift Relation

- Example: object of known physical size $\lambda = a(t)\Lambda$ ("standard ruler") subtending an (observed) angle on the sky $\alpha$

\[
\alpha = \frac{\Lambda}{D_A(z)} = \frac{\lambda}{aR \sin(D(z)/R)}
\]

\[
= \frac{\lambda}{R \sin(D(z)/R)} (1 + z) \equiv \frac{\lambda}{d_A(z)}
\]

- Example: object of known luminosity $L$ ("standard candle") with a measured flux $S$. Comoving surface area $4\pi D_A^2$, frequency/energy $(1 + z)$, time-dilation or arrival rate of photons (crests) $(1 + z)$:

\[
S = \frac{L}{4\pi D_A^2 (1 + z)^2}
\]

\[
\equiv \frac{L}{4\pi d_L^2} \quad (d_L = (1 + z)D_A = (1 + z)^2d_A)
\]
Relative Measures

- If absolute calibration of standards unknown, then absolute distance (or Hubble constant) unknown

\[ d_A(z) = \lambda / \alpha(z); \quad d_L(z) = \sqrt{L / 4\pi S(z)} \]

- Ratio at two different redshifts drops out the unknown standards \( \lambda, L \) and measures evolution of the distance-redshift relation \( H_0 D(z) \):

\[ \frac{d_{A,L}(z_2)}{d_{A,L}(z_1)} \approx \frac{H_0}{z_1} d_{A,L}(z_2) \quad [z_1 \ll 1] \]

- Alternately, distances & curvature are measured in units of \( h^{-1} \) Mpc.
**Fundamental Observable**

- Fundamental dependence (aside from \((1 + z)\) factors)

\[
H_0 D_A(z) = H_0 R \sin(D(z)/\tilde{R})
\]

\[
= \tilde{R} \sin(H_0 D(z)/\tilde{R}), \quad \tilde{R} = H_0 R
\]

\[
H_0 D(z) = \int \frac{da}{a^2} \frac{H_0}{H(a)}
\]

- Maps out the kinematics of the expansion

- Current best standard ruler: acoustic oscillations; current best standard candle supernovae type Ia

- Adding in the dynamics of the expansion, measurements of \(D(z)\) indicate a flat universe whose expansion is accelerating
Evolution of Scale Factor

- FRW cosmology is fully specified if the function $a(t)$ is given.
- General relativity relates the scale factor with the matter content of the universe.
- Build the Einstein tensor $G_{\mu \nu}$ out of the metric and use Einstein equation.

$$G_{\mu \nu} = -8\pi G T_{\mu \nu}$$

$$G^0_0 = -\frac{3}{a^2} \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{1}{R^2} \right]$$

$$G^i_j = -\frac{1}{a^2} \left[ 2\frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 + \frac{1}{R^2} \right] \delta^i_j$$
Einstein Equations

- Isotropy demands that the stress-energy tensor take the form

\[ T^0_0 = \rho \]
\[ T^i_j = -p\delta^i_j \]

where \( \rho \) is the energy density and \( p \) is the pressure

- So Einstein equations become

\[
\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{R^2} = \frac{8\pi G}{3}a^2\rho \\
2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{R^2} = -8\pi Ga^2p \\
\text{or} \quad \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -\frac{4\pi G}{3}a^2(\rho + 3p)
\]
Friedman Equations

- More usual to see Einstein equations expressed in time not conformal time
  \[
  \frac{\dot{a}}{a} = \frac{1}{\eta a} = \frac{da}{dt} = aH(a)
  \]
  \[
  \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{\eta} \left(\frac{\dot{a}}{a}\right) = a \frac{d}{dt} \left(\frac{da}{dt}\right) = a \frac{d^2a}{dt^2}
  \]

- Friedmann equations:
  \[
  H^2(a) + \frac{1}{a^2 R^2} = \frac{8\pi G}{3} \rho
  \]
  \[
  \frac{1}{a} \frac{d^2a}{dt^2} = -\frac{4\pi G}{3} (\rho + 3p)
  \]

- Convenient fiction to describe curvature as an energy density component
  \[
  \rho_K = -\frac{3}{(8\pi G a^2 R^2)} \propto a^{-2} \text{ that does not accelerate the expansion}, \ p_K = -\rho_K/3
  \]
Critical Density

- Friedmann equation for $H$ then reads

$$H^2(a) = \frac{8\pi G}{3} (\rho + \rho_K) \equiv \frac{8\pi G}{3} \rho_c$$

defining a critical density today $\rho_c$ in terms of the expansion rate

- In particular, its value today is given by the Hubble constant as

$$\rho_c(z = 0) = 3H_0^2 / 8\pi G = 1.8788 \times 10^{-29} h^2 g \text{ cm}^{-3}$$

- Energy density today is given as a fraction of critical

$$\Omega \equiv \rho/\rho_c |_{z=0}.$$ Radius of curvature then given by

$$R^{-2} = H_0^2 (\Omega - 1)$$

- If $\Omega \approx 1$, $\rho \approx \rho_c$, then $\rho_K \ll \rho_c$ or $H_0 R \ll 1$, universe is flat across the Hubble distance. $\Omega < 1$ negatively curved; $\Omega > 1$ positively curved
Newtonian Interpretation

• Consider a test particle of mass $m$ in expanding spherical region of radius $r$ and total mass $M$. Energy conservation

\[ E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{const} \]

\[ \frac{1}{2} \left( \frac{dr}{dt} \right)^2 - \frac{GM}{r} = \text{const} \]

\[ \frac{1}{2} \left( \frac{1}{r} \frac{dr}{dt} \right)^2 - \frac{GM}{r^3} = \frac{\text{const}}{r^2} \]

\[ H^2 = \frac{8\pi G \rho}{3} - \frac{\text{const}}{a^2} \]

• Constant determines whether the system is bound and in the Friedmann equation is associated with curvature – not general since neglects pressure
Conservation Law

- Second Friedmann equation, or acceleration equation, simply expresses energy conservation (why: stress energy is automatically conserved in GR via Bianchi identity)

\[ d\rho V + pdV = 0 \]
\[ d\rho a^3 + pda^3 = 0 \]
\[ \dot{\rho}a^3 + 3\frac{\dot{a}}{a}\rho a^3 + 3\frac{\dot{a}}{a}pa^3 = 0 \]
\[ \frac{\dot{\rho}}{\rho} = -3(1 + w)\frac{\dot{a}}{a} \quad w \equiv \frac{p}{\rho} \]

- If \( w = \text{const.} \) then the energy density depends on the scale factor as \( \rho \propto a^{-3(1+w)} \).
Multicomponent Universe

- The total energy density can be composed of a sum of components with differing equations of state

\[ \rho(a) = \sum_i \rho_i(a) = \sum_i \rho_i(a = 1)a^{-3(1+w_i)}, \quad \Omega_i \equiv \rho_i/\rho_c|_{a=1} \]

- Important cases: nonrelativistic matter \( \rho_m = mn_m \propto a^{-3} \), \( w_m = 0 \); relativistic radiation \( \rho_r = En_r \propto \nu n_r \propto a^{-4} \), \( w_r = 1/3 \); “curvature” \( \rho_K \propto a^{-2} \), \( w_K = -1/3 \); constant energy density or cosmological constant \( \rho_\Lambda \propto a^0, w_\Lambda = -1 \)

- Or generally with \( w_c = p_c/\rho_c = (p + p_K)/(\rho + \rho_K) \)

\[ \rho_c(a) = \rho_c(a = 1)e^{-\int d\ln a 3(1+w_c(a))} \]

\[ H^2(a) = H_0^2e^{-\int d\ln a 3(1+w_c(a))} \]
Acceleration Equation

- Time derivative of (first) Friedman equation

\[
2 \frac{1}{a} \frac{da}{dt} \left[ \frac{1}{a} \frac{d^2 a}{dt^2} - H^2(a) \right] = \frac{8\pi G}{3} \frac{d\rho_c}{dt}
\]

\[
\left[ \frac{1}{a} \frac{d^2 a}{dt^2} - \frac{8\pi G}{3} \rho_c \right] = \frac{4\pi G}{3} \left[ -3(1 + w_c) \rho_c \right]
\]

\[
\frac{1}{a} \frac{d^2 a}{dt^2} = -\frac{4\pi G}{3} \left[ (1 + 3w_c) \rho_c \right]
\]

\[
= -\frac{4\pi G}{3} \left( \rho + \rho_K + 3p + 3p_K \right)
\]

\[
= -\frac{4\pi G}{3} \left( 1 + 3w \right) \rho
\]

- Acceleration equation says that universe decelerates if \( w > -1/3 \)
Friedmann equations “predict” the expansion of the universe. Non-expanding conditions $\frac{da}{dt} = 0$ and $\frac{d^2a}{dt^2} = 0$ require

$$\rho = -\rho_K \quad \rho = -3p$$

i.e. a positive curvature and a total equation of state

$w \equiv \frac{p}{\rho} = -\frac{1}{3}$

Since matter is known to exist, one can in principle achieve this with

$$\rho = \rho_m + \rho_\Lambda = -\rho_K = -3p = 3\rho_\Lambda$$

$$\rho_\Lambda = -\frac{1}{3}\rho_K \quad \rho_m = -\frac{2}{3}\rho_K$$

Einstein introduced $\rho_\Lambda$ for exactly this reason – “biggest blunder”; but note that this balance is unstable: $\rho_m$ can be perturbed but $\rho_\Lambda$, a true constant cannot.
Dark Energy

• Distance redshift relation depends on energy density components

\[ H_0 D(z) = \int \frac{da}{a^2} \frac{H_0}{H(a)} \]

\[ = \int \frac{da}{a^2} e^{\int d\ln a} \frac{3}{2} (1+w_c(a)) \]

• Distant supernova Ia as standard candles imply that \( w_c < -1/3 \) so that the expansion is accelerating

• Consistent with a cosmological constant that is

\[ \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{\text{crit}}} = \frac{2}{3} \] of the total energy density

• Coincidence problem: different components of matter scale differently with \( a \). Why are (at least) two components comparable today? – Anthropic?
Dark Matter

- Since Zwicky in the 1930’s non-luminous or dark matter has been known to dominate over luminous matter in stars (and hot gas)
- Arguments are basically from a balance of gravitational force against “pressure” from internal motions: rotation velocity in galactic disks, velocity dispersion of galaxies in clusters, gas pressure in clusters, radiation pressure in CMB
- Assuming that the object is somehow typical in its non-luminous to luminous density, these measures are converted to an overall dark matter density through a “mass-to-light ratio”
- From galaxy surveys the luminosity density in solar units is

\[ \rho_L = 2 \pm 0.7 \times 10^8 h \, L_\odot Mpc^{-3} \]

(h’s: distances in \( h^{-1} \) Mpc; luminosity inferred from flux \( L \propto Sd^2 \propto h^{-2} \); inverse volume \( \propto h^3 \))
Dark Matter

- Critical density in solar units is $\rho_c = 2.7754 \times 10^{11} h^2 \, M_\odot \, \text{Mpc}^{-3}$ so that the critical mass-to-light ratio in solar units is

$$\left( \frac{M}{L} \right) \approx 1400h$$

- Flat rotation curves: $GM/r^2 \approx v^2/r \rightarrow M \approx v^2r/G$, so the observed flat rotation curve implies $M \propto r$ out to $30h^{-1}$ kpc, beyond the light. Implies $M/L > 30h$ and perhaps more – closure if flat out to $\sim 1$ Mpc.

- Similar argument holds in clusters of galaxies where velocity dispersion replaces circular velocity and centripetal force is replaced by a “pressure gradient” $T/m = \sigma^2$ and $p = \rho T/m = \rho \sigma^2$ – generalization of hydrostatic equilibrium: Zwicky got $M/L \approx 300h$. 
Hydrostatic Equilibrium

- Evidence for dark matter in X-ray clusters also comes from direct hydrostatic equilibrium inference from the gas: balance radial pressure gradient with gravitational potential gradient

- Infinitesimal volume of area $dA$ and thickness $dr$ at radius $r$ and interior mass $M(r)$: pressure difference supports the gas

\[
[p_g(r) - p_g(r + dr)]dA = \frac{GmM}{r^2} = \frac{G\rho_g M}{r^2}dV
\]

\[
\frac{dp_g}{dr} = -\rho_g \frac{d\Phi}{dr}
\]

with $p_g = \rho_g T_g/m$ becomes

\[
\frac{GM}{r} = -\frac{T_g}{m} \left( \frac{d\ln \rho_g}{d\ln r} + \frac{d\ln T_g}{d\ln r} \right)
\]

- $\rho_g$ from X-ray luminosity; $T_g$ sometimes taken as isothermal
Gravitational Lensing

- Mass can be directly measured in the gravitational lensing of sources behind the cluster
- Strong lensing (giant arcs) probes central region of clusters
- Weak lensing (1-10%) elliptical distortion to galaxy image probes outer regions of cluster and total mass
- All techniques agree on the necessity of dark matter and are roughly consistent with a dark matter density $\Omega_m \sim 0.2 - 0.4$
- $\Omega_m + \Omega_\Lambda \approx 1$ from matter density + dark energy
- CMB provides a test of $D_A \neq D$ through the standard rulers of the acoustic peaks and shows that the universe is close to flat $\Omega \approx 1$
- Consistency has lead to the standard model for the cosmological matter budget
Astro 282
Thermal History
Thermal & Diffusive Equilibrium

- A gas in thermal & diffusive contact with a reservoir at temperature $T$

- Probability of system being in state of energy $E_i$ and number $N_i$ (Gibbs Factor)

\[ P(E_i, N_i) \propto \exp\left[-\left(E_i - \mu N_i\right) / T\right] \]

where $\mu$ is the chemical potential (defines the free energy “cost” for adding a particle at fixed temperature and volume)

- Chemical potential appears when particles are conserved

- CMB photons can carry chemical potential if creation and annihilation processes inefficient, as they are after $t \sim 1\text{yr.}$
Distribution Function

- Mean occupation of the state in thermal equilibrium

\[ f \equiv \frac{\sum N_i P(E_i, N_i)}{\sum P(E_i, N_i)} \]

where the total energy is related to the particle energy as
\[ E_i = N_i E \] (ignoring zero pt)

- Density of (energy) states in phase space makes the net spatial density of particles

\[ n = g \int \frac{d^3 p}{(2\pi)^3} f \]

where \( g \) is the number of spin states
Fermi-Dirac Distribution

• For fermions, the occupancy can only be $N_i = 0, 1$

\[ f = \frac{P(E, 1)}{P(0, 0) + P(E, 1)} \]

\[ = \frac{e^{-(E-\mu)/T}}{1 + e^{-(E-\mu)/T}} \]

\[ = \frac{1}{e^{(E-\mu)/T} + 1} \]

• In the non-relativistic limit

\[ E = (p^2 + m^2)^{1/2} \approx m + \frac{1}{2m}p^2 \]

and $m \gg T$ so the distribution is Maxwell-Boltzmann

\[ f = e^{-(m-\mu)/T} e^{-p^2/2mT} = e^{-(m-\mu)/T} e^{-mv^2/2T} \]
Bose-Einstein Distribution

- For bosons each state can have multiple occupation,

\[
f = \frac{\frac{d}{d\mu/T} \sum_{N=0}^{\infty} (e^{-(E-\mu)/T})^N}{\sum_{N=0}^{\infty} (e^{-(E-\mu)/T})^N}
\]

with \( \sum_{N=0}^{\infty} x^N = \frac{1}{1 - x} \)

\[
f = \frac{1}{e^{(E-\mu)/T} - 1}
\]

- Again, non relativistic distribution is Maxwell-Boltzmann

\[
f = e^{-(m-\mu)/T} e^{-p^2/2mT} = e^{-(m-\mu)/T} e^{-mv^2/2T}
\]

with a spatial number density

\[
n = ge^{-(m-\mu)/T} \int \frac{d^3p}{(2\pi)^3} e^{-p^2/2mT}
\]

\[
= ge^{-(m-\mu)/T} \left( \frac{mT}{2\pi} \right)^{3/2}
\]
Recombination

- Maxwell-Boltzmann distribution determines the equilibrium distribution for reactions, e.g. big-band nucleosynthesis, recombination:

\[ p + e^- \leftrightarrow H + \gamma \]

\[
\frac{n_p n_e}{n_H} \approx e^{-B/T} \left( \frac{m_e T}{2\pi} \right)^{3/2} e^{(\mu_p + \mu_e - \mu_H)/T}
\]

where \( B = m_p + m_e - m_H = 13.6 \text{eV} \) is the binding energy, \( g_p = g_e = \frac{1}{2} g_H = 2 \), and \( \mu_p + \mu_e = \mu_H \) in equilibrium

- Define ionization fraction

\[ n_p = n_e = x_e n_b \]

\[ n_H = n_{\text{tot}} - n_b = (1 - x_e)n_b \]
Recombination

- Saha Equation

\[
\frac{n_en_p}{n_Hn_b} = \frac{x_e^2}{1 - x_e}
\]
\[
= \frac{1}{n_b} \left( \frac{m_e T}{2\pi} \right)^{3/2} e^{-B/T}
\]

- Naive guess of \( T_* = B \) wrong due to the low baryon-photon ratio
  - \( T_* \approx 0.3 \text{eV} \) so recombination at \( z_* \approx 1000 \)

- But the photon-baryon ratio is very low

\[
\eta_{b\gamma} \equiv \frac{n_b}{n_\gamma} \approx 3 \times 10^{-8} \Omega_b h^2
\]
Recombination

- **Eliminate** in favor of $\eta_{b\gamma}$ and $B/T$ through

$$n_\gamma = 0.244 T^3, \quad \frac{m_e}{B} = 3.76 \times 10^4$$

- Big coefficient

$$\frac{x_e^2}{1 - x_e} = 3.16 \times 10^{15} \left( \frac{B}{T} \right)^{3/2} e^{-B/T}$$

$$T = 1/3\text{eV} \rightarrow x_e = 0.7, \quad T = 0.3\text{eV} \rightarrow x_e = 0.2$$

- **Further delayed** by inability to maintain equilibrium since net is through $2\gamma$ process and redshifting out of line
Astro 282
Acoustic Kinematics
Temperature Fluctuations

- Observe blackbody radiation with a temperature that differs at $10^{-5}$ coming from the surface of last scattering, with distribution function (specific intensity $I_{\nu} = 4\pi\nu^3 f(\nu)$ each polarization)

\[ f(\nu) = [\exp(2\pi\nu/T(\hat{n})) - 1]^{-1} \]

- Decompose the temperature perturbation in spherical harmonics

\[ T(\hat{n}) = \sum_{\ell m} T_{\ell m} Y_{\ell m}(\hat{n}) \]

- For Gaussian random fluctuations, the statistical properties of the temperature field are determined by the power spectrum

\[ \langle T_{\ell m}^* T_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_{\ell} \]

where the $\delta$-function comes from statistical isotropy
Spatial vs Angular Power

- Take the radiation distribution at last scattering to also be described by an isotropic temperature field $T(x)$ and recombination to be instantaneous

$$T(\hat{n}) = \int dD T(x) \delta(D - D_*)$$

where $D$ is the comoving distance and $D_*$ denotes recombination.

- Describe the temperature field by its Fourier moments

$$T(x) = \int \frac{d^3k}{(2\pi)^3} T(k) e^{ik \cdot x}$$

with a power spectrum

$$\langle T(k)^* T(k') \rangle = (2\pi)^3 \delta(k - k') P_T(k)$$
Spatial vs Angular Power

- Note that the variance of the field

\[
\langle T(\mathbf{x})T(\mathbf{x}) \rangle = \int \frac{d^3 k}{(2\pi)^3} P(k)
\]

\[
= \int d \ln k \frac{k^3 P(k)}{2\pi^2} \equiv \int d \ln k \Delta_T^2(k)
\]

so it is more convenient to think in the log power spectrum \(\Delta_T^2(k)\)

- Temperature field

\[
T(\hat{n}) = \int \frac{d^3 k}{(2\pi)^3} T(k) e^{i \mathbf{k} \cdot \mathbf{D}_* \hat{n}}
\]

- Expand out plane wave in spherical coordinates

\[
e^{i \mathbf{k} \cdot \mathbf{D}_* \hat{n}} = 4\pi \sum_{\ell m} i^\ell j_\ell(k D_*) Y^*_\ell m(\mathbf{k}) Y_\ell m(\hat{n})
\]
Spatial vs Angular Power

- Multipole moments

\[ T_{\ell m} = \int \frac{d^3 k}{(2\pi)^3} T(k) 4\pi i^\ell j^\ell(k D_*) Y_{\ell m}(k) \]

- Power spectrum

\[ \langle T^*_{\ell m} T_{\ell' m'} \rangle = \int \frac{d^3 k}{(2\pi)^3} (4\pi)^2 (i)^{\ell - \ell'} j^\ell(k D_*) j^\ell(k D_*) Y^*_{\ell m}(k) Y_{\ell' m'}(k) P_T(k) \]

\[ = \delta_{\ell \ell'} \delta_{mm'} 4\pi \int d \ln k j^2_\ell(k D_*) \Delta^2_T(k) \]

with \( \int_0^\infty j^2_\ell(x) d \ln x = 1/(2\ell(\ell + 1)) \), slowly varying \( \Delta^2_T \)

\[ C_\ell = \frac{4\pi \Delta^2_T(\ell/D_*)}{2\ell(\ell + 1)} = \frac{2\pi}{\ell(\ell + 1)} \Delta^2_T(\ell/D_*) \]

so \( \ell(\ell + 1) C_\ell/2\pi = \Delta^2_T \) is commonly used log power
Scale Invariant Fluctuations

- Scale invariant temperature fluctuations have $\Delta^2_T = \text{const}$

- Equal contributions to the rms temperature fluctuation per decade in frequency $k$

- Observed angular fluctuations then have $\ell(\ell + 1)C_\ell / 2\pi = \text{const}$

- Weaker assumption of scale free initial temperature fluctuations $\Delta^2_T \propto k^{n-1}$, where $n$ is called the tilt.

- $n = 1$ is scale invariant for historical reasons.

- However fluctuations evolve from their initial conditions due to gravitational and pressure forces
Thomson Scattering

- Thomson scattering of photons off of free electrons is the most important CMB process with a cross section (averaged over polarization states) of

\[ \sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25}\text{cm}^2 \]

- Density of free electrons in a fully ionized $x_e = 1$ universe

\[ n_e = (1 - Y_p/2)x_e n_b \approx 10^{-5}\Omega_b h^2(1 + z)^3\text{cm}^{-3}, \]

where $Y_p \approx 0.24$ is the Helium mass fraction, creates a high (comoving) Thomson opacity

\[ \dot{\tau} \equiv n_e \sigma_T a \]

where dots are conformal time $\eta \equiv \int dt/a$ derivatives and $\tau$ is the optical depth.
Tight Coupling Approximation

- Near recombination $z \approx 10^3$ and $\Omega_b h^2 \approx 0.02$, the (comoving) mean free path of a photon

$$\lambda_C \equiv \frac{1}{\dot{\tau}} \sim 2.5\text{Mpc}$$

small by cosmological standards!

- On scales $\lambda \gg \lambda_C$ photons are tightly coupled to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions

- Specifically, their bulk velocities are defined by a single fluid velocity $v_\gamma = v_b$ and the photons carry no anisotropy in the rest frame of the baryons

- $\rightarrow$ No heat conduction or viscosity (anisotropic stress) in fluid
Tight Coupling Approximation

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- → No heat conduction or viscosity (anisotropic stress) in fluid
Zeroth Order Approximation

- **Momentum density** of a fluid is \((\rho + p)v\), where \(p\) is the pressure.

- **Neglect** the momentum density of the baryons

\[
R \equiv \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} v_b = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}
\]

\[
\approx 0.6 \left( \frac{\Omega_b h^2}{0.02} \right) \left( \frac{a}{10^{-3}} \right)
\]

since \(\rho_\gamma \propto T^4\) is fixed by the CMB temperature \(T = 2.73(1 + z)K\).

- **Neglect** radiation in the expansion

\[
\frac{\rho_m}{\rho_r} = 3.6 \left( \frac{\Omega_m h^2}{0.15} \right) \left( \frac{a}{10^{-3}} \right)
\]
Number Continuity

- Photons are not created or destroyed. Without expansion

\[ \dot{n}_\gamma + \nabla \cdot (n_\gamma \mathbf{v}_\gamma) = 0 \]

but the expansion or Hubble flow causes \( n_\gamma \propto a^{-3} \) or

\[ \dot{n}_\gamma + 3n_\gamma \frac{\dot{a}}{a} + \nabla \cdot (n_\gamma \mathbf{v}_\gamma) = 0 \]

- Linearize \( \delta n_\gamma = n_\gamma - \bar{n}_\gamma \)

\[
(\delta n_\gamma) \cdot = -3\delta n_\gamma \frac{\dot{a}}{a} - n_\gamma \nabla \cdot \mathbf{v}_\gamma
\]

\[
\left( \frac{\delta n_\gamma}{n_\gamma} \right) \cdot = -\nabla \cdot \mathbf{v}_\gamma
\]
Continuity Equation

- Number density \( n_\gamma \propto T^3 \) so define temperature fluctuation \( \Theta \)

\[
\frac{\delta n_\gamma}{n_\gamma} = 3 \frac{\delta T}{T} \equiv 3\Theta
\]

- Real space continuity equation

\[
\dot{\Theta} = -\frac{1}{3} \nabla \cdot \mathbf{v}_\gamma
\]

- Fourier space

\[
\dot{\Theta} = -\frac{1}{3} i\mathbf{k} \cdot \mathbf{v}_\gamma
\]
Momentum Conservation

- No expansion: $\dot{q} = F$

- De Broglie wavelength stretches with the expansion

$$\dot{q} + \frac{\dot{a}}{a}q = F$$

for photons this the redshift, for non-relativistic particles expansion drag on peculiar velocities

- Collection of particles: momentum $\rightarrow$ momentum density $(\rho_\gamma + p_\gamma)v_\gamma$ and force $\rightarrow$ pressure gradient

$$[(\rho_\gamma + p_\gamma)v_\gamma] = -4\frac{\dot{a}}{a}(\rho_\gamma + p_\gamma)v_\gamma - \nabla p_\gamma$$

$$\frac{4}{3}\rho_\gamma \dot{v}_\gamma = \frac{1}{3}\nabla \rho_\gamma$$

$$\dot{v}_\gamma = -\nabla \Theta$$
Euler Equation

- Fourier space
  \[ \dot{v}_\gamma = -i k \Theta \]

- Pressure gradients (any gradient of a scalar field) generates a curl-free flow

- For convenience define velocity amplitude:
  \[ v_\gamma \equiv -i v_\gamma \hat{k} \]

- Euler Equation:
  \[ \dot{v}_\gamma = k \Theta \]

- Continuity Equation:
  \[ \dot{\Theta} = -\frac{1}{3} kv_\gamma \]
Oscillator: Take One

• Combine these to form the simple harmonic oscillator equation

\[ \ddot{\Theta} + c_s^2 k^2 \Theta = 0 \]

where the adiabatic sound speed is defined through

\[ c_s^2 \equiv \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma} \]

here \( c_s^2 = 1/3 \) since we are photon-dominated

• General solution:

\[ \Theta(\eta) = \Theta(0) \cos(ks) + \frac{\dot{\Theta}(0)}{kc_s} \sin(ks) \]

where the sound horizon is defined as \( s \equiv \int c_s d\eta \)
Harmonic Extrema

- All modes are frozen in at recombination (denoted with a subscript \(\ast\)) yielding temperature perturbations of different amplitude for different modes. For the adiabatic (curvature mode) \(\dot{\Theta}(0) = 0\)

\[\Theta(\eta\ast) = \Theta(0) \cos(k s\ast)\]

- Modes caught in the extrema of their oscillation will have enhanced fluctuations

\[k_n s\ast = n\pi\]

yielding a fundamental scale or frequency, related to the inverse sound horizon

\[k_A = \pi / s\ast\]

and a harmonic relationship to the other extrema as 1 : 2 : 3...
Peak Location

- The fundamental physical scale is translated into a fundamental angular scale by simple projection according to the angular diameter distance $D_A$

$$\theta_A = \frac{\lambda_A}{D_A}$$
$$\ell_A = k_A D_A$$

- In a flat universe, the distance is simply $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$, the horizon distance, and $k_A = \pi/s_* = \sqrt{3\pi}/\eta_*$ so

$$\theta_A \approx \frac{\eta_*}{\eta_0}$$

- In a matter-dominated universe $\eta \propto a^{1/2}$ so $\theta_A \approx 1/30 \approx 2^\circ$ or

$$\ell_A \approx 200$$
Curvature

- In a **curved universe**, the apparent or **angular diameter distance** is no longer the conformal distance \( D_A = R \sin(D/R) \neq D \)

- Objects in a **closed universe** are **further** than they appear! gravitational **lensing** of the background...

- Curvature scale of the universe must be substantially **larger than current horizon**

- **Flat universe** indicates critical density and implies missing energy given local measures of the matter density “**dark energy**”

- \( D \) also depends on **dark energy density** \( \Omega_{DE} \) and **equation of state** \( w = p_{DE}/\rho_{DE} \).

- Expansion rate at recombination or **matter-radiation ratio** enters into calculation of \( k_A \).
Doppler Effect

- Bulk motion of fluid changes the observed temperature via Doppler shifts

\[
\left( \frac{\Delta T}{T} \right)_{\text{dop}} = \mathbf{n} \cdot \mathbf{v}_\gamma
\]

- Averaged over directions

\[
\left( \frac{\Delta T}{T} \right)_{\text{rms}} = \frac{v_\gamma}{\sqrt{3}}
\]

- Acoustic solution

\[
\frac{v_\gamma}{\sqrt{3}} = -\frac{\sqrt{3}}{k} \dot{\Theta} = \frac{\sqrt{3}}{k} k c_s \Theta(0) \sin(ks)
\]

\[
= \Theta(0) \sin(ks)
\]
Doppler Peaks?

- Doppler effect for the photon dominated system is of equal amplitude and \( \pi/2 \) out of phase: extrema of temperature are turning points of velocity

- Effects add in quadrature:

\[
\left( \frac{\Delta T}{T} \right)^2 = \Theta^2(0)[\cos^2(ks) + \sin^2(ks)] = \Theta^2(0)
\]

- No peaks in \( k \) spectrum! However the Doppler effect carries an angular dependence that changes its projection on the sky

\( \hat{n} \cdot \mathbf{v}_\gamma \propto \hat{n} \cdot \hat{k} \)

- Coordinates where \( \hat{z} \parallel \hat{k} \)

\[
Y_{10}Y_{\ell 0} \rightarrow Y_{\ell \pm 1 0}
\]

recoupling \( j'_\ell Y_{\ell 0} \): no peaks in Doppler effect
Restoring Gravity: Continuity

- Take a simple photon dominated system with gravity
- Continuity altered since a gravitational potential represents a stretching of the spatial fabric that dilutes number densities – formally a spatial curvature perturbation
- Think of this as a perturbation to the scale factor $a \rightarrow a(1 + \Phi)$ so that the cosmological redshift is generalized to

$$\frac{\dot{a}}{a} \rightarrow \frac{\dot{a}}{a} + \dot{\Phi}$$

$$(\delta n_{\gamma})' = -3\delta n_{\gamma} \frac{\dot{a}}{a} - 3n_{\gamma} \dot{\Phi} - n_{\gamma} \nabla \cdot \mathbf{v}_{\gamma}$$

so that the continuity equation becomes

$$\dot{\Theta} = -\frac{1}{3} k v_{\gamma} - \dot{\Phi}$$
Restoring Gravity: Euler

- **Gravitational force** in momentum conservation \( F = -m \nabla \Psi \) generalized to momentum density modifies the **Euler equation** to

\[
\dot{v_\gamma} = k (\Theta + \Psi)
\]

- General relativity says that \( \Phi \) and \( \Psi \) are the relativistic analogues of the **Newtonian potential** and that \( \Phi \approx -\Psi \).

- In our matter-dominated approximation, \( \Phi \) represents matter density fluctuations through the cosmological **Poisson equation**

\[
k^2 \Phi = 4\pi G a^2 \rho_m \Delta_m
\]

where the difference comes from the use of comoving coordinates for \( k \) (\( a^2 \) factor), the removal of the **background density** into the background expansion \( (\rho_m \Delta_m) \) and finally a coordinate subtlety that enters into the definition of \( \Delta_m \).
Constant Potentials

• In the matter dominated epoch potentials are constant because infall generates velocities as \( v_m \sim k\eta \Psi \).

• Velocity divergence generates density perturbations as \( \Delta_m \sim -k\eta v_m \sim -(k\eta)^2 \Psi \).

• And density perturbations generate potential fluctuations as \( \Phi \sim \Delta_m/(k\eta)^2 \sim -\Psi \), keeping them constant. Note that because of the expansion, density perturbations must grow to keep potentials constant.

• Here we have used the Friedman equation \( H^2 = \frac{8\pi G \rho_m}{3} \) and \( \eta = \int d\ln a/(aH) \sim 1/(aH) \).

• More generally, if stress perturbations are negligible compared with density perturbations (\( \delta p \ll \delta \rho \)) then potential will remain roughly constant – more specifically a variant called the Bardeen or comoving curvature \( \zeta \) is constant.
Oscillator: Take Two

- Combine these to form the **simple harmonic oscillator equation**

\[ \ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - \dot{\Phi} \]

- In a CDM dominated expansion \( \dot{\Phi} = \dot{\Psi} = 0 \). Also for photon domination \( c_s^2 = 1/3 \) so the oscillator equation becomes

\[ \ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0 \]

- Solution is just an **offset version** of the original

\[ [\Theta + \Psi](\eta) = [\Theta + \Psi](0) \cos(ks) \]

- \( \Theta + \Psi \) is also the **observed temperature fluctuation** since photons lose energy climbing out of **gravitational potentials** at recombination
Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or effective temperature

\[ \Theta + \Psi \]

- Effective temperature oscillates around zero with amplitude given by the initial conditions
- Note: initial conditions are set when the perturbation is outside of horizon, need inflation or other modification to matter-radiation FRW universe.
- GR says that initial temperature is given by initial potential
Sachs-Wolfe Effect and the Magic 1/3

- A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]

\[
\frac{\delta t}{t} = \Psi
\]

- Convert this to a perturbation in the scale factor,

\[
t = \int \frac{da}{aH} \propto \int \frac{da}{a\rho^{1/2}} \propto a^{3(1+w)/2}
\]

where \( w \equiv p/\rho \) so that during matter domination

\[
\frac{\delta a}{a} = \frac{2}{3} \frac{\delta t}{t}
\]

- CMB temperature is cooling as \( T \propto a^{-1} \) so

\[
\Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3} \Psi
\]
Baryon Loading

- Baryons add extra mass to the photon-baryon fluid
- Controlling parameter is the momentum density ratio:

\[ R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30 \Omega_b h^2 \left( \frac{a}{10^{-3}} \right) \]

of order unity at recombination

- Momentum density of the joint system is conserved

\[ (\rho_\gamma + p_\gamma) v_\gamma + (\rho_b + p_b) v_b \approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma) v_\gamma = (1 + R)(\rho_\gamma + p_\gamma) v_{\gamma b} \]

where the controlling parameter is the momentum density ratio:

\[ R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30 \Omega_b h^2 \left( \frac{a}{10^{-3}} \right) \]

of order unity at recombination
New Euler Equation

- Momentum density ratio enters as

\[
[(1 + R)(\rho \gamma + p \gamma)\mathbf{v}_\gamma b] \cdot = -4\frac{\ddot{a}}{a}(1 + R)(\rho \gamma + p \gamma)\mathbf{v}_\gamma b
- \nabla p \gamma - (1 + R)(\rho \gamma + p \gamma)\nabla \Psi
\]

same as before except for \((1 + R)\) terms so

\[
[(1 + R)\mathbf{v}_\gamma b] \cdot = k\Theta + (1 + R)k\Psi
\]

- Photon continuity remains the same

\[
\dot{\Theta} = -\frac{k}{3}\mathbf{v}_\gamma b - \Phi
\]

- Modification of oscillator equation

\[
[(1 + R)\dot{\Theta}] \cdot + \frac{1}{3}k^2\Theta = -\frac{1}{3}k^2(1 + R)\Psi - [(1 + R)\dot{\Phi}] \cdot
\]
Oscillator: Take Three

- Combine these to form the not-quite-so simple harmonic oscillator equation

\[ c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi}) \]

where \( c_s^2 \equiv \frac{\dot{p}_{\gamma b}}{\dot{\rho}_{\gamma b}} \)

\[ c_s^2 = \frac{1}{3} \frac{1}{1 + R} \]

- In a CDM dominated expansion \( \dot{\Phi} = \dot{\Psi} = 0 \) and the adiabatic approximation \( \dot{R}/R \ll \omega = k c_s \)

\[ [\Theta + (1 + R) \Psi](\eta) = [\Theta + (1 + R) \Psi](0) \cos(k s) \]
Baryon Peak Phenomenology

- Photon-baryon ratio enters in three ways
- Overall larger amplitude:

\[ \left[ \Theta + (1 + R)\Psi \right](0) = \frac{1}{3}(1 + 3R)\Psi(0) \]

- Even-odd peak modulation of effective temperature

\[ \left[ \Theta + \Psi \right]_{\text{peaks}} = \left[ \pm(1 + 3R) - 3R \right] \frac{1}{3}\Psi(0) \]

\[ \left[ \Theta + \Psi \right]_1 - \left[ \Theta + \Psi \right]_2 = [-6R] \frac{1}{3}\Psi(0) \]

- Shifting of the sound horizon down or \( \ell_A \) up

\[ \ell_A \propto \sqrt{1 + R} \]

- Actual effects smaller since \( R \) evolves
Photon Baryon Ratio Evolution

- Oscillator equation has time evolving mass

\[ c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = 0 \]

- Effective mass is \( m_{\text{eff}} = 3c_s^{-2} = (1 + R) \)

- Adiabatic invariant

\[ \frac{E}{\omega} = \frac{1}{2} m_{\text{eff}} \omega A^2 = \frac{1}{2} 3c_s^{-2} k c_s A^2 \propto A^2 (1 + R)^{1/2} = \text{const.} \]

- Amplitude of oscillation \( A \propto (1 + R)^{-1/4} \) decays adiabatically as the photon-baryon ratio changes
Oscillator: Take Three and a Half

- The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator

\[ c_s^2 \frac{d}{d\eta}(c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2} \Phi) \]

changes in the gravitational potentials alter the form of the acoustic oscillations

- If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator

- Term involving \( \Psi \) is the ordinary gravitational force

- Term involving \( \Phi \) involves the \( \dot{\Phi} \) term in the continuity equation as a (curvature) perturbation to the scale factor
Potential Decay

- Matter-to-radiation ratio
  \[
  \frac{\rho_m}{\rho_r} \approx 24 \Omega_m \hbar^2 \left( \frac{a}{10^{-3}} \right)
  \]
  of order unity at recombination in a low \( \Omega_m \) universe

- Radiation is not stress free and so impedes the growth of structure
  \[
  k^2 \Phi = 4\pi G a^2 \rho_r \Delta_r
  \]
  \( \Delta_r \sim 4\Theta \) oscillates around a constant value, \( \rho_r \propto a^{-4} \) so the Newtonian curvature decays.

- General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale
Radiation Driving

- Decay is timed precisely to **drive** the oscillator - close to fully coherent

\[
[\Theta + \Psi](\eta) = [\Theta + \Psi](0) + \Delta \Psi - \Delta \Phi
\]

\[
= \frac{1}{3} \Psi(0) - 2\Psi(0) = \frac{5}{3} \Psi(0)
\]

- **5×** the amplitude of the Sachs-Wolfe effect!

- Coherent approximation is **exact** for a photon-baryon fluid but reality is reduced to \(\sim 4\times\) because of neutrino contribution to radiation

- Actual **initial conditions** are \(\Theta + \Psi = \Psi/2\) for radiation domination but comparison to matter dominated SW correct
External Potential Approach

- Solution to homogeneous equation

\[ (1 + R)^{-1/4} \cos(ks), \quad (1 + R)^{-1/4} \sin(ks) \]

- Give the general solution for an external potential by propagating impulsive forces

\[ (1 + R)^{1/4} \Theta(\eta) = \Theta(0) \cos(ks) + \frac{\sqrt{3}}{k} \left[ \dot{\Theta}(0) + \frac{1}{4} \dot{R}(0) \Theta(0) \right] \sin ks \]

\[ + \frac{\sqrt{3}}{k} \int_0^{\eta} d\eta' (1 + R')^{3/4} \sin[ks - ks'] F(\eta') \]

where

\[ F = -\ddot{\Phi} - \frac{\dot{R}}{1 + R} \dot{\Phi} - \frac{k^2}{3} \Psi \]

- Useful if general form of potential evolution is known
Damping

- Tight coupling equations assume a perfect fluid: no viscosity, no heat conduction.

- Fluid imperfections are related to the mean free path of the photons in the baryons,
  \[ \lambda_C = \frac{1}{\dot{\tau}} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a \]

  where \( \dot{\tau} \) is the conformal opacity to Thompson scattering.

- Dissipation is related to the diffusion length: random walk approximation,
  \[ \lambda_D = \sqrt{N} \lambda_C = \sqrt{\frac{\eta}{\lambda_C}} \lambda_C = \sqrt{\eta \lambda_C} \]

  the geometric mean between the horizon and mean free path.

- \( \lambda_D / \eta_* \sim \text{few \%} \), so expect the peaks \( \gg 3 \) to be affected by dissipation.
Equations of Motion

- Continuity

\[ \dot{\Theta} = -\frac{k}{3} v_\gamma - \dot{\Phi} , \quad \dot{\delta}_b = -k v_b - 3\dot{\Phi} \]

where the photon equation remains unchanged and the baryons follow number conservation with \( \rho_b = m_b n_b \)

- Euler

\[ \dot{v}_\gamma = k(\Theta + \Psi) - \frac{k}{6} \pi_\gamma - \dot{\tau}(v_\gamma - v_b) \]

\[ \dot{v}_b = -\frac{\dot{a}}{a} v_b + k\Psi + \dot{\tau}(v_\gamma - v_b)/R \]

where the photons gain an anisotropic stress term \( \pi_\gamma \) from radiation viscosity and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation
Viscosity

- **Viscosity** is generated from radiation **streaming** from hot to cold regions
- Expect

\[ \pi_\gamma \sim v_\gamma \frac{k}{\dot{T}} \]

generated by streaming, suppressed by **scattering** in a wavelength of the fluctuation. **Radiative transfer** says

\[ \pi_\gamma \approx 2A_v v_\gamma \frac{k}{\dot{T}} \]

where \( A_v = 16/15 \)

\[ \dot{v}_\gamma = k(\Theta + \Psi) - \frac{k}{3} A_v \frac{k}{T} v_\gamma \]
Oscillator: Penultimate Take

- Adiabatic approximation \((\omega \gg \dot{a}/a)\)

\[
\dot{\Theta} \approx -\frac{k}{3} \nu_{\gamma}
\]

- Oscillator equation contains a \(\dot{\Theta}\) damping term

\[
c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\tau} A_v \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})
\]

- Heat conduction term similar in that it is proportional to \(\nu_{\gamma}\) and is suppressed by scattering \(k/\tau\). Expansion of Euler equations to leading order in \(k/\tau\) gives

\[
A_h = \frac{R^2}{1 + R}
\]

since the effects are only significant if the baryons are dynamically important
Oscillator: Final Take

- **Final oscillator equation**

\[ c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} [A_v + A_h] \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi}) \]

- **Solve in the adiabatic approximation**

\[ \Theta \propto \exp(i \int \omega d\eta) \]

\[ -\omega^2 + \frac{k^2 c_s^2}{\dot{\tau}} (A_v + A_h) i\omega + k^2 c_s^2 = 0 \quad (1) \]
Dispersion Relation

• Solve

\[ \omega^2 = k^2 c_s^2 \left[ 1 + i \frac{\omega}{\dot{T}} (A_v + A_h) \right] \]

\[ \omega = \pm k c_s \left[ 1 + \frac{i \omega}{2 \dot{T}} (A_v + A_h) \right] \]

\[ = \pm k c_s \left[ 1 \pm \frac{i k c_s}{2 \dot{T}} (A_v + A_h) \right] \]

• Exponentiate

\[ \exp(i \int \omega d\eta) = e^{\pm ik_s} \exp[-k^2 \int d\eta \frac{1}{2} \frac{c_s^2}{\dot{T}} (A_v + A_h)] \]

\[ = e^{\pm ik_s} \exp[-(k/k_D)^2] \quad (2) \]

• Damping is exponential under the scale \( k_D \)
Diffusion Scale

- Diffusion wavenumber

\[ k_D^{-2} = \int d\eta \frac{1}{\dot{\tau}} \frac{1}{6(1 + R)} \left( \frac{16}{15} + \frac{R^2}{(1 + R)} \right) \]

- Limiting forms

\[ \lim_{R \to 0} k_D^{-2} = \frac{1}{6} \frac{16}{15} \int d\eta \frac{1}{\dot{\tau}} \]

\[ \lim_{R \to \infty} k_D^{-2} = \frac{1}{6} \int d\eta \frac{1}{\dot{\tau}} \]

- Geometric mean between horizon and mean free path as expected from a random walk

\[ \lambda_D = \frac{2\pi}{k_D} \sim \frac{2\pi}{\sqrt{6}} (\eta \dot{\tau}^{-1})^{1/2} \]
Astro 282
Idealized Data Analysis
Gaussian Statistics

- Statistical isotropy says two-point correlation depends only on the power spectrum

\[ \Theta(\hat{n}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m}(\hat{n}) \]

\[ \langle \Theta^*_{\ell m} \Theta_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{mm'} C^\Theta\Theta_{\ell} \]

- Reality of field says \( \Theta_{\ell m} = (-1)^m \Theta_{\ell(-m)} \)

- For a Gaussian random field, power spectrum defines all higher order statistics, e.g.

\[ \langle \Theta_{\ell_1 m_1} \Theta_{\ell_2 m_2} \Theta_{\ell_3 m_3} \Theta_{\ell_4 m_4} \rangle \]

\[ = (-1)^{m_1 + m_2} \delta_{\ell_1 \ell_3} \delta_{m_1(-m_3)} \delta_{\ell_2 \ell_4} \delta_{m_2(-m_4)} C^\Theta\Theta_{\ell_1} C^\Theta\Theta_{\ell_2} + \text{all pairs} \]
Idealized Statistical Errors

- Take a noisy estimator of the multipoles in the map

\[ \hat{\Theta}_{\ell m} = \Theta_{\ell m} + N_{\ell m} \]

and take the noise to be statistically isotropic

\[ \langle N_{\ell m}^* N_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_{\ell}^{NN} \]

- Construct an unbiased estimator of the power spectrum

\[ \langle \hat{C}_{\ell}^{\Theta \Theta} \rangle = C_{\ell}^{\Theta \Theta} \]

\[ \hat{C}_{\ell}^{\Theta \Theta} = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} \hat{\Theta}_{\ell m}^* \hat{\Theta}_{\ell m} - C_{\ell}^{NN} \]

- Variance in estimator

\[ \langle \hat{C}_{\ell}^{\Theta \Theta} \hat{C}_{\ell}^{\Theta \Theta} \rangle - \langle \hat{C}_{\ell}^{\Theta \Theta} \rangle^2 = \frac{2}{2\ell + 1} (C_{\ell}^{\Theta \Theta} + C_{\ell}^{NN})^2 \]
Cosmic and Noise Variance

- RMS in estimator is simply the total power spectrum reduced by \( \sqrt{2/N_{\text{modes}}} \) where \( N_{\text{modes}} \) is the number of \( m \)-mode measurements.
- Even a perfect experiment where \( C_{\ell}^{NN} = 0 \) has statistical variance due to the Gaussian random realizations of the field. This cosmic variance is the result of having only one realization to measure.
- Noise variance is often approximated as white detector noise. Removing the beam to place the measurement on the sky

\[
N^\Theta\Theta_\ell = \left( \frac{T}{dt} \right)^2 \ell(\ell+1)\sigma^2 = \left( \frac{T}{dt} \right)^2 e^{\ell(\ell+1)\text{FWHM}^2/8 \ln 2}
\]

where \( dt \) can be thought of as a noise level per steradian of the temperature measurement, \( \sigma \) is the Gaussian beam width, FWHM is the full width at half maximum of the beam.
Idealized Parameter Forecasts

- A crude propagation of errors is often useful for estimation purposes.
- Suppose $C_{\alpha\beta}$ describes the covariance matrix of the estimators for a given parameter set $\pi_\alpha$.
- Define $F = C^{-1}$ [formalized as the Fisher matrix later]. Making an infinitesimal transformation to a new set of parameters $p_\mu$

$$F_{\mu\nu} = \sum_{\alpha\beta} \frac{\partial \pi_\alpha}{\partial p_\mu} F_{\alpha\beta} \frac{\partial \pi_\beta}{\partial p_\nu}$$

- In our case $\pi_\alpha$ are the $C_\ell$ the covariance is diagonal and $p_\mu$ are cosmological parameters

$$F_{\mu\nu} = \sum_{\ell} \frac{2\ell + 1}{2(C_\ell^{\Theta\Theta} + C_\ell^{NN})^2} \frac{\partial C_\ell^{\Theta\Theta}}{\partial p_\mu} \frac{\partial C_\ell^{\Theta\Theta}}{\partial p_\nu}$$
Idealized Parameter Forecasts

- Polarization handled in same way (requires covariance)
- Fisher matrix represents a local approximation to the transformation of the covariance and hence is only accurate for well constrained directions in parameter space
- Derivatives evaluated by finite difference
- Fisher matrix identifies parameter degeneracies but only the local direction – i.e. all errors are ellipses not bananas
Beyond Idealizations: Time Ordered Data

- For the data analyst the starting point is a string of “time ordered” data coming out of the instrument (post removal of systematic errors!)

- Begin with a model of the time ordered data as

\[ d_t = P_{ti} \Theta_i + n_t \]

where \( i \) denotes pixelized positions indexed by \( i \), \( d_t \) is the data in a time ordered stream indexed by \( t \). Number of time ordered data will be of the order \( 10^{10} \) for a satellite! number of pixels \( 10^6 - 10^7 \).

- The noise \( n_t \) is drawn from a distribution with a known power spectrum

\[ \langle n_t n_{t'} \rangle = C_{d,tt'} \]
Pointing Matrix

- The pointing matrix \( P \) is the mapping between pixel space and the time ordered data.
- Simplest incarnation: row with all zeros except one column which just says what point in the sky the telescope is pointing at that time.

\[
P = \begin{pmatrix}
0 & 0 & 1 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 1 & \ldots & 0 \\
\end{pmatrix}
\]

- More generally incorporates differencing, beam, rotation (for polarization).
Maximum Likelihood Mapmaking

- What is the best estimator of the underlying map $\Theta_i$?
- Likelihood function: the probability of getting the data given the theory $\mathcal{L} \equiv P[\text{data}|\text{theory}]$. In this case, the theory is the set of parameters $\Theta_i$.

$$
\mathcal{L}_{\Theta}(d_t) = \frac{1}{(2\pi)^{N_t/2} \sqrt{\det C_d}} \exp \left[ -\frac{1}{2} (d_t - P_{ti}\Theta_i) C_{d,tt'}^{-1} (d_{t'} - P_{t'j}\Theta_j) \right].
$$

- Bayes theorem says that $P[\Theta_i|d_t]$, the probability that the temperatures are equal to $\Theta_i$ given the data, is proportional to the likelihood function times a prior $P(\Theta_i)$, taken to be uniform

$$
P[\Theta_i|d_t] \propto P[d_t|\Theta_i] \equiv \mathcal{L}_{\Theta}(d_t)
$$
Maximum Likelihood Mapmaking

- Maximizing the likelihood of $\Theta_i$ is simple since the log-likelihood is quadratic.

- Differentiating the argument of the exponential with respect to $\Theta_i$ and setting to zero leads immediately to the estimator

$$\hat{\Theta}_i = C_{N,i,j} P_{jt} C_{d,tt'}^{-1} d_{t'},$$

where $C_N \equiv (P^{\text{tr}} C_d^{-1} P)^{-1}$ is the covariance of the estimator.

- Given the large dimension of the time ordered data, direct matrix manipulation is unfeasible. A key simplifying assumption is the stationarity of the noise, that $C_{d,tt'}$ depends only on $t - t'$ (temporal statistical homogeneity).
Foregrounds

- Maximum likelihood mapmaking can be applied to the time streams of multiple observations frequencies $N_\nu$ and hence obtain multiple maps.

- A cleaned CMB map can be obtained by modeling the maps as

$$\hat{\Theta}_i^\nu = A_i^\nu \Theta_i + n_i^\nu + f_i^\nu$$

where $A_i^\nu = 1$ if all the maps are at the same resolution (otherwise, embed the beam as in the pointing matrix; $f_i^\nu$ is the noise contributed by the foregrounds).

- Again, a map making problem. Given a covariance matrix for foregrounds noise (a prior from other data), same solution. Alternately, can derive weights from stats of the recovered maps.

- 5 foregrounds: synchrotron, free-free, radio pt sources, at low frequencies and dust and IR pt sources at high frequencies.
The next step in the chain of inference is the power spectrum extraction. Here the correlation between pixels is modelled through the power spectrum

\[ C_{S,ij} \equiv \langle \Theta_i \Theta_j \rangle = \sum_{\ell} \Delta_{T,\ell}^2 W_{\ell,ij} \]

Here \( W_\ell \), the window function, is derived by writing down the expansion of \( \Theta(\hat{n}) \) in harmonic space, including smoothing by the beam and pixelization.

For example in the simple case of a gaussian beam of width \( \sigma \) it is proportional to the Legendre polynomial \( P_\ell(\hat{n}_i \cdot \hat{n}_j) \) for the pixel separation multiplied by \( b_\ell^2 \propto e^{-\ell(\ell+1)\sigma^2} \)
• In principle the underlying theory to extract from maximum likelihood is the power spectrum at every $\ell$

• However with a finite patch of sky, it is not possible to extract multipoles separated by $\Delta \ell < 2\pi / L$ where $L$ is the dimension of the survey

• So consider instead a theory parameterization of $\Delta^2_{T, \ell}$ constant in bands of $\Delta \ell$ chosen to match the survey forming a set of band powers $B_a$

• The likelihood of the bandpowers given the pixelized data is

$$L_B(\Theta_i) = \frac{1}{(2\pi)^{N_p/2} \sqrt{\det C_\Theta}} \exp \left( -\frac{1}{2} \Theta_i C^{-1}_{\Theta, ij} \Theta_j \right)$$

where $C_\Theta = C_S + C_N$ and $N_p$ is the number of pixels in the map.
Band Power Estimation

- As before, $L_B$ is Gaussian in the anisotropies $\Theta_i$, but in this case $\Theta_i$ are not the parameters to be determined; the theoretical parameters are the $B_a$, upon which the covariance matrix depends.

- The likelihood function is not Gaussian in the parameters, and there is no simple, analytic way to find the maximum likelihood bandpowers.

- Iterative approach to maximizing the likelihood: take a trial point $B_a^{(0)}$ and improve estimate based a Newton-Rhapson approach to finding zeros

\[
\hat{B}_a = \hat{B}_a^{(0)} + \hat{F}_{B,ab}^{-1} \frac{\partial \ln L_B}{\partial B_b}
\]

\[
= \hat{B}_a^{(0)} + \frac{1}{2} \hat{F}_{B,ab}^{-1} \left( \Theta_i C_{\Theta,ij}^{-1} \frac{\partial C_{\Theta,kl}}{\partial B_b} \Theta_l - C_{\Theta,ij}^{-1} \frac{\partial C_{\Theta,ji}}{\partial B_b} \right),
\]
The expectation value of the local curvature is the Fisher matrix

\[ F_{B,ab} \equiv \left\langle -\frac{\partial^2 \ln \mathcal{L}_B}{\partial B_a \partial B_b} \right\rangle \]

\[ = \frac{1}{2} C_{\Theta,ij}^{-1} \frac{\partial C_{\Theta,jk}}{\partial B_a} C_{\Theta,kl}^{-1} \frac{\partial C_{\Theta,li}}{\partial B_b}. \]

This is a general statement: for a gaussian distribution the Fisher matrix

\[ F_{ab} = \frac{1}{2} \text{Tr}[C^{-1}C_{,a}C^{-1}C_{,b}] \]

Kramer-Rao identity says that the best possible covariance matrix on a set of parameters is \( C = F^{-1} \)

Thus, the iteration returns an estimate of the covariance matrix of the estimators \( C_B \)
Cosmological Parameters

- The probability distribution of the bandpowers given the cosmological parameters $c_i$ is not Gaussian but it is often an adequate approximation

$$\mathcal{L}_c(\hat{B}_a) \approx \frac{1}{(2\pi)^{N_c/2} \sqrt{\det C_B}} \exp \left[ -\frac{1}{2} (\hat{B}_a - B_a) C_B^{-1} (\hat{B}_b - B_b) \right]$$

- Grid based approaches evaluate the likelihood in cosmological parameter space and maximize

- Faster approaches monte carlo the exploration of the likelihood space intelligently ("Monte Carlo Markov Chains")

- Since the number of cosmological parameters in the working model is $N_c \sim 10$ this represents a final radical compression of information in the original timestream which recall has up to $N_t \sim 10^{10}$ data points.
MCMC

- Monte Carlo Markov Chain: a random walk in parameter space
- Start with a set of cosmological parameters $c^m$, compute likelihood
- Take a random step in parameter space to $c^{m+1}$ of size drawn from a multivariate Gaussian (a guess at the parameter covariance matrix) $C_c$ (e.g. from the crude Fisher approximation. Compute likelihood.
- Draw a random number between 0,1 and if the likelihood ratio exceeds this value take the step (add to Markov chain); if not then do not take the step (add the original point to the Markov chain). Goto 3.
MCMC

- With a complete chain of $N_M$ elements, compute the mean of the chain and its variance

$$\bar{c}_i = \frac{1}{N_M} \sum_{m=1}^{N_M} c_i^m$$

$$\sigma^2(c_i) = \frac{1}{N_M - 1} \sum_{m=1}^{N_M} (c_i^m - \bar{c}_i)^2$$

- Trick is in assuring burn in (not sensitive to initial point), step size, and convergence

- Usually requires running multiple chains. Typically tens of thousands of elements per chain.
Radical Compression

- Started with time ordered data $\sim 10^{10}$ numbers for a satellite experiment
- Compressed to a map assuming a CMB spectrum (and time independent fluctuations) $\sim 10^7$ numbers
- Compressed to a power spectrum (Gaussian statistics) independent of $m$ (statistical isotropy) $\sim 10^3$ numbers
- Compressed to cosmological parameters (a cosmological model) $\sim 10^3$
- A factor of $10^9$ reduction in the representation. Nature is very efficient.
Parameter Forecasts

- The Fisher matrix of the cosmological parameters becomes

\[ F_{c,ij} = \frac{\partial B_a}{\partial c_i} C_{B,ab}^{-1} \frac{\partial B_b}{\partial c_j}. \]

which is the error propagation formula discussed above

- The Fisher matrix can be more accurately defined for an experiment by taking the pixel covariance and using the general formula for the Fisher matrix of gaussian data

\[ F_{ij} = \sum_\ell \frac{(2\ell + 1) f_{\text{sky}}}{2(C_\ell^{\Theta \Theta} + C_\ell^{NN})^2} \frac{\partial C_\ell^{\Theta \Theta}}{\partial c_i} \frac{\partial C_\ell^{\Theta \Theta}}{\partial c_j} \]

where the sky fraction \( f_{\text{sky}} \) quantifies the loss of independent modes due to the sky cut
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Polarization
Stokes Parameters

• Polarization state of radiation in direction $\hat{n}$ described by the intensity matrix $\langle E_i(\hat{n}) E^*_j(\hat{n}) \rangle$, where $E$ is the electric field vector and the brackets denote time averaging.

• As a hermitian matrix, it can be decomposed into the Pauli basis

$$P = C \langle E(\hat{n}) E^\dagger(\hat{n}) \rangle$$

$$= \Theta(\hat{n}) \sigma_0 + Q(\hat{n}) \sigma_3 + U(\hat{n}) \sigma_1 + V(\hat{n}) \sigma_2,$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

• Stokes parameters recovered as $\text{Tr}(\sigma_i P)/2$
Monochromatic Wave

- A pure monochromatic wave is fully polarized

\[ \mathbf{E} = E_1 \mathbf{e}_1 + E_2 \mathbf{e}_2 \]

where

\[ E_{1,2} = \text{Re}[A_{1,2} e^{i \phi_{1,2}} e^{i(k \cdot \mathbf{x} - \omega t)}] \]

- Implies \( \Theta^2 = Q^2 + U^2 + V^2 \)

- However a finite bandwidth leads to a sum of components

\[ \mathbf{E} = \sum_{\alpha} \mathbf{E}^{\alpha} \]
Partial Polarization

- A signal of finite bandwidth is only partially polarized since the time averaging will destroy the correlation between the frequency components

\[ \langle E E^\dagger \rangle = \sum_\alpha \langle E^\alpha E^{\alpha\dagger} \rangle \]

- Stokes parameters then add

\[ \Theta = \sum_\alpha \Theta^\alpha, \quad Q = \sum_\alpha Q^\alpha, \quad U = \sum_\alpha U^\alpha, \quad V = \sum_\alpha V^\alpha \]

- Result is \( \Theta^2 > Q^2 + U^2 + V^2 \) (since \( Q, U, V \) have either sign) or partially polarized radiation - like a mixed state in quantum mechanics)
Linear Polarization

• $Q \propto \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle$, $U \propto \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle$.

• Counterclockwise rotation of axes by $\theta = 45^\circ$

$$E_1 = (E_1' - E_2')/\sqrt{2}, \quad E_2 = (E_1' + E_2')/\sqrt{2}$$

• $U \propto \langle E_1' E_1'^* \rangle - \langle E_2' E_2'^* \rangle$, difference of intensities at $45^\circ$ or $Q'$

• More generally, $P$ transforms as a tensor under rotations and

$$Q' = \cos(2\theta)Q + \sin(2\theta)U$$

$$U' = -\sin(2\theta)Q + \cos(2\theta)U$$

• or

$$Q' \pm iU' = e^{\mp 2i\theta} [Q \pm iU]$$

acquires a phase under rotation and is a spin $\pm 2$ object
Coordinate Independent Representation

- Two directions: orientation of polarization and change in amplitude, i.e. $Q$ and $U$ in the basis of the Fourier wavevector (pointing with angle $\phi_l$) for small sections of sky are called $E$ and $B$ components

$$E(l) \pm iB(l) = - \int d\hat{n}[Q'(\hat{n}) \pm iU'(\hat{n})]e^{-i\hat{l} \cdot \hat{n}}$$

$$= -e^{\mp 2i\phi_l} \int d\hat{n}[Q(\hat{n}) \pm iU(\hat{n})]e^{-i\hat{l} \cdot \hat{n}}$$

- For the $B$-mode to not vanish, the polarization must point in a direction not related to the wavevector - not possible for density fluctuations in linear theory

- Generalize to all-sky: plane waves are eigenmodes of the Laplace operator on the tensor $\mathbf{P}$. 
Spin Harmonics

- Laplace Eigenfunctions

\[ \nabla^2_{\pm 2} Y_{\ell m}[\sigma_3 \mp i\sigma_1] = -[l(l + 1) - 4]_{\pm 2} Y_{\ell m}[\sigma_3 \mp i\sigma_1] \]

- Spin $s$ spherical harmonics: orthogonal and complete

\[ \int d\hat{n}_s Y_{\ell m}^*(\hat{n})_s Y_{\ell m}(\hat{n}) = \delta_{\ell \ell'} \delta_{m m'} \]

\[ \sum_{\ell m} s Y_{\ell m}^*(\hat{n})_s Y_{\ell m}(\hat{n}') = \delta(\phi - \phi') \delta(\cos \theta - \cos \theta') \]

where the ordinary spherical harmonics are $Y_{\ell m} = 0 Y_{\ell m}$

- Given in terms of the rotation matrix

\[ s Y_{\ell m}(\beta \alpha) = (-1)^m \sqrt{\frac{2\ell + 1}{4\pi}} D_{-m s}^{\ell}(\alpha \beta 0) \]
Statistical Representation

- All-sky decomposition

\[
[Q(\hat{n}) \pm iU(\hat{n})] = \sum_{\ell m} [E_{\ell m} \pm iB_{\ell m}] \pm 2Y_{\ell m}(\hat{n})
\]

- Power spectra

\[
\langle E_{\ell m}^* E_{\ell m} \rangle = \delta_{\ell \ell'}\delta_{mm'}C_{\ell}^{EE}
\]
\[
\langle B_{\ell m}^* B_{\ell m} \rangle = \delta_{\ell \ell'}\delta_{mm'}C_{\ell}^{BB}
\]

- Cross correlation

\[
\langle E_{\ell m}^* E_{\ell m} \rangle = \delta_{\ell \ell'}\delta_{mm'}C_{\ell}^{EE}
\]

others vanish if parity is conserved
Thomson Scattering

- Differential cross section

\[ \frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{E}' \cdot \hat{E}|^2 \sigma_T, \]

where \( \sigma_T = \frac{8\pi \alpha^2}{3m_e} \) is the Thomson cross section, \( \hat{E}' \) and \( \hat{E} \) denote the incoming and outgoing directions of the electric field or polarization vector.

- Summed over angle and incoming polarization

\[ \sum_{i=1,2} \int \hat{n}' \frac{d\sigma}{d\Omega} = \sigma_T \]
Polarization Generation

- Heuristic: incoming radiation shakes an electron in direction of electric field vector $\hat{E}'$
- Radiates photon with polarization also in direction $\hat{E}'$
- But photon cannot be longitudinally polarized so that scattering into $90^\circ$ can only pass one polarization
- Linearly polarized radiation like polarization by reflection
- Unlike reflection of sunlight, incoming radiation is nearly isotropic
- Missing linear polarization supplied by scattering from direction orthogonal to original incoming direction
- Only quadrupole anisotropy generates polarization by Thomson scattering
Acoustic Polarization

- Break down of tight-coupling leads to quadrupole anisotropy of
  \[ \pi_{\gamma} \approx \frac{k}{\dot{\tau}} v_{\gamma} \]

- Scaling \( k_D = (\dot{\tau}/\eta_*)^{1/2} \rightarrow \dot{\tau} = k_D^2 \eta_* \)

- Know: \( k_D s_* \approx k_D \eta_* \approx 10 \)

- So:
  \[ \pi_{\gamma} \approx \frac{k}{k_D} \frac{1}{10} v_{\gamma} \]
  \[ \Delta_P \approx \frac{\ell}{\ell_D} \frac{1}{10} \Delta_T \]
Acoustic Polarization

• Gradient of velocity is along direction of wavevector, so polarization is pure $E$-mode

• Velocity is $90^\circ$ out of phase with temperature – turning points of oscillator are zero points of velocity:

$$\Theta + \Psi \propto \cos(ks); \quad v_\gamma \propto \sin(ks)$$

• Polarization peaks are at troughs of temperature power
Cross Correlation

- Cross correlation of temperature and polarization

\[(\Theta + \Psi)(\nu) \propto \cos(ks) \sin(ks) \propto \sin(2ks)\]

- Oscillation at twice the frequency

- Correlation: radial or tangential around hot spots

- Partial correlation: easier to measure if polarization data is noisy, harder to measure if polarization data is high S/N or if bands do not resolve oscillations

- Good check for systematics and foregrounds

- Comparison of temperature and polarization is proof against features in initial conditions mimicking acoustic features
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Formal CMB Theory & CMBFAST
Boltzmann Equation

- CMB radiation is generally described by the phase space distribution function for each polarization state $f_a(x, q, \eta)$, where $x$ is the comoving position and $q$ is the photon momentum.
- Boltzmann equation describes the evolution of the distribution function under gravity and collisions.
- Low order moments of the Boltzmann equation are simply the covariant conservation equations.
- Higher moments provide the closure condition to the conservation law (specification of stress tensor) and the CMB observable – fine scale anisotropy.
- Higher moments mainly describe the simple geometry of source projection.
Liouville Equation

- In absence of scattering, the phase space distribution of photons is conserved along the propagation path
- Rewrite variables in terms of the photon propagation direction \( q = q \hat{n} \), so \( f_a(x, \hat{n}, q, \eta) \) and

\[
\frac{d}{d\eta} f_a(x, \hat{n}, q, \eta) = 0
\]

\[
= \left( \frac{\partial}{\partial \eta} + \frac{dx}{d\eta} \cdot \frac{\partial}{\partial x} + \frac{d\hat{n}}{d\eta} \cdot \frac{\partial}{\partial \hat{n}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a
\]

- For simplicity, assume spatially flat universe \( K = 0 \) then \( d\hat{n}/d\eta = 0 \) and \( dx = \hat{n} d\eta \)

\[
\dot{f}_a + \hat{n} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0
\]
Correspondence to Einstein Eqn.

- Geodesic equation gives the redshifting term

\[
\frac{\dot{q}}{q} = -\frac{\dot{a}}{a} - \frac{1}{2} n^i n^j \dot{H}_{Tij} - \dot{H}_L + n^i \dot{B}_i - \hat{n} \cdot \nabla A
\]

- which is incorporated in the conservation and gauge transformation equations

- Stress energy tensor involves integrals over the distribution function the two polarization states

\[
T^{\mu\nu} = \int \frac{d^3q}{(2\pi)^3} \frac{q^\mu q^{\nu}}{E} (f_a + f_b)
\]

- Components are simply the low order angular moments of the distribution function
Angular Moments

- Define the angularly dependent temperature perturbation

\[ \Theta(x, \hat{n}, \eta) = \frac{1}{4\rho_\gamma} \int \frac{q^3 dq}{2\pi^2} (f_a + f_b) - 1 \]

and likewise for the linear polarization states \(Q\) and \(U\)

- Decompose into normal modes: plane waves for spatial part and spherical harmonics for angular part

\[ G_m^\ell(k, x, \hat{n}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} Y_m^\ell(\hat{n}) \exp(ik \cdot x) \]

\[ \pm 2G_m^\ell(k, x, \hat{n}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} \pm 2Y_m^\ell(\hat{n}) \exp(ik \cdot x) \]

- In a spatially curved universe generalize the plane wave part
Normal Modes

• Temperature and polarization fields

\[ \Theta(x, \hat{n}, \eta) = \int \frac{d^3 k}{(2\pi)^3} \sum_{\ell m} \Theta_{\ell}^{(m)} G_{\ell}^{m} \]

\[ [Q \pm iU](x, \hat{n}, \eta) = \int \frac{d^3 k}{(2\pi)^3} \sum_{\ell m} [E_{\ell}^{(m)} \pm iB_{\ell}^{(m)}]_{\pm 2} G_{\ell}^{m} \]

• For each \( k \) mode, work in coordinates where \( k \parallel z \) and so \( m = 0 \) represents scalar modes, \( m = \pm 1 \) vector modes, \( m = \pm 2 \) tensor modes, \( |m| > 2 \) vanishes. Since modes add incoherently and \( Q \pm iU \) is invariant up to a phase, rotation back to a fixed coordinate system is trivial.
Scalar, Vector, Tensor

Normalization of modes is chosen so that the lowest angular mode for scalars, vectors, and tensors are normalized in the same way as the mode function

\[ G_0^0 = Q^{(0)} \]
\[ G_1^0 = n^i Q_i^{(0)} \]
\[ G_2^0 \propto n^i n^j Q_{ij}^{(0)} \]
\[ G_1^{\pm 1} = n^i Q_i^{(\pm 1)} \]
\[ G_2^{\pm 1} \propto n^i n^j Q_{ij}^{(\pm 1)} \]
\[ G_2^{\pm 2} = n^i n^j Q_{ij}^{(\pm 2)} \]

where recall

\[ Q^{(0)} = \exp(ik \cdot x) \]
\[ Q_i^{(\pm 1)} = \frac{-i}{\sqrt{2}} (\hat{e}_1 \pm i\hat{e}_2)_i \exp(ik \cdot x) \]
\[ Q_{ij}^{(\pm 2)} = -\sqrt{\frac{3}{8}} (\hat{e}_1 \pm i\hat{e}_2)_i (\hat{e}_1 \pm i\hat{e}_2)_j \exp(ik \cdot x) \]
Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies.

- Spatial gradient term hits plane wave:

\[
\hat{n} \cdot \nabla e^{ik \cdot x} = i\hat{n} \cdot k e^{ik \cdot x} = i\sqrt{\frac{4\pi}{3}} k Y_0^0(\hat{n}) e^{ik \cdot x}
\]

- Dipole term adds to angular dependence through the addition of angular momentum:

\[
\sqrt{\frac{4\pi}{3}} Y_1^0 Y_{\ell}^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell + 1)(2\ell - 1)}} Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell + 1)(2\ell + 3)}} Y_{\ell+1}^m
\]

where \(\kappa_\ell^m = \sqrt{\ell^2 - m^2}\) is given by Clebsch-Gordon coefficients.
Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

\[ \dot{\Theta}^{(m)}_{\ell} = k \left[ \frac{k^m_{\ell}}{2\ell + 1} \Theta^{(m)}_{\ell-1} - \frac{k^m_{\ell+1}}{2\ell + 3} \Theta^{(m)}_{\ell+1} \right] - \dot{\tau} \Theta^{(m)}_{\ell} + S_{\ell}^{(m)} \]

where \( S_{\ell}^{(m)} \) are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic \( \ell = 0 \) temperature perturbation will eventually become a high order anisotropy by “free streaming” or simple projection

- Original CMB codes solved the full hierarchy equations out to the \( \ell \) of interest.
Integral Solution

- Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface.

- In general, the solution describes the decomposition of the source $S^{(m)}_\ell$ with its local angular dependence as seen at a distance $x = D\hat{n}$.

- Proceed by decomposing the angular dependence of the plane wave

$$e^{ik\cdot x} = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell + 1)} j_\ell(kD)Y^0_\ell(\hat{n})$$

- Recouple to the local angular dependence of $G^m_\ell$

$$G^m_{\ell_s} = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell + 1)}\alpha^m_{\ell_s\ell}(kD)Y^m_\ell(\hat{n})$$
Integral Solution

- Projection kernels:
  \[ \ell_s = 0, \quad m = 0 \]
  \[ \alpha_0^{(0)} \equiv j_\ell \]
  \[ \ell_s = 1, \quad m = 0 \]
  \[ \alpha_1^{(0)} \equiv j_\ell' \]

- Integral solution:
  \[
  \frac{\Theta_{\ell}^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S_{\ell_s}^{(m)} \alpha_{\ell_s \ell}^{(m)} (k(\eta_0 - \eta))
  \]

- Power spectrum:
  \[
  C_\ell = \frac{2}{\pi} \int \frac{dk}{k} \sum_m k^3 \langle \Theta_{\ell}^{(m)*} \Theta_{\ell}^{(m)} \rangle \frac{1}{(2\ell + 1)^2}
  \]

- Solving for \( C_\ell \) reduces to solving for the behavior of a handful of sources
Polarization Hierarchy

• In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hierarchy:

\[
\dot{E}_\ell^{(m)} = k \left[ \frac{2\kappa^m_\ell}{2\ell - 1} E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell + 1)} B_\ell^{(m)} - \frac{2\kappa^m_{\ell+1}}{2\ell + 3} \right] - \dot{\tau} E_\ell^{(m)} + \mathcal{E}_\ell^{(m)}
\]

\[
\dot{B}_\ell^{(m)} = k \left[ \frac{2\kappa^m_\ell}{2\ell - 1} B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell + 1)} B_\ell^{(m)} - \frac{2\kappa^m_{\ell+1}}{2\ell + 3} \right] - \dot{\tau} E_\ell^{(m)} + \mathcal{B}_\ell^{(m)}
\]

where \(2\kappa^m_\ell = \sqrt{(\ell^2 - m^2)(\ell^2 - 4)/\ell^2}\) is given by the Clebsch-Gordon coefficients and \(\mathcal{E}, \mathcal{B}\) are the sources (scattering only).

• Note that for vectors and tensors \(|m| > 0\) and \(B\) modes may be generated from \(E\) modes by projection. Cosmologically \(\mathcal{B}_\ell^{(m)} = 0\)
Polarization Integral Solution

• Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

\[
\frac{E_{\ell}^{(m)}(k, \eta_0)}{2\ell + 1} = \int_{0}^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{s\ell}^{(m)}(k) e_{s\ell}^{(m)}(k(\eta_0 - \eta))
\]

\[
\frac{B_{\ell}^{(m)}(k, \eta_0)}{2\ell + 1} = \int_{0}^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{s\ell}^{(m)}(k) \beta_{s\ell}^{(m)}(k(\eta_0 - \eta))
\]

• The only source to the polarization is from the quadrupole anisotropy so we only need \( \ell_s = 2 \), e.g. for scalars

\[
\epsilon_{2\ell}^{(0)}(x) = \sqrt{\frac{3 (\ell + 2)!}{8 (\ell - 2)!}} \frac{j_{\ell}(x)}{x^2} \quad \beta_{2\ell}^{(0)} = 0
\]
Truncated Hierarchy

- CMBFast uses the integral solution and relies on a fast $j_\ell$ generator.
- However sources are not external to system and are defined through the Boltzmann hierarchy itself.
- Solution: recall that we used this technique in the tight coupling regime by applying a closure condition from tight coupling.
- CMBFast extends this idea by solving a truncated hierarchy of equations, e.g. out to $\ell = 25$ with non-reflecting boundary conditions.
Thomson Collision Term

• Full Boltzmann equation

\[
\frac{d}{d\eta} f_{a,b} = C[f_a, f_b]
\]

• Collision term describes the scattering out of and into a phase space element

• Thomson collision based on differential cross section

\[
\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{E}' \cdot \hat{E}|^2 \sigma_T ,
\]

where \( \hat{E}' \) and \( \hat{E} \) denote the incoming and outgoing directions of the electric field or polarization vector.
Scattering Calculation

- Start in the electron rest frame and in a coordinate system fixed by the scattering plane, spanned by incoming and outgoing directional vectors $-\mathbf{n}' \cdot \mathbf{n} = \cos \beta$, where $\beta$ is the scattering angle.

- $\Theta_{\parallel}$: in-plane polarization state; $\Theta_{\perp}$: $\perp$-plane polarization state

- Transfer probability (constant set by $\hat{\tau}$)

  $$\Theta_{\parallel} \propto \cos^2 \beta \Theta'_{\parallel}, \quad \Theta_{\perp} \propto \Theta'_{\perp}$$

- and with the $45^\circ$ axes as

  $$\hat{\mathbf{E}}_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\parallel} + \hat{\mathbf{E}}_{\perp}), \quad \hat{\mathbf{E}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\parallel} - \hat{\mathbf{E}}_{\perp})$$
Stokes Parameters

- Define the temperature in this basis

\[ \Theta_1 \propto |\hat{E}_1 \cdot \hat{E}_1|^2 \Theta'_1 + |\hat{E}_1 \cdot \hat{E}_2|^2 \Theta'_2 \]
\[ \propto \frac{1}{4} (\cos \beta + 1)^2 \Theta'_1 + \frac{1}{4} (\cos \beta - 1)^2 \Theta'_2 \]
\[ \Theta_2 \propto |\hat{E}_2 \cdot \hat{E}_2|^2 \Theta'_2 + |\hat{E}_2 \cdot \hat{E}_1|^2 \Theta'_1 \]
\[ \propto \frac{1}{4} (\cos \beta + 1)^2 \Theta'_2 + \frac{1}{4} (\cos \beta - 1)^2 \Theta'_1 \]

or \( \Theta_1 - \Theta_2 \propto \cos \beta (\Theta'_1 - \Theta'_2) \)

- Define \( \Theta, Q, U \) in the scattering coordinates

\[ \Theta \equiv \frac{1}{2} (\Theta_\parallel + \Theta_\perp), \quad Q \equiv \frac{1}{2} (\Theta_\parallel - \Theta_\perp), \quad U \equiv \frac{1}{2} (\Theta_1 - \Theta_2) \]
Scattering Matrix

- Transfer of Stokes states, e.g.
  \[ \Theta = \frac{1}{2}(\Theta_\parallel + \Theta_\perp) \propto \frac{1}{4}(\cos^2 \beta + 1)\Theta' + \frac{1}{4}(\cos^2 \beta - 1)Q' \]

- Transfer matrix of Stokes state \( T \equiv (\Theta, Q + iU, Q - iU) \)

\[ T \propto S(\beta)T' \]

\[ S(\beta) = \frac{3}{4} \begin{pmatrix}
\cos^2 \beta + 1 & -\frac{1}{2}\sin^2 \beta & -\frac{1}{2}\sin^2 \beta \\
-\frac{1}{2}\sin^2 \beta & \frac{1}{2}(\cos \beta + 1)^2 & \frac{1}{2}(\cos \beta - 1)^2 \\
-\frac{1}{2}\sin^2 \beta & \frac{1}{2}(\cos \beta - 1)^2 & \frac{1}{2}(\cos \beta + 1)^2
\end{pmatrix} \]

normalization factor of 3 is set by photon conservation in scattering
Scattering Matrix

- Transform to a fixed basis, by a rotation of the incoming and outgoing states $T = R(\psi)T$ where

$$R(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & e^{2i\psi} \end{pmatrix}$$

giving the scattering matrix

$$R(-\gamma)S(\beta)R(\alpha) =$$

$$\frac{1}{2} \sqrt{\frac{4\pi}{5}} \begin{pmatrix} Y_2^0(\beta, \alpha) + 2\sqrt{5}Y_0^0(\beta, \alpha) & -\sqrt{\frac{3}{2}}Y_2^{-2}(\beta, \alpha) & -\sqrt{\frac{3}{2}}Y_2^2(\beta, \alpha) \\ -\sqrt{6}Y_2^0(\beta, \alpha)e^{2i\gamma} & 3_2Y_2^{-2}(\beta, \alpha)e^{2i\gamma} & 3_2Y_2^2(\beta, \alpha)e^{2i\gamma} \\ -\sqrt{6}Y_2^0(\beta, \alpha)e^{-2i\gamma} & 3_2Y_2^{-2}(\beta, \alpha)e^{-2i\gamma} & 3_2Y_2^2(\beta, \alpha)e^{-2i\gamma} \end{pmatrix}$$
Addition Theorem for Spin Harmonics

- Spin harmonics are related to rotation matrices as

\[
s Y^m_\ell (\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} D^\ell_{-m s} (\phi, \theta, 0)
\]

Note: for explicit evaluation sign convention differs from usual (e.g. Jackson) by \((-1)^m\)

- Multiplication of rotations

\[
\sum_m D^\ell_{m m''} (\alpha_2, \beta_2, \gamma_2) D^\ell_{m'' m} (\alpha_1, \beta_1, \gamma_1) = D^\ell_{m m'} (\alpha, \beta, \gamma)
\]

- Implies

\[
\sum_m s_1 Y^{m*}_\ell (\theta', \phi') s_2 Y^m_\ell (\theta, \phi) = (-1)^{s_1-s_2} \sqrt{\frac{2\ell + 1}{4\pi}} s_2 Y^{-s_1}_\ell (\beta, \alpha) e^{i s_2 \gamma}
\]
Sky Basis

- Scattering into the state (rest frame)

\[
C_{\text{in}}[T] = \dot{\tau} \int \frac{d\hat{n}'}{4\pi} R(-\gamma) S(\beta) R(\alpha) T(\hat{n}') ,
\]

\[
= \dot{\tau} \int \frac{d\hat{n}'}{4\pi} (\Theta', 0, 0) + \frac{1}{10} \dot{\tau} \int d\hat{n}' \sum_{m=-2}^{2} P^{(m)}(\hat{n}, \hat{n}') T(\hat{n}') .
\]

where the quadrupole coupling term is \( P^{(m)}(\hat{n}, \hat{n}') = \)

\[
\begin{pmatrix}
            Y_2^{m*}(\hat{n}') Y_2^m(\hat{n}) & -\sqrt{\frac{3}{2}} Y_2^{m*}(\hat{n}') Y_2^m(\hat{n}) & -\sqrt{\frac{3}{2}} Y_2^{m*}(\hat{n}') Y_2^m(\hat{n}) \\
-\sqrt{6} Y_2^{m*}(\hat{n}') Y_2^m(\hat{n}) & 3 Y_2^{m*}(\hat{n}') Y_2^m(\hat{n}) & 3 Y_2^{m*}(\hat{n}') Y_2^m(\hat{n}) \\
-\sqrt{6} Y_2^{m*}(\hat{n}') Y_2^m(\hat{n}) & 3 Y_2^{m*}(\hat{n}') Y_2^m(\hat{n}) & 3 Y_2^{m*}(\hat{n}') Y_2^m(\hat{n})
\end{pmatrix} ,
\]

expression uses angle addition relation above. We call this term \( C_Q \).
Scattering Matrix

- Full scattering matrix involves difference of scattering into and out of state

\[ C[T] = C_{\text{in}}[T] - C_{\text{out}}[T] \]

- In the electron rest frame

\[ C[T] = \dot{\tau} \int \frac{d\hat{n}'}{4\pi} (\Theta', 0, 0) - \dot{\tau}T + C_Q[T] \]

which describes isotropization in the rest frame. All moments have \( e^{-\tau} \) suppression except for isotropic temperature \( \Theta_0 \). Transformation into the background frame simply induces a dipole term

\[ C'[T] = \dot{\tau} \left( \hat{n} \cdot v_b + \int \frac{d\hat{n}'}{4\pi} \Theta', 0, 0 \right) - \dot{\tau}T + C_Q[T] \]
Source Terms

- Temperature source terms $S^{(m)}_l$ (rows $\pm|m|$; flat assumption)

\[
\begin{pmatrix}
\dot{\tau}\Theta^{(0)}_0 - \dot{H}^{(0)}_L & \dot{\tau}v_b^{(0)} + \dot{B}^{(0)} & \dot{\tau}P^{(0)} - \frac{2}{3}\ddot{H}^{(0)}_T \\
0 & \dot{\tau}v_b^{(\pm1)} + \dot{B}^{(\pm1)} & \dot{\tau}P^{(\pm1)} - \frac{\sqrt{3}}{3}\ddot{H}^{(\pm1)}_T \\
0 & 0 & \dot{\tau}P^{(\pm2)} - \ddot{H}^{(\pm2)}_T
\end{pmatrix}
\]

where

\[P^{(m)} = \frac{1}{10}(\Theta^{(m)}_2 - \sqrt{6}E^{(m)}_2)\]

- Polarization source term

\[\mathcal{E}^{(m)}_\ell = -\dot{\tau}\sqrt{6}P^{(m)}\delta_{\ell,2}\]
\[\mathcal{B}^{(m)}_\ell = 0\]
Astro 282
Secondaries
Reionization

- Ionization depth during reionization

\[
\tau(z) = \int d\eta n_e \sigma_T a = \int d \ln a \frac{n_e \sigma_T}{H(a)} \propto (\Omega_b h^2)(\Omega_m h^2)^{-1/2}(1 + z)^{3/2}
\]

\[
= \left( \frac{\Omega_b h^2}{0.02} \right) \left( \frac{\Omega_m h^2}{0.15} \right)^{-1/2} \left( \frac{1 + z}{61} \right)^{3/2}
\]

- Quasars say \( z_{ri} \geq 7 \) so \( \tau > 0.04 \)

- During reionization, cosmic quadrupole of \( \sim 30 \mu K \) from the Sachs-Wolfe effect scatters into \( E \)-polarization

- Few percent optical depth leads to fraction of a \( \mu K \) signal

- Peaks at horizon scale at recombination: quadrupole source \( j_2(k D_*) \) maximal at \( k D_* \approx k \eta \approx 2 \)
Breaking degeneracies

• First objects, breaking degeneracy of initial amplitude vs optical depth in the peak heights

\[ C_\ell \propto e^{-2\tau} \]

only below horizon scale at reionization

• Breaks degeneracies in angular diameter distance by removing an ambiguity for ISW-dark energy measure, helps in \( \Omega_{DE} - w_{DE} \) plane
Gravitational Wave

- Gravitational waves produce a quadrupolar distortion in the temperature of the CMB like effect on a ring of test particles.
- Like ISW effect, source is a metric perturbation with time dependent amplitude.
- After recombination, is a source of observable temperature anisotropy – but is therefore confined to low order multipoles.
- Generated during inflation by quantum fluctuations.
Gravitational Wave Polarization

- In the tight coupling regime, quadrupole anisotropy suppressed by scattering

\[
\pi \gamma \approx \frac{\dot{h}}{\dot{\tau}}
\]

- Since gravitational waves oscillate and decay at horizon crossing, the polarization peaks at the horizon scale at recombination not the damping scale

- More distinct signature in the $B$-mode polarization since symmetry of plane wave is broken by the transverse nature of gravity wave polarization
Secondary Anisotropy

- CMB photons traverse the large-scale structure of the universe from $z = 1000$ to the present.

- With the nearly scale-invariant adiabatic fluctuations observed in the CMB, structures form from the bottom up, i.e. small scales first, a.k.a. hierarchical structure formation.

- First objects reionize the universe between $z \sim 7 - 30$

- Main sources of secondary anisotropy

  - Gravitational: Integrated Sachs-Wolfe effect (gravitational redshift) and gravitational lensing

  - Scattering: peak suppression, large-angle polarization, Doppler effect(s), inverse Compton scattering
Transfer Function

- **Transfer function** transfers the initial Newtonian curvature to its value today (linear response theory)

\[ T(k) = \frac{\Phi(k, a = 1)}{\Phi(k, a_{\text{init}})} \times \frac{\Phi(k_{\text{norm}}, a_{\text{init}})}{\Phi(k_{\text{norm}}, a = 1)} \]

- Conservation of Bardeen curvature: Newtonian curvature is a constant when stress perturbations are negligible: above the horizon during radiation and dark energy domination, on all scales during matter domination

- When stress fluctuations dominate, perturbations are stabilized by the Jeans mechanism

- **Hybrid Poisson equation**: Newtonian curvature, comoving density perturbation \( \Delta \equiv (\delta \rho / \rho)_{\text{com}} \) implies \( \Phi \) decays

\[ (k^2 - 3K)\Phi = 4\pi G \rho \Delta \sim \eta^{-2} \Delta \]
Transfer Function

- Matter-radiation example: Jeans scale is horizon scale and $\Delta$ freezes into its value at horizon crossing $\Delta_H \approx \Phi_{\text{init}}$

- Freezing of $\Delta$ stops at $\eta_{eq}$

$$
\Phi \sim (k\eta_{eq})^{-2} \Delta_H \sim (k\eta_{eq})^{-2}\Phi_{\text{init}}
$$

- Conventionally $k_{\text{norm}}$ is chosen as a scale between the horizon at matter radiation equality and dark energy domination.

- Small correction since growth with a smooth radiation component is logarithmic not frozen

- Run CMBfast to get transfer function or use fits
Transfer Function

- Transfer function has a $k^{-2}$ fall-off beyond $k_{eq} \sim \eta_{eq}^{-1}$

- Additional baryon wiggles are due to acoustic oscillations at recombination – an interesting means of measuring distances
Growth Function

- Same physics applies to the dark energy dominated universe
- Under the dark energy sound horizon, dark energy density frozen. Potential decays at the same rate for all scales

\[ g(a) = \frac{\Phi(k_{\text{norm}}, a)}{\Phi(k_{\text{norm}}, a_{\text{init}})} \]

- Pressure growth suppression: \( \delta \equiv \frac{\delta \rho_m}{\rho_m} \propto a g \)

\[ \frac{d^2 g}{d \ln a^2} + \left[ \frac{5}{2} - \frac{3}{2} w(z) \Omega_{DE}(z) \right] \frac{dg}{d \ln a} + \frac{3}{2} [1 - w(z)] \Omega_{DE}(z) g = 0, \]

where \( w \equiv \frac{p_{DE}}{\rho_{DE}} \) and \( \Omega_{DE} \equiv \frac{\rho_{DE}}{(\rho_m + \rho_{DE})} \) with initial conditions \( g = 1, \frac{dg}{d \ln a} = 0 \)

- As \( \Omega_{DE} \to 0 \) \( g = \text{const.} \) is a solution. The other solution is the decaying mode, eliminated by initial conditions
ISW effect

- Potential decay leads to gravitational redshifts through the integrated Sachs-Wolfe effect

- Intrinsically a large effect since $2\Delta \Phi = 6\Psi_{\text{init}}/3$

- But net redshift is integral along along line of sight

$$\frac{\Theta_\ell(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} [2\dot{\Phi}(k, \eta)] j_\ell(k(\eta_0 - \eta))$$

$$= 2\Phi(k, \eta_{MD}) \int_0^{\eta_0} d\eta e^{-\tau} \dot{g}(D) j_\ell(kD)$$

- On small scales where $k \gg \dot{g}/g$, can pull source out of the integral

$$\int_0^{\eta_0} d\eta \dot{g}(D) j_\ell(kD) \approx \dot{g}(D = \ell/k) \frac{1}{k} \sqrt{\frac{\pi}{2\ell}}$$

evaluated at peak, where we have used $\int dx j_\ell(x) = \sqrt{\pi/2\ell}$
ISW effect

- Power spectrum

\[
C_\ell = \frac{2}{\pi} \int \frac{dk}{k} \frac{k^3 \langle \Theta^{*}_\ell(k, \eta_0) \Theta_\ell(k, \eta_0) \rangle}{(2\ell + 1)^2} = \frac{2\pi^2}{l^3} \int d\eta D \dot{g}^2(\eta) \Delta^2_\Phi(\ell/D, \eta_{MD})
\]

- Or \(l^2C_\ell/2\pi \propto 1/\ell\) for scale invariant potential. This is the Limber equation in spherical coordinates. Projection of 3D power retains only the transverse piece. For a general dark energy model, add in the scale dependence of growth rate on large scales.

- Cancellation of redshifts and blueshifts as the photon traverses many crests and troughs of a small scale fluctuation during decay. Enhancement of the \(\ell < 10\) multipoles. Difficult to extract from cosmic variance and galaxy. Current ideas: cross correlation with other tracers of structure
Gravitational Lensing

- Lensing is a surface brightness conserving remapping of source to image planes by the gradient of the projected potential

\[
\phi(\hat{n}) = 2 \int_{\eta_*}^{\eta_0} d\eta \frac{(D_\ast - D)}{D D_\ast} \Phi(D\hat{n}, \eta) .
\]

such that the fields are remapped as

\[
x(\hat{n}) \rightarrow x(\hat{n} + \nabla \phi) ,
\]

where \( x \in \{\Theta, Q, U\} \) temperature and polarization.

- Taylor expansion leads to product of fields and Fourier mode-coupling
Flat-sky Treatment

- Taylor expand

\[ \Theta(\hat{n}) = \tilde{\Theta}(\hat{n} + \nabla \phi) \]

\[ = \tilde{\Theta}(\hat{n}) + \nabla_i \phi(\hat{n}) \nabla^i \tilde{\Theta}(\hat{n}) + \frac{1}{2} \nabla_i \phi(\hat{n}) \nabla_j \phi(\hat{n}) \nabla^i \nabla^j \tilde{\Theta}(\hat{n}) + \ldots \]

- Fourier decomposition

\[ \phi(\hat{n}) = \int \frac{d^2 l}{(2\pi)^2} \phi(l) e^{il \cdot \hat{n}} \]

\[ \tilde{\Theta}(\hat{n}) = \int \frac{d^2 l}{(2\pi)^2} \tilde{\Theta}(l) e^{il \cdot \hat{n}} \]
Flat-sky Treatment

- Mode coupling of harmonics

\[ \Theta(l) = \int d\hat{n} \Theta(\hat{n}) e^{-il \cdot \hat{n}} \]

\[ = \tilde{\Theta}(l) - \int \frac{d^2l_1}{(2\pi)^2} \tilde{\Theta}(l_1) L(l, l_1), \]

where

\[ L(l, l_1) = \phi(l - l_1) (l - l_1) \cdot l_1 \]

\[ + \frac{1}{2} \int \frac{d^2l_2}{(2\pi)^2} \phi(l_2) \phi^*(l_2 + l_1 - l) (l_2 \cdot l_1)(l_2 + l_1 - l) \cdot l_1. \]

- Represents a coupling of harmonics separated by \( L \approx 60 \) peak of deflection power
Power Spectrum

- Power spectra

\[
\langle \Theta^*(l) \Theta(l') \rangle = (2\pi)^2 \delta(l - l') \, C_{l}^{\Theta \Theta},
\]
\[
\langle \phi^*(l) \phi(l') \rangle = (2\pi)^2 \delta(l - l') \, C_{l}^{\phi \phi},
\]

becomes

\[
C_{l}^{\Theta \Theta} = (1 - l^2 R) \, \tilde{C}_{l}^{\Theta \Theta} + \int \frac{d^2 l_1}{(2\pi)^2} \tilde{C}_{|l-l_1|}^{\Theta \Theta} |C_{l_1}^{\phi \phi}|^2 [(l - l_1) \cdot l_1]^2,
\]

where

\[
R = \frac{1}{4\pi} \int \frac{dl}{l} \, l^4 C_{l}^{\phi \phi}.
\]
Smoothing Power Spectrum

• If $\tilde{C}_l^{\Theta \Theta}$ slowly varying then two term cancel

$$\tilde{C}_l^{\Theta \Theta} \int \frac{d^2 l_1}{(2\pi)^2} C_l^{\phi \phi} (l \cdot l_1)^2 \approx (l^2 R \tilde{C}_l^{\Theta \Theta}).$$

• So lensing acts to smooth features in the power spectrum. Smoothing kernel is $L \sim 60$ the peak of deflection power spectrum

• Because acoustic feature appear on a scale $l_A \sim 300$, smoothing is a subtle effect in the power spectrum.

• Lensing generates power below the damping scale which directly reflect power in deflections on the same scale
Polarization Lensing

- Polarization field harmonics lensed similarly

\[
[Q \pm iU](\hat{n}) = -\int \frac{d^2 l}{(2\pi)^2} [E \pm iB](l)e^{\pm 2i\phi_1}e^{l.n}
\]

so that

\[
[Q \pm iU](\hat{n}) = [\tilde{Q} \pm i\tilde{U}](\hat{n} + \nabla \phi)
\]

\[
\approx [\tilde{Q} \pm i\tilde{U}](\hat{n}) + \nabla_i \phi(\hat{n}) \nabla^i [\tilde{Q} \pm i\tilde{U}](\hat{n})
\]

\[
+ \frac{1}{2} \nabla_i \phi(\hat{n}) \nabla_j \phi(\hat{n}) \nabla^i \nabla^j [\tilde{Q} \pm i\tilde{U}](\hat{n})
\]
Polarization Power Spectra

- Carrying through the algebra

\[
C_l^{EE} = (1 - l^2 R) \tilde{C}_l^{EE} + \frac{1}{2} \int \frac{d^2 l_1}{(2\pi)^2} [(l - l_1) \cdot l_1]^2 C_{|l-l_1|}^{\phi \phi} \times [(\tilde{C}_l^{EE} + \tilde{C}_l^{BB}) + \cos(4\varphi_l)(\tilde{C}_l^{EE} - \tilde{C}_l^{BB})],
\]

\[
C_l^{BB} = (1 - l^2 R) \tilde{C}_l^{BB} + \frac{1}{2} \int \frac{d^2 l_1}{(2\pi)^2} [(l - l_1) \cdot l_1]^2 C_{|l-l_1|}^{\phi \phi} \times [(\tilde{C}_l^{EE} + \tilde{C}_l^{BB}) - \cos(4\varphi_l)(\tilde{C}_l^{EE} - \tilde{C}_l^{BB})],
\]

\[
C_l^{\Theta E} = (1 - l^2 R) \tilde{C}_l^{\Theta E} + \int \frac{d^2 l_1}{(2\pi)^2} [(l - l_1) \cdot l_1]^2 C_{|l-l_1|}^{\phi \phi} \times \tilde{C}_l^{\Theta E} \cos(2\varphi_l),
\]

- Lensing generates \(B\)-modes out of the acoustic polarization \(E\)-modes contaminates gravitational wave signature if \(E_i < 10^{16}\text{GeV}\).
Reconstruction from the CMB

- Correlation between Fourier moments reflect lensing potential

$$\langle x(l)x'(l') \rangle_{\text{CMB}} = f_\alpha(l,l') \phi(l + l') ,$$

where \( x \in \text{temperature, polarization fields} \) and \( f_\alpha \) is a fixed weight that reflects geometry

- Each pair forms a noisy estimate of the potential or projected mass - just like a pair of galaxy shears

- Minimum variance weight all pairs to form an estimator of the lensing mass
Scattering Secondaries

- Optical depth during reionization
  \[ \tau \approx 0.066 \left( \frac{\Omega_b h^2}{0.02} \right) \left( \frac{\Omega_m h^2}{0.15} \right)^{-1/2} \left( \frac{1 + z}{10} \right)^{3/2} \]

- Anisotropy suppressed as \( e^{-\tau} \). Integral solution
  \[ \frac{\Theta_\ell(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} S_0^{(0)} j_\ell(k(\eta_0 - \eta)) + \ldots \]

- Isotropic (large scale) fluctuations not suppressed since suppression represents isotropization by scattering

- Quadrupole from the Sachs-Wolfe effect scatters into a large angle polarization bump
Doppler Effects

- Velocity fields of $10^{-3}$ and optical depths of $10^{-2}$ would imply large Doppler effect due to reionization.

- Limber approximation says only fluctuations transverse to line of sight survive.

- In linear theory, transverse fluctuations have no line of sight velocity and so Doppler effect is highly suppressed.

- Beyond linear theory: modulate the optical depth in the transverse direction using density fluctuations or ionization fraction fluctuations. Generate a modulated Doppler effect.

- Linear fluctuations: Vishniac effect; Clusters: kinetic SZ effect; ionization patches: inhomogeneous reionization effect.
Thermal SZ Effect

- Thermal velocities also lead to Doppler effect but first order contribution cancels because of random directions.

- Residual effect is of order $v^2\tau \approx T_e/m_e \tau$ and can reach a sizeable level for clusters with $T_e \approx 10$ keV.

- Raleigh-Jeans decrement and Wien enhancement described by second order collision term in Boltzmann equation: Kompaneets equation.

- Clusters are rare objects so contribution to power spectrum suppressed, but may have been detected by CBI/BIMA: extremely sensitive to power spectrum normalization $\sigma_8$.

- White noise on large-scales ($l < 2000$), turnover as cluster profile is resolved.