Supplement
Inflationary Perturbations

Field perturbations

- Let's define the perturbed scalar field as $\phi = \phi_0 + \phi_1$ where ϕ_0 is the unperturbed field (i.e. $\phi_1 = \delta\phi$)
- Field fluctuations obey a damped harmonic oscillator equation (with dots referring to conformal time derivatives)

$$\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + k^2\phi_1 \approx 0$$

- We want a simple harmonic oscillator that we can then quantize so define $u \equiv a\phi_1$

$$\dot{u} = \dot{a}\phi_1 + a\dot{\phi}_1$$

$$\ddot{u} = \ddot{a}\phi_1 + 2\dot{a}\dot{\phi}_1 + a\ddot{\phi}_1$$

$$\ddot{u} + [k^2 - \frac{\ddot{a}}{a}]u = 0$$

- Note Friedmann equations say if $p = -\rho$, $\ddot{a}/a = 2(\dot{a}/a)^2$

Harmonic Oscillator

- Now let's look at the oscillator equation

$$\ddot{u} + [k^2 - 2\left(\frac{\dot{a}}{a}\right)^2]u = 0$$

or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\rm end}$$

$$\tilde{\eta} = \int_{a_{\rm end}}^{a} \frac{da}{Ha^2} \approx -\frac{1}{aH}$$

- So we can rewrite this as

$$\ddot{u} + [k^2 - \frac{2}{\tilde{\eta}^2}]u = 0$$

Quantum Fluctuations

- Simple harmonic oscillator $\ll$ Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k, \tilde{\eta})\hat{a} + u^*(k, \tilde{\eta})\hat{a}^\dagger$$

where $u(k, \tilde{\eta})$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \qquad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\tilde{\eta}] = i$

$$u(k, \eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

Quantum Fluctuations

- Zero point fluctuations of ground state

$$\begin{aligned}
\langle u^2 \rangle &= \langle 0 | u^\dagger u | 0 \rangle \\
&= \langle 0 | (u^* \hat{a}^\dagger + u \hat{a})(u \hat{a} + u^* \hat{a}^\dagger) | 0 \rangle \\
&= \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle | u(k, \tilde{\eta}) |^2 \\
&= \langle 0 | [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} | 0 \rangle | u(k, \tilde{\eta}) |^2 \\
&= | u(k, \tilde{\eta}) |^2 = \frac{1}{2k}
\end{aligned}$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in.

Slow Roll Limit

- Classical equation of motion then has the exact solution

$$u = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\tilde{\eta}} \right) e^{-ik\tilde{\eta}}$$

- For $|k\tilde{\eta}| \ll 1$ (late times, $\gg$ Hubble length) fluctuation freezes in

$$\lim_{|k\tilde{\eta}| \to 0} u = -\frac{1}{\sqrt{2k}} \frac{i}{k\tilde{\eta}} \approx \frac{iHa}{\sqrt{2k^3}}$$

$$\phi_1 = \frac{iH}{\sqrt{2k^3}}$$

- Power spectrum of field fluctuations

$$\Delta^2_{\phi_1} = \frac{k^3 |\phi_1|^2}{2\pi^2} = \frac{H^2}{(2\pi)^2}$$