Astro 321

Set 3: Inhomogeneous Fields

Wayne Hu

Inhomogeneous Universe

- Inhomogeneities in the matter-energy distribution of the universe grow via gravitational instability
- In expanding universe, growth rate is power law not exponential
- Must be a source in the early universe to explain structure in the universe
- Follow general principles of the FRW/Thermal History discussion but drop homogeneity and isotropy

  Matter moves in the perturbed geometry (automatically conserving stress-energy)

    Closure requires more than just the relation between average pressure and energy density w

  Matter curves the geometry - cosmological Poisson equation generates gravitational potential from density perturbations

Inhomogeneous Fields

- Like homogeneous cosmology, a full description of the matter fields is given through their phase space distribution function

$$f(\mathbf{x}, \mathbf{q}, t)$$

where the momentum dependence $\mathbf{q}$ describes the bulk motion of the particles

- Energy density and pressure are functions of position

$$\rho(\mathbf{x}, t) = g \int \frac{d^3q}{(2\pi)^3} f(\mathbf{x}, \mathbf{q}, t) E$$
$$p(\mathbf{x}, t) = g \int \frac{d^3q}{(2\pi)^3} f(\mathbf{x}, \mathbf{q}, t) \frac{|\mathbf{q}|^2}{3E}$$

and can be considered as low order moments of the distribution function

Inhomogeneous Boltzmann Equation

- Evolution of density inhomogeneities is governed by the Boltzmann equation. Switch over to comoving representation: η, comoving $\mathbf{x}$, retain physical momentum $\mathbf{q}$
- For non-interacting species, Liouville equation

$$\dot{f} + \dot{\mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{q}} + \dot{\mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0$$

- Momentum $\mathbf{q} = q\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a directional unit vector and in a flat universe $\dot{\mathbf{q}} = \dot{q}\hat{\mathbf{n}}$
- Particle velocity $\dot{\mathbf{x}} = \mathbf{q}/E$

$$\dot{f} + \dot{q}\frac{\partial f}{\partial q} + \frac{\mathbf{q}}{E} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0$$

Gravitational Change of Momentum

- Momentum term: carries two contributions
- Consider the perturbed FRW line element to take the form

$$d\tau^2 = a^2[(1+2\Psi)d\eta^2 - (1+2\Phi)(dD^2 + D_A^2 d\Omega)]$$

where $|\Phi| \ll 1$ and $\Psi \ll 1$

- Just as the background scale factor changes the de Broglie wavelength of particles, a perturbation to the scale factor (or spatial curvature)

$$a(\mathbf{x}) = a(1+\Phi)$$

Gravitational Change of Momentum

- So Φ gives a time dependence to the momentum through

$$\dot{a}(\mathbf{x}) = \dot{a}(1+\Phi) + a\dot{\Phi}$$

$$\frac{\dot{a}(\mathbf{x})}{a(\mathbf{x})} \approx \frac{\dot{a}}{a} + \frac{\dot{\Phi}}{1+\Phi} \approx \frac{\dot{a}}{a} + \dot{\Phi}$$

- Contribution from the spatial metric (independent of direction)

$$\dot{q} = -\left(\frac{\dot{a}}{a} + \dot{\Phi}\right) q$$

- Second term comes from Ψ which plays the role of the gravitational potential
- Non-relativistic: gravitational force changes momentum

$$\dot{\mathbf{q}} = \mathbf{F} = -m\nabla\Psi \quad \rightarrow \quad \dot{q} = \hat{\mathbf{n}} \cdot \dot{\mathbf{q}} = -m(\hat{\mathbf{n}} \cdot \nabla\Psi)$$

Gravitational Change of Momentum

- Ultra-Relativistic: time dilation implies shift of frequency or gravitational redshift and hence momentum

$$\frac{\Delta q}{q} = -\Delta\Psi$$

Rate of change from moving through a Ψ gradient is

$$\frac{\dot{q}}{q} = -\dot{\mathbf{x}} \cdot \nabla\Psi = -\hat{\mathbf{n}} \cdot \nabla\Psi$$

- In both relativistic and non-relativistic cases

$$\dot{q} = -E(\hat{\mathbf{n}} \cdot \nabla\Psi)$$

- Combining the two momentum terms

$$\dot{q} = -\left(\frac{\dot{a}}{a} + \dot{\Phi}\right) q - (\hat{\mathbf{n}} \cdot \nabla\Psi) E$$

Energy or Continuity Equation

- Integrate Boltzmann equation over

$$g \int \frac{d^3 q}{(2\pi)^3} E \left(\dot{f} + \dot{q} \frac{\partial f}{\partial q} + \frac{\mathbf{q}}{E} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0 \right)$$

- Time term

$$g \int \frac{d^3 q}{(2\pi)^3} E \dot{f} = \dot{\rho}$$

- Momentum terms

$$g \int \frac{d^3 q}{(2\pi)^3} \dot{q} E \frac{\partial f}{\partial q} = g \int \frac{d^3 q}{(2\pi)^3} \left[- \left(\frac{\dot{a}}{a} + \dot{\Phi} \right) q - (\hat{\mathbf{n}} \cdot \nabla \Psi) E \right] E \frac{\partial f}{\partial q}$$

  second term vanishes by symmetry integrating over momenta direction

Energy or Continuity Equation

- First term is identical to background derivation

$$g \int \frac{d^3q}{(2\pi)^3} \dot{q} E \frac{\partial f}{\partial q} = -\left(\frac{\dot{a}}{a} + \dot{\Phi}\right) g \int \frac{d^3q}{(2\pi)^3} q E \frac{\partial f}{\partial q} = 3\left[\frac{\dot{a}}{a} + \dot{\Phi}\right](\rho + p)$$

- Position term: define average momentum as momentum density

$$\nabla \cdot g \int \frac{d^3q}{(2\pi)^3} \mathbf{q} f \equiv \nabla \cdot (\rho + p)\mathbf{v}$$

- Linearized energy/continuity equation

$$\dot{\rho} = -3\left[\frac{\dot{a}}{a} + \dot{\Phi}\right](\rho + p) - \nabla \cdot (\rho + p)\mathbf{v}$$

- Local energy density changes due to: global expansion, local change in expansion, flows of particles into/out of volume

Momentum or Navier-Stokes Equation

- Integrate Boltzmann equation over

$$g \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} \left(\dot{f} + \dot{q} \frac{\partial f}{\partial q} + \frac{\mathbf{q}}{E} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0 \right)$$

- Time term

$$\frac{\partial}{\partial \eta} \to \frac{\partial}{\partial \eta} [(\rho + p)\mathbf{v}]$$

- Momentum term: de Broglie redshift

$$-[\frac{\dot{a}}{a} + \dot{\Phi}] g \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} \frac{\partial f}{\partial q} = 4[\frac{\dot{a}}{a} + \dot{\Phi}] g \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} f$$

$$= 4 \left[\frac{\dot{a}}{a} + \dot{\Phi} \right] (\rho + p)\mathbf{v} \approx 4 \frac{\dot{a}}{a} (\rho + p)\mathbf{v}$$

Momentum Equation

- Momentum term: gravitational potential jth component

$$-\partial_i \Psi \cdot g \int \frac{d^3 q}{(2\pi)^3} q E n_j n^i \frac{\partial f}{\partial q} \approx \partial_j \Psi (\rho + p)$$

where angle averaged $\langle n^i n_j \rangle = \frac{1}{3}\delta^i{}_j$ and used relation from homogeneous energy equation

- Spatial term: recall stress tensor divided into isotropic and anisotropic pieces

$$g \int \frac{d^3 q}{(2\pi)^3} \frac{q^i q_j}{E} f \equiv p \delta^i{}_j + \pi^i{}_j$$

- Combined momentum terms

$$\frac{\partial}{\partial \eta}[(\rho + p) v^i] = -4 \frac{\dot{a}}{a} (\rho + p) v^i - \partial^i p - \partial^j \pi^i{}_j - (\rho + p) \partial^i \Psi$$

Boltzmann Hierarchy

- Momentum equation is Navier-Stokes equation. Unless stress tensor is specified, equation is not closed
- In general, the time derivative of a low order moment of the Boltzmann equation is given by the spatial gradient of higher order moments (here anisotropic stress)
- Microphysics closes the Boltzmann equation. Energy and momentum equations simply reflect conservation of the stress energy tensor and is valid for *any* component of matter – even things like cosmological defects.
- For non-relativistic particles, the velocity dispersion tensor characterizes the pressure and anisotropic stress
- Thus closure relation for pressure and anisotropic stress need not be linear in the momentum or density variables

Linear Perturbation Theory

- Note that we have not yet assumed that the fluctuations in the density, pressure or momentum are small compared to spatial average
- We have assumed that metric fluctuations are small to simplify some equations
- Same equations as would be derived from stress energy conservation $\nabla_\mu T^{\mu\nu} = 0$
- Equations are valid for components with no background term, e.g. cosmological defects, other trace components
- Nonetheless though valid, these equations cannot in general be solved directly due to unknown and not necessarily linear closure relation

Linear Perturbation Theory

- For most components it is useful to subtract off the background and consider the dimensionless (fractional) density fluctuation

$$\delta(\mathbf{x}, t) = \frac{\rho(\mathbf{x}, t) - \langle \rho(\mathbf{x}, t) \rangle}{\langle \rho(\mathbf{x}, t) \rangle}$$

  where $\delta \ll 1$ on large scales

- Likewise assume that relative pressure fluctuation, bulk velocity and anisotropic stress are all of the same order (or smaller)
- Thus we can think of the Boltzmann equation and its moment expansion as a linear equation relating the fluctuation variables
- For a linear equation, eigenmodes of the spatial fluctuations evolve independently
- Partial differential equation in space becomes a set of ordinary differential equations in harmonic modes

Harmonic Decomposition

- In general, the eigenvectors of the Laplace operator form a complete basis to expand a general function of $\mathbf{x}$
- In a spatially flat cosmology these are simply plane waves

$$\nabla^2 Q = -k^2 Q \quad \rightarrow \quad Q = e^{i\mathbf{k}\cdot\mathbf{x}}$$

- In a curved geometry, these functions are somewhat more complicated but on scales smaller than the curvature scale behave like plane waves
- Empirically we know that the curvature scale if non-zero is at least several times the Hubble scale
- Fourier decomposition suffices for observable modes in the universe until near the Hubble scale
- When done properly, corrections take the form, e.g. $k^2 \rightarrow k^2 - 3K$ where $K = H_0^2(\Omega - 1)$ is the curvature

Fourier Conventions

- Often required to relate harmonics in a finite (e.g. survey) volume to infinite volume
- Periodicity: assume a 1D field $F(x)$ periodic in finite volume of length L

$$
\begin{aligned}
F(x+L) &= F(x) \\
&= \sum_n F(k_n) e^{-ik_n x - ik_n L} \\
&= \sum_n F(k_n) e^{-ik_n x} = F(x) \quad \text{if } k_n = \frac{2\pi}{L} n
\end{aligned}
$$

Fourier Conventions

- Reality:

$$F^*(x) = \sum_n F^*(k_n) e^{ik_n x} = F(x) = \sum_n F(k_n) e^{-ik_n x}$$

$$F^*(k_n) = F(-k_n)$$

Fourier Conventions

- Band limited: function has no high frequency structure, e.g. because smoothed

$$k_n < k_{\rm max} \equiv \frac{2\pi}{L}\frac{N}{2}$$

$$F(x) = \sum_{n=-N/2}^{N/2} F(k_n) e^{-ik_n x}$$

- Sampling theorem: sampling at a rate $\Delta = L/N$ is sufficient to reconstruct field exactly. Inverse relation

$$F(k_n) = \frac{1}{N}\sum_{m=0}^{N-1} F(x_m) e^{ik_n x_m}\,, \qquad x_m = m\Delta$$

Fourier Conventions

- δ (Kronecker) function

$$F(k_n) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n'=-N/2}^{N/2} F(k_{n'}) e^{-i(k_{n'}-k_n)x_m}$$

if $n' = n$ then

$$\frac{1}{N} \sum_{m=0}^{N-1} e^{-i(k_{n'}-k_n)x_m} = \frac{1}{N} \sum_{m=0}^{N-1} 1 = 1$$

if $n' \neq n$ then

$$e^{-i(k_{n'}-k_n)x_m} = \cos(k_n - k_{n'})x_m + i \sin(k_n - k_{n'})x_m$$

Fourier Conventions

$$\sum_{m=0}^{N-1} \cos[(k_n - k_{n'})\frac{2\pi m}{N}] = \frac{\sin[(N-\frac{1}{2})(n-n')\frac{2\pi}{N}]}{2\sin[(n-n')\frac{\pi}{N}]} + \frac{1}{2}$$

$$(N(n-n')2\pi/N = 2\pi(n-n') \text{ where } n-n' \text{ integer})$$

$$= -\frac{\sin[(n-n')\frac{\pi}{N}]}{2\sin[(n-n')\frac{\pi}{N}]} + \frac{1}{2} = 0$$

$$\sum_{m=0}^{N-1} \sin[(k_n - k_{n'})\frac{2\pi m}{N}] = \frac{\sin[(n-n')\pi]\ \sin[\frac{N-1}{N}(n-n')\pi]}{\sin[(n-n')\frac{\pi}{N}]} = 0$$

$$\rightarrow \frac{1}{N}\sum_{m=0}^{N-1} e^{-i(k_{n'}-k_n)x_m} = \delta_{nn'}$$

Fourier Conventions

- Two point correlation

$$\langle F(x)F(x')\rangle = \sum_{nn'} \langle F^*(k_n)F(k_{n'})\rangle e^{ik_n x - ik_{n'} x'}$$

- Translational invariance

$$\langle F(x+d)F(x'+d)\rangle = \langle F(x)F(x')\rangle$$

$$\sum_{nn'} \langle F^*(k_n)F(k_{n'})\rangle e^{ik_n x - ik_{n'} x'} e^{i(k_n - k_{n'})d} = \sum_{nn'} \langle F^*(k_n)F(k_{n'})\rangle e^{ik_n x - ik_{n'} x'}$$

$$\langle F^*(k_n)F(k_{n'})\rangle = \delta_{nn'} P_F(k_n)$$

two point statistical properties are given by the power spectrum P_F and correlation function depends only on separation

$$\langle F(x)F(x')\rangle = \xi(x - x')$$

Fourier Conventions

- Continuous conventions: let $L \to \infty$, density of k_n states gets high

$$\sum_n \to \int dn$$

- Forward and inverse transform

$$F(x) = \sum_{n=-N/2}^{N/2} F(k_n) e^{-ik_n x} = \int_{-N/2}^{N/2} dn F(k_n) e^{-ik_n x},$$

$$= L \int_{-k_{\max}}^{k_{\max}} \frac{dk}{2\pi} F(k) e^{-ikx} \qquad (dk_n = dn \frac{2\pi}{L})$$

$$F(k) = \frac{1}{N} \sum_{m=0}^{N-1} F(x_m) e^{ikx_m} = \frac{1}{L} \int dx F(x) e^{ikx} \qquad dx_m = \frac{L}{N} dm$$

Fourier Conventions

- The (Dirac) δ function

$$\delta_{nn'} = \frac{1}{N}\sum_{m=0}^{N-1} e^{-i(k_{n'}-k_n)x_m} = \frac{1}{L}\int dx e^{-i(k_{n'}-k_n)x_m}$$

$$F(k_n) = \sum_{n'} F(k_{n'})\delta_{nn'} = \int dn' F(k_{n'})\delta_{nn'} = \frac{L}{2\pi}\int dk'_{n'} F(k_{n'})\delta_{nn'}$$

- Define the δ function as

$$\int dk' F(k')\delta(k-k') = F(k)$$

$$\text{then} \quad \delta(k-k') = \frac{L}{2\pi}\delta_{nn'} = \frac{1}{2\pi}\int dx e^{i(k-k')x}$$

$$\langle F^*(k)F(k')\rangle = \frac{2\pi}{L}\delta(k-k')P_F(k)$$

Fourier Conventions

- 3D Fields

$$F(\mathbf{x}) = V \int \frac{d^3k}{(2\pi)^3} F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$F(\mathbf{k}) = \frac{1}{V} \int d^3x F(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$(2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') = \int d^3x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}$$

$$\langle F^*(\mathbf{k}) F(\mathbf{k}')\rangle = \frac{(2\pi)^3}{V} \delta(\mathbf{k} - \mathbf{k}') P_F(\mathbf{k})$$

- Statistical isotropy: $P_F(\mathbf{k}) = P_F(k)$

Fourier Conventions

- Suppress volume terms by making Fourier representation dimensionful $\tilde{F}(\mathbf{k}) \equiv V F(\mathbf{k})$, $\tilde{P}_F = V P_F$

$$
\begin{aligned}
F(\mathbf{x}) &= \int \frac{d^3k}{(2\pi)^3} \tilde{F}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\
\tilde{F}(\mathbf{k}) &= \int d^3x F(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} \\
\langle \tilde{F}^*(\mathbf{k}) \tilde{F}(\mathbf{k}') \rangle &= (2\pi)^3 V \delta(\mathbf{k}-\mathbf{k}') P_F(\mathbf{k}) \\
&= (2\pi)^3 \delta(\mathbf{k}-\mathbf{k}') \tilde{P}_F(\mathbf{k})
\end{aligned}
$$

- Hereafter, suppress $\sim$, power spectra have dimensions of volume
- So: what does it mean to have a large fluctuation in power?

Fourier Conventions

- Variance

$$\sigma_F^2 \equiv \langle F(\mathbf{x})F(\mathbf{x})\rangle = \int \frac{d^3k}{(2\pi)^3} P_F(k)$$
$$= \int \frac{k^2 dk}{2\pi^2} \int \frac{d\Omega}{4\pi} P_F(k)$$
$$= \int d\ln k \frac{k^3}{2\pi^2} P_F(k)$$

- Define power per logarithmic interval

$$\Delta_F^2(k) \equiv \frac{k^3 P_F(k)}{2\pi^2}$$

- This quantity is dimensionless in all representations. Serves as a definition of the linear regime k's where $\Delta_\delta^2 \ll 1$

Linearity

- Fields related by a linear equation obey equation independent equations

$$F(\mathbf{x}) = AG(\mathbf{x}) + B \quad \to \quad F(\mathbf{k}) = AG(\mathbf{k}) \qquad (k > 0)$$

includes linear differential equation

$$F(\mathbf{x}) = A\nabla G(\mathbf{k}) + B$$

$$F(\mathbf{k}) = A \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} \nabla \int \frac{d^3k'}{(2\pi)^3} e^{-i\mathbf{k}'\cdot\mathbf{x}} G(\mathbf{k}')$$

$$= A \int \frac{d^3k'}{(2\pi)^3} \int d^3x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} (-i\mathbf{k}') G(\mathbf{k}') = A(-i\mathbf{k}) G(\mathbf{k})$$

converts differential equations to algebraic relations

Convolution

- Convolution in real space often occurs – smoothing of field by finite resolution and normalization $\int d^3x W(\mathbf{x}) = 1$

$$F_W(\mathbf{x}) = \int d^3y W(\mathbf{x}-\mathbf{y}) F(\mathbf{y})$$

$$= \int d^3y \int \frac{d^3k}{(2\pi)^3} W(\mathbf{k}) e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \int \frac{d^3k'}{(2\pi)^3} F(\mathbf{k}') e^{-i\mathbf{k}'\cdot\mathbf{y}}$$

$$= \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} W(\mathbf{k}) F(\mathbf{k}') \int d^3y e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{y}}$$

$$= \int \frac{d^3k}{(2\pi)^3} W(\mathbf{k}) F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$F_W(\mathbf{k}) = W(\mathbf{k}) F(\mathbf{k})$$

- Smoothing acts as a low pass filter: if $W(\mathbf{x})$ is a broad function of width L, $W(\mathbf{k})$ suppressed for $k > 2\pi/L$

Convolution

- Filtered Variance

$$\langle F_W(\mathbf{x})F_W(\mathbf{x})\rangle = \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\langle F^*(\mathbf{k})F(\mathbf{k}')\rangle W^*(\mathbf{k})W(\mathbf{k}')$$

$$= \int \frac{d^3k}{(2\pi)^3} P_F(k)|W(\mathbf{k})|^2$$

- Common filter is the spherical tophat:

$$W_R(\mathbf{x}) = V_R^{-1} \qquad x < R$$

$$W_R(\mathbf{x}) = 0 \qquad x > R$$

- Fourier transform

$$W_R(\mathbf{k}) = \frac{3}{y^3}(\sin y - y\cos y)\,, \qquad (y = kR)$$

Normalization

- Normalization is often quoted as the top hat rms of the density field

$$\sigma_R^2 = \int d\ln k\, \Delta_\delta^2(k)|W_R(k)|^2$$

where observationally $\sigma_{8h^{-1}\mathrm{Mpc}} \equiv \sigma_8 \approx 1$

- Note that $\Delta_\delta^2(k)$ itself can be thought of as the variance of the field with a filter that has sharp high and low pass filters in k-space
- Convention is that σ_R is defined against the linear density field, not the true non-linear density field

Linear Perturbation Theory

- Energy (continuity) and momentum (Navier-Stokes) equations are linearized and hence Fourier modes obey

$$\frac{\partial}{\partial \eta}[(\rho + p)v^i] = -4\frac{\dot{a}}{a}(\rho + p)v^i + +ikp + ik^j \pi^i{}_j + ik^i(\rho + p)\Psi$$

- If the source of perturbations is from the (scalar) gravitational potential, directional dependence of velocity and anisotropic stress follows the direction of the plane wave, so define scalar velocity and anisotropic stress as

$$\mathbf{v}(\mathbf{k}) = i\hat{\mathbf{k}}v$$

$$\pi^i{}_j(\mathbf{k}) = \left(-\hat{k}^i\hat{k}_j + \frac{1}{3}\delta^i{}_j\right) p\pi$$

Linear Perturbation Theory

- Navier-Stokes equation

$$\frac{\partial}{\partial\eta}[(\rho+p)v] = -4\frac{\dot{a}}{a}(\rho+p)v^i + kp - \frac{2}{3}kp\pi + (\rho+p)k\Psi$$

$$(w = p/\rho\,, \quad c_s^2 = \delta p/\delta\rho\,, \quad \dot{\rho}/\rho = -3(1+w)\dot{a}/a)$$

$$\dot{v} = -(1-3w)\frac{\dot{a}}{a}v - \frac{\dot{w}}{1+w}v + \frac{kc_s^2}{1+w}\delta - \frac{2}{3}\frac{w}{1+w}k\pi + k\Psi$$

- Continuity Equation

$$\dot{\rho} = -3\left[\frac{\dot{a}}{a} + \dot{\Phi}\right](\rho+p) + i\mathbf{k}\cdot(\rho+p)\mathbf{v}$$

$$\dot{\rho} = -3\left[\frac{\dot{a}}{a} + \dot{\Phi}\right](\rho+p) - k(\rho+p)v$$

$$\dot{\delta} = -3\frac{\dot{a}}{a}(c_s^2 - w)\delta - (1+w)(kv + 3\dot{\Phi})$$

Poisson Equation

- Naive expectation: $\Phi = -\Psi$ and

$$\nabla^2 \Phi = -4\pi G a^2 \delta\rho$$
$$k^2 \Phi = 4\pi G a^2 \rho\delta$$

  where a^2 comes from physical $\rightarrow$ comoving and $\delta\rho$ since background density goes into scale factor evolution

- Einstein equations put in a relativistic correction (flat universe)

$$k^2 \Phi = 4\pi G a^2 \rho[\delta + 3\frac{\dot{a}}{a}(1+w)v/k]$$
$$k^2(\Phi + \Psi) = -8\pi G a^2 p\pi$$

  convenient to call combination

$$\Delta \equiv \delta + 3\frac{\dot{a}}{a}(1+w)v/k$$

Constancy of Potential & Growth Rate

- Given the Poisson equation relates a redshifting total density ρ and the comoving derivative factor a the density perturbation must grow as $\Delta \propto (a^2\rho)^{-1} \propto a^{1-3w}$ to maintain a constant potential.
- Density perturbations are stabilized by the expanding universe (expansion drag) and do not grow exponentially. Presents a new version of the horizon problem.
- Naive (Newtonian) argument: in the absence of stress perturbations the Euler equation takes the form $\dot{v} \sim k\Psi$
- Given an initial potential perturbation Ψ_i a velocity perturbation $v \sim (k\eta)\Psi_i$
- Given a velocity perturbation continuity grows a density fluctuation as $\dot{\Delta} \sim -kv$ or $\Delta = -(k\eta)^2\Psi_i$.

Constancy of Potential & Growth Rate

- The growing density perturbation is exactly that required to maintain the potential constant

$$\Psi \approx -\frac{4\pi G a^2 \rho}{k^2}\Delta \approx \frac{4\pi G a^2 \rho}{k^2}(k\eta)^2 \Psi_i$$

$\eta \propto a^{(1+3w)/2}$, $a^2\rho \propto a^{-(1+3w)}$

- Under gravity alone, the density fluctuations grow just fast enough to maintain constant potentials
- Stress fluctuations only decrease the rate of growth of the potential. Starting from an unperturbed $\Psi_i = 0$ universe, where do the fluctuations that form large scale structure come from

Bardeen Curvature

- A proper relativistic generalization involves the $(\dot{a}/a)v/k$ corrections, called the Bardeen (or comoving) curvature

$$\mathcal{R} \equiv \Phi - \frac{\dot{a}}{a} v/k \,.$$

- Geometric meaning: space curvature fluctuation on comoving (velocity-orthogonal-isotropic) time slicing
- Same time slicing gives Δ as the density perturbation

Bardeen Curvature

- Continuity equation becomes

$$\dot{\Delta} = -3\frac{\dot{a}}{a}\left(C_s^2 - w\right)\Delta - (1+w)(kv + 3\dot{\mathcal{R}})\,,$$

where the transformed sound speed

$$C_s^2 \equiv \frac{\Delta p}{\Delta \rho}$$

$$\Delta p \equiv \delta p - \dot{p}v/k$$

- Euler equation becomes

$$\dot{\mathcal{R}} = \frac{\dot{a}}{a}\xi$$

$$\xi = -\frac{C_s^2}{1+w}\Delta + \frac{2}{3}\frac{w}{1+w}\pi\,.$$

Bardeen Curvature

- So that the Bardeen curvature only changes in the presence of stress fluctuations – scales below the horizon
- Extremely useful result (proven in problem set) says that calculated $\mathcal{R}$ once and for all – e.g. during formation in an inflationary epoch
- Relationship to gravitational potential: (from Poisson & conservation equations)

$$\frac{\dot{a}}{a}\Psi - \dot{\Phi} = 4\pi G a^2 (\rho + p) v/k$$

so that if Φ constant and $\Psi = -\Phi$ then

$$-\left(\frac{\dot{a}}{a}\right)^2 \Phi = 4\pi G a^2 \rho (1+w) \frac{\dot{a}}{a} v/k$$

$$= \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 (1+w) \frac{\dot{a}}{a} v/k$$

Bardeen Curvature

- Relationship between the curvature Φ and v

$$\frac{\dot{a}}{a} v/k = -\frac{2}{3(1+w)}\Phi \qquad \rightarrow \mathcal{R} = 1 + \frac{2}{3(1+w)}\Phi$$

- Matter dominated $\Phi = 3\mathcal{R}/5$, radiation dominated $\Phi = 2\mathcal{R}/3$, Λ dominated $\Phi \rightarrow 0$.

- So: put these pieces together assuming dark energy is smooth

$$\begin{aligned}
\frac{k^3}{2\pi^2} P_\Delta(k) &= \left(\frac{k^2}{4\pi G a^2 \rho_m} \right)^2 \frac{k^3}{2\pi^2} P_\Phi(k) \\
&= \left(\frac{k^2}{4\pi G a^2 \rho_m} \right)^2 \frac{k^3}{2\pi^2} P_\Phi(k) \\
&= \frac{4}{9} \frac{a^2 k^4}{\Omega_m^2 H_0^4} \frac{k^3}{2\pi^2} P_\Phi(k)
\end{aligned}$$

Bardeen Curvature

- Assume initial curvature power spectrum

$$\frac{k^3}{2\pi^2} P_{\mathcal{R}}(k) = A_S \left(\frac{k}{k_{\rm norm}} \right)^{n_S - 1}$$

  and a transfer function $T(k)$ that defines the subhorizon evolution which is influenced by pressure effects during radiation domination

- Finally normalize to the matter dominated expectation and take $\Phi = [3G(a)/5]\,\mathcal{R}$ where $G(a)$ is the modification to the growth rate of Φ due to the dark energy and curvature

$$\Phi(a, k) = \frac{3}{5} G(a) T(k) \mathcal{R}(0, k)$$

$$\frac{k^3}{2\pi^2} P_{\Delta}(k) = \frac{4}{25} A_S \left(\frac{G(a)a}{\Omega_m} \right)^2 \left(\frac{k}{H_0} \right)^4 \left(\frac{k}{k_{\rm norm}} \right)^{n_S - 1} T^2(k)$$