

Ast 321: Cosmology II

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Inhomogeneous Cosmology

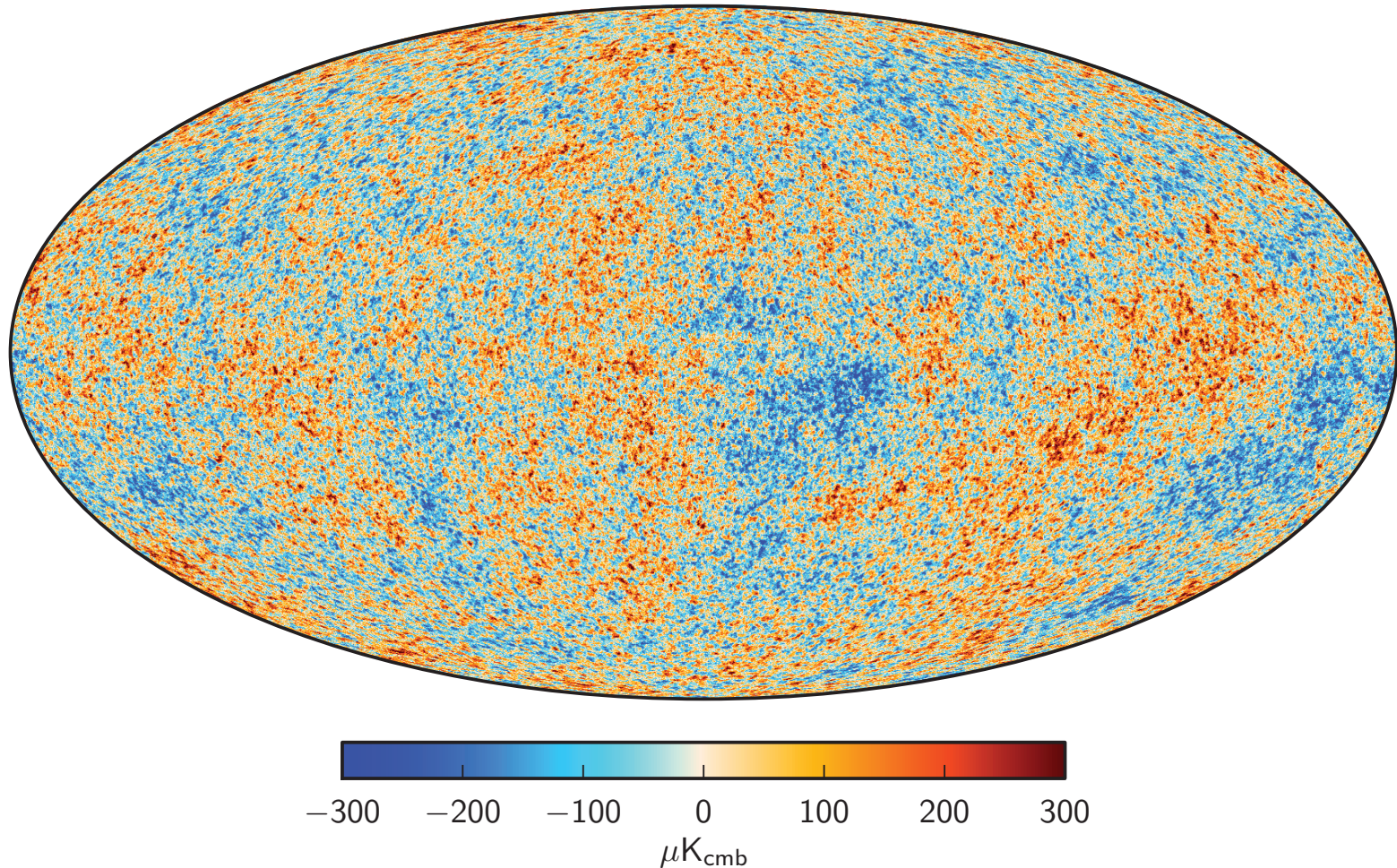
- Prerequisites: homogeneous cosmology at Ast 310 level
- Requirements: final project presented in class
- Gravitational Instability
- CMB Temperature Anisotropy (aka Ast 448)
- Large Scale Structure
- Dark Energy Models
- CMB Polarization (optional)
- Gravitational Lensing (optional)
- Ast 408: Formal Relativistic Perturbation Theory (Winter 2026)

Structure Formation

- Small perturbations from inflation over the course of the 14Gyr life of the universe are gravitationally enhanced into all of the structure seen today
- Cosmic microwave background shows a snapshot at a few hundred thousand years old at recombination
- Discovery in 1992 of cosmic microwave background anisotropy provided the observational breakthrough - convincing support for adiabatic initial density fluctuations of amplitude 10^{-5}
- Combine with galaxy clustering - large scale structure seen in galaxy surveys - right amplitude given cold dark matter

CMB Temperature Anisotropy

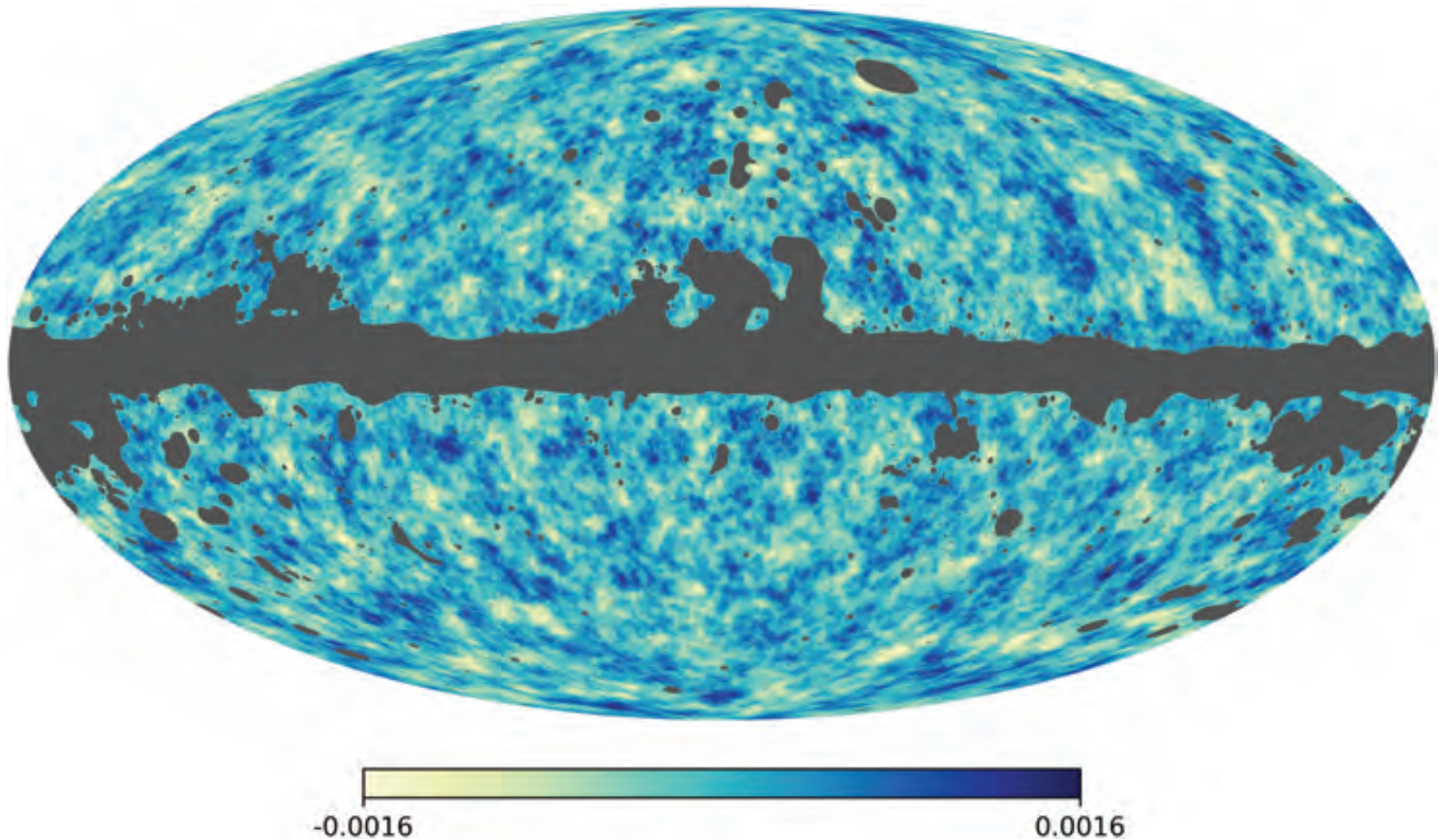
- Planck map of the temperature anisotropy (first discovered by COBE) from recombination:



Gravitational Lensing

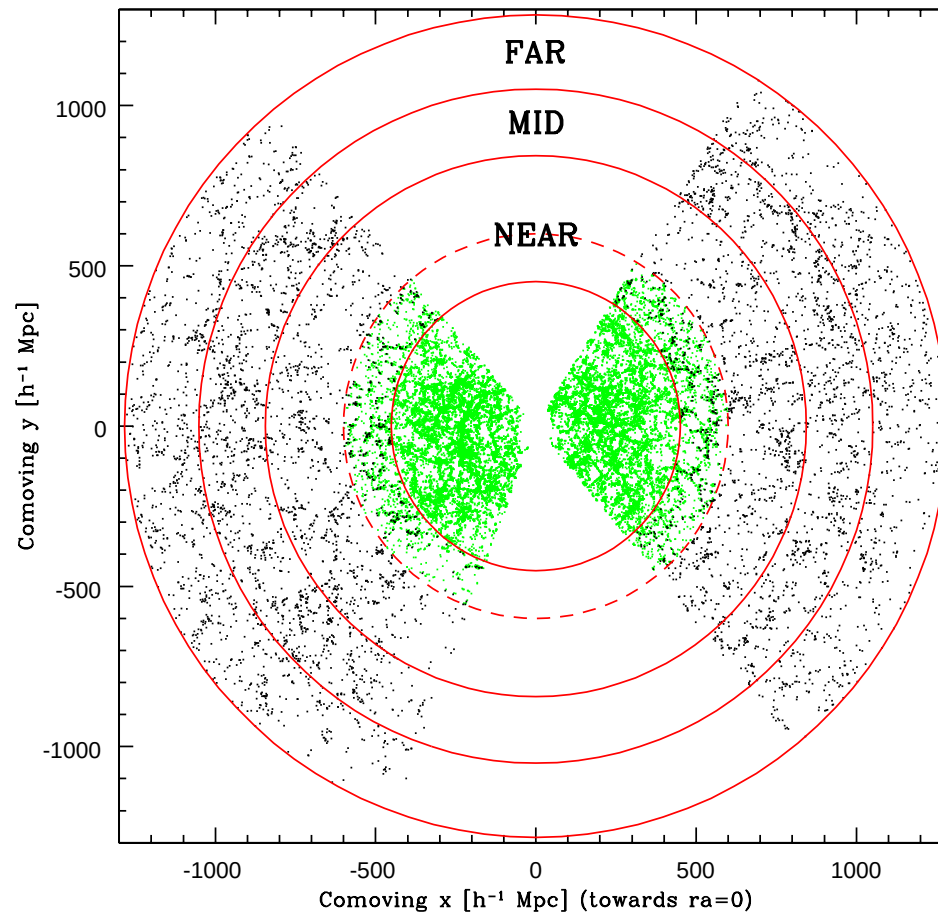
- Gravitational Lensing measures projected mass
- Planck CMB lensing map

lensing



Galaxy Redshift Surveys

- Galaxy redshift surveys measure the three dimensional distribution of galaxies today:

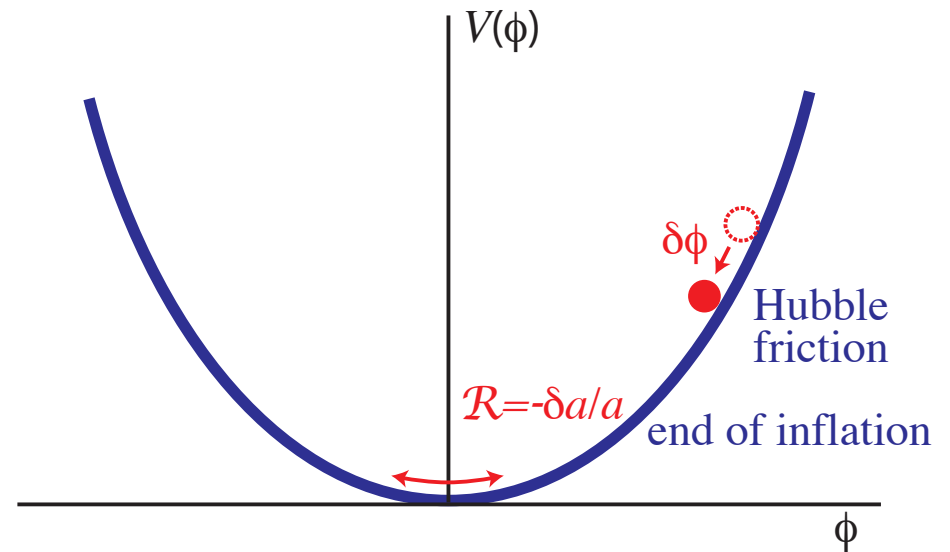


Curvature Fluctuations

- All structure originates from initial curvature fluctuations
- In the inflationary Λ CDM model these curvature fluctuations come from quantum field fluctuations during inflation: $\delta\phi_{\text{rms}} = H/2\pi$
- Field fluctuations change the scale factor at which inflation ends

$$\mathcal{R} = -\delta \ln a = -\frac{d \ln a}{dt} \frac{dt}{d\phi} \delta\phi = -H \frac{dt}{d\phi} \frac{H}{2\pi}$$

- Using the equation of state of ϕ we can convert $d\phi/dt$ to ϵ_H



Derivation in Slow Roll

- Rescaling the field fluctuation to absorb the scale factor, like with comoving coordinates $u = a\delta\phi$ brings the wave equation $\square\delta\phi = 0$ to

$$\ddot{u} + \left[k^2 - \frac{2}{\tilde{\eta}^2}\right]u = 0$$

where $\tilde{\eta}$ is the conformal time normalized to zero at the end of inflation such that $k\tilde{\eta} = -1$ denotes horizon exit

- Simple harmonic oscillator well inside the horizon

$$\ddot{u} + k^2u = 0$$

Derivation in Slow Roll

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k, \tilde{\eta})\hat{a} + u^*(k, \tilde{\eta})\hat{a}^\dagger$$

where $u(k, \tilde{\eta})$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \quad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\tilde{\eta}] = i$

$$u(k, \eta) = \frac{1}{\sqrt{2k}}e^{-ik\tilde{\eta}}$$

Quantum Fluctuations

- Zero point fluctuations of ground state

$$\begin{aligned}\langle u^2 \rangle &= \langle 0 | u^\dagger u | 0 \rangle \\ &= \langle 0 | (u^* \hat{a}^\dagger + u \hat{a})(u \hat{a} + u^* \hat{a}^\dagger) | 0 \rangle \\ &= \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle |u(k, \tilde{\eta})|^2 \\ &= \langle 0 | [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} | 0 \rangle |u(k, \tilde{\eta})|^2 \\ &= |u(k, \tilde{\eta})|^2 = \frac{1}{2k}\end{aligned}$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in.

Slow Roll Limit

- Classical equation of motion then has the exact solution

$$u = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\tilde{\eta}} \right) e^{-ik\tilde{\eta}}$$

- For $|k\tilde{\eta}| \ll 1$ (late times, $\gg$ Hubble length) fluctuation freezes in

$$\lim_{|k\tilde{\eta}| \rightarrow 0} u = -\frac{1}{\sqrt{2k}} \frac{i}{k\tilde{\eta}} \approx \frac{iHa}{\sqrt{2k^3}}$$
$$\delta\phi = \frac{iH}{\sqrt{2k^3}}$$

- Power spectrum of field fluctuations

$$\Delta_{\delta\phi}^2 = \frac{k^3 |\delta\phi|^2}{2\pi^2} = \frac{H^2}{(2\pi)^2}$$

Curvature Fluctuation

- Equation of state characterized by slow roll parameter $\epsilon_H \ll 1$

$$w_\phi \equiv \frac{2}{3}\epsilon_H - 1 = \frac{p_\phi}{\rho_\phi} = \frac{(d\phi/dt)^2/2 - V}{(d\phi/dt)^2/2 + V} \\ \approx \frac{(d\phi/dt)^2}{V} - 1$$

and $H^2 \approx 8\pi GV/3$ from Friedmann

$$\epsilon_H \approx \frac{3}{2} \frac{(d\phi/dt)^2}{V} \approx 4\pi G \frac{(d\phi/dt)^2}{H^2}$$

and the variance of fluctuations per log wavenumber $d \ln k$

$$\Delta_{\mathcal{R}}^2 \equiv \langle \mathcal{R}^2 \rangle \approx \left(\frac{H}{2\pi} \right)^2 \frac{4\pi G}{\epsilon_H} = \frac{H^2}{8\pi^2 \epsilon_H M_{\text{pl}}^2}$$

Gravitational Waves

- Gravitational wave amplitude satisfies wave equation same as scalar field $\square h_{+,\times} = 0$

$$\ddot{h}_{+,\times} + 2\frac{\dot{a}}{a}\dot{h}_{+,\times} + k^2 h_{+,\times} = 0.$$

- Acquires quantum fluctuations in same manner as ϕ .

Einstein-Hilbert action sets the canonical normalization: ADM

$^{(4)}R/16\pi G = (\dot{h}_{ij}/2)^2/16\pi G + \dots$ and $h_{ij} = h_+e_{+ij} + h_\times e_{\times ij}$ so

$$\Delta_{+,\times}^2 = 16\pi G \Delta_{\delta\phi}^2 = 16\pi G \frac{H^2}{(2\pi)^2}$$

Gravitational Waves

- Quantum fluctuations in gravitational waves follow a similar prescription but are not enhanced by $1/\epsilon_H$
- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation
- Tensor-scalar ratio is therefore generally small

$$r \equiv 4 \frac{\Delta_+^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon_H$$

- Gravitational waves from inflation can be measured via its imprint on the polarization of the CMB (current upper limits $r < 0.032$ 95% CL)

Tilt

- Curvature power spectrum is scale invariant to the extent that H and ϵ_H are constant

$$\Delta_{\mathcal{R}}^2 \propto H^2 / \epsilon_H \approx \text{const}$$

- But with a small tilt that indicates inflation must end in ~ 60 efolds

$$\frac{d \ln \Delta_{\mathcal{R}}^2}{d \ln k} \equiv n_S - 1 = 2 \frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon_H}{d \ln k}$$

- Evaluate at horizon crossing where fluctuation freezes $k = aH$

$$\frac{d \ln H}{d \ln k} \approx \frac{d \ln H}{d \ln a} = -\epsilon_H$$

$$\frac{d \ln \epsilon_H}{d \ln k} \approx \frac{d \ln \epsilon_H}{d \ln a} = 2(\delta_1 + \epsilon_H)$$

$$\text{so } n_S = 1 = -4\epsilon_H - 2\delta_1$$

Power Spectrum: A_S, n_S

- Tilt in the slow-roll approximation

$$n_S - 1 = -4\epsilon_H - 2\delta_1$$

- Power spectrum parameters:

$$\Delta_{\mathcal{R}}^2 = A_S \left(\frac{k}{0.05 \text{Mpc}^{-1}} \right)^{n_S - 1}$$

with pivot scale 0.05 Mpc^{-1} chosen to be approximately where the data constrains inflation

- A_S, n_S are two of the 6 Λ CDM parameters

CMB Parameter Inferences

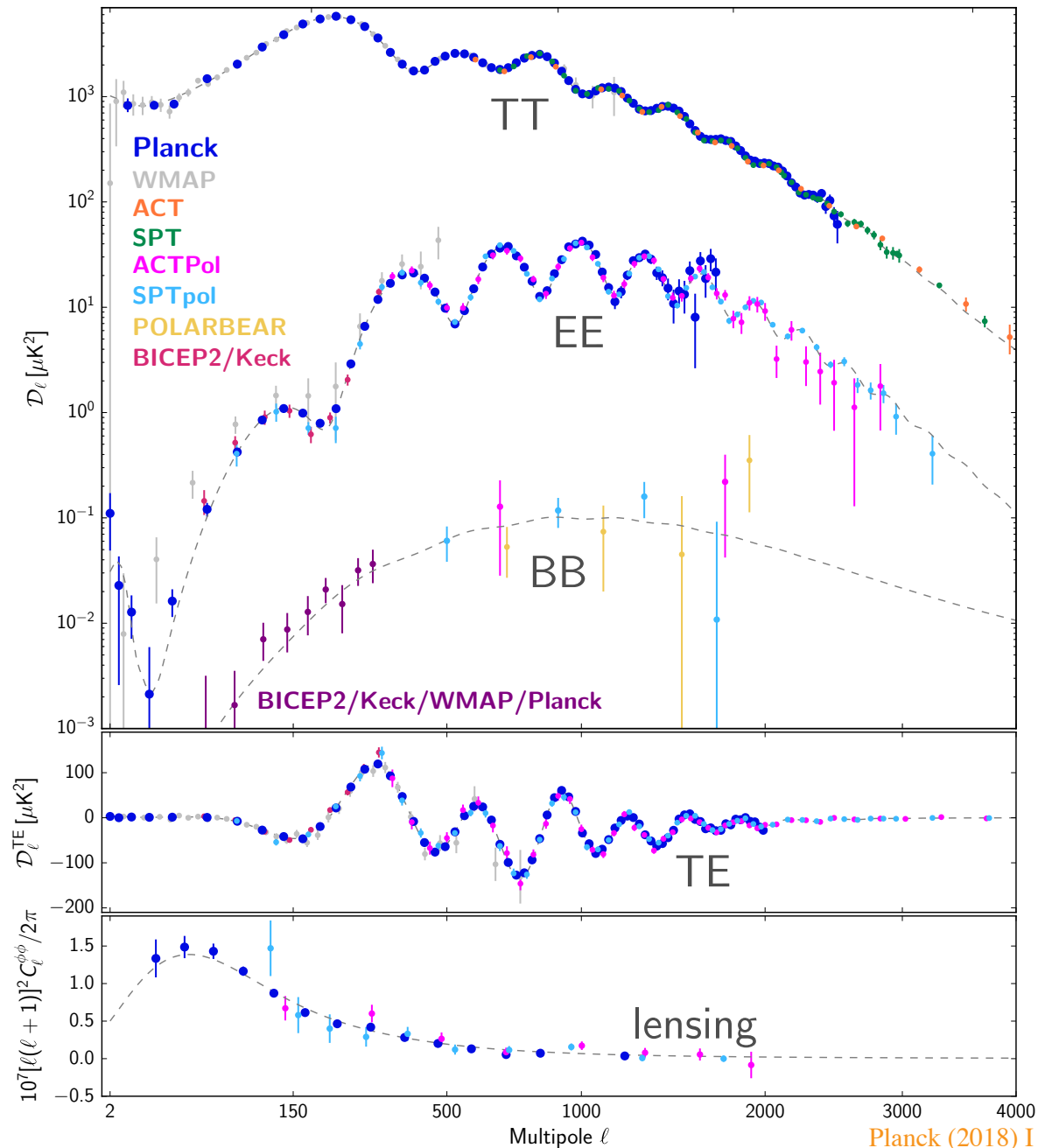
- Spectrum constrains the matter-energy contents of the universe
- Planck 2018 results [arXiv:1807.06209]

Parameter	TT+lowE 68% limits	TE+lowE 68% limits	EE+lowE 68% limits	TT,TE,EE+lowE 68% limits	TT,TE,EE+lowE+lensing 68% limits
$\Omega_b h^2$	0.02212 ± 0.00022	0.02249 ± 0.00025	0.0240 ± 0.0012	0.02236 ± 0.00015	0.02237 ± 0.00015
$\Omega_c h^2$	0.1206 ± 0.0021	0.1177 ± 0.0020	0.1158 ± 0.0046	0.1202 ± 0.0014	0.1200 ± 0.0012
$100\theta_{MC}$	1.04077 ± 0.00047	1.04139 ± 0.00049	1.03999 ± 0.00089	1.04090 ± 0.00031	1.04092 ± 0.00031
τ	0.0522 ± 0.0080	0.0496 ± 0.0085	0.0527 ± 0.0090	$0.0544^{+0.0070}_{-0.0081}$	0.0544 ± 0.0073
$\ln(10^{10} A_s)$	3.040 ± 0.016	$3.018^{+0.020}_{-0.018}$	3.052 ± 0.022	3.045 ± 0.016	3.044 ± 0.014
n_s	0.9626 ± 0.0057	0.967 ± 0.011	0.980 ± 0.015	0.9649 ± 0.0044	0.9649 ± 0.0042
H_0 [km s ⁻¹ Mpc ⁻¹] . .	66.88 ± 0.92	68.44 ± 0.91	69.9 ± 2.7	67.27 ± 0.60	67.36 ± 0.54

- Since 2018, CMB improvements from ground based instruments mainly on lensing, polarization (including upper limits on r)
- Note the low value of H_0 compared with the Cepheid-SN distance ladder

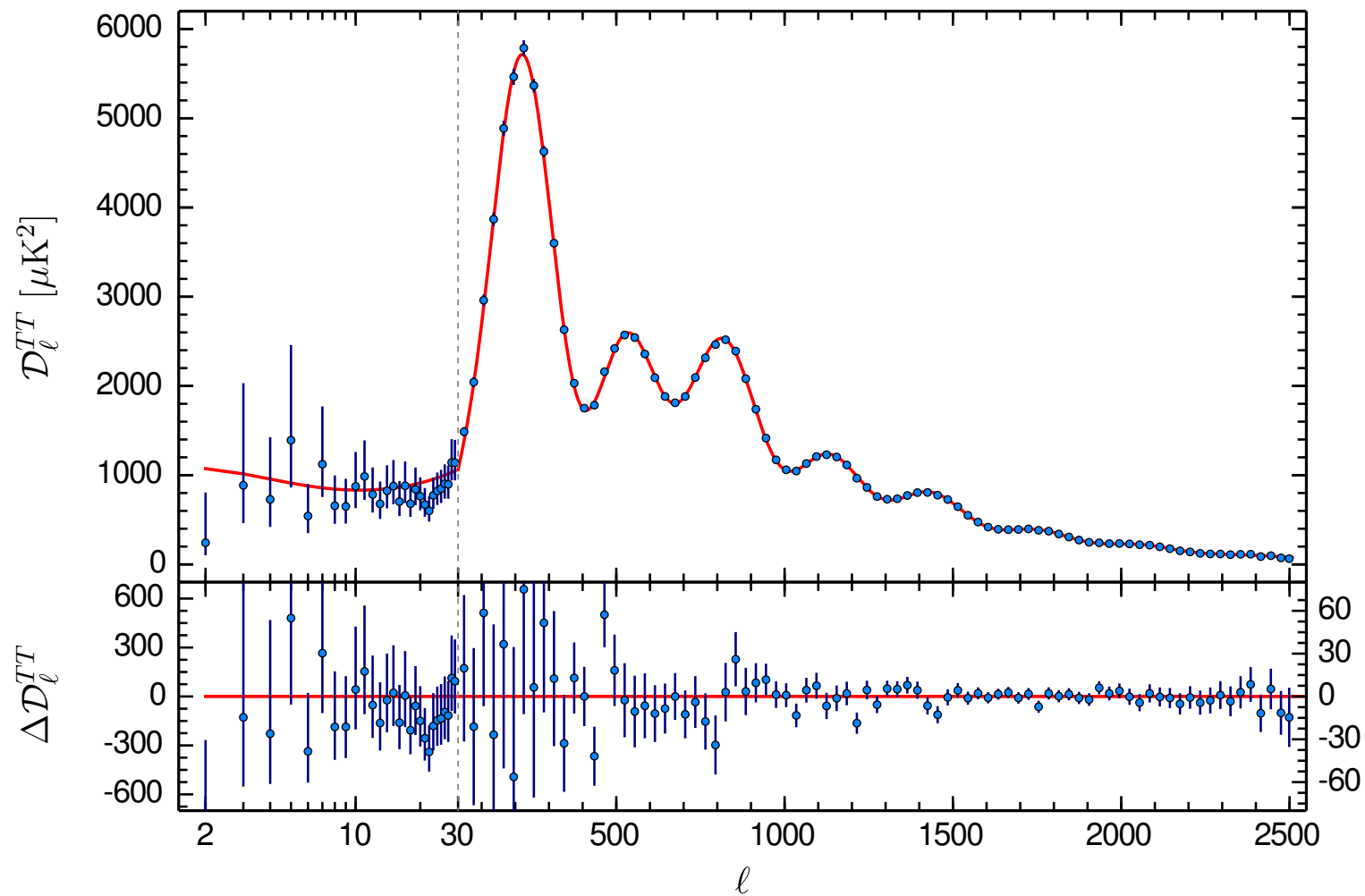
CMB Power Spectra

- Power spectra of CMB
 - temperature
 - polarization
 - lensing



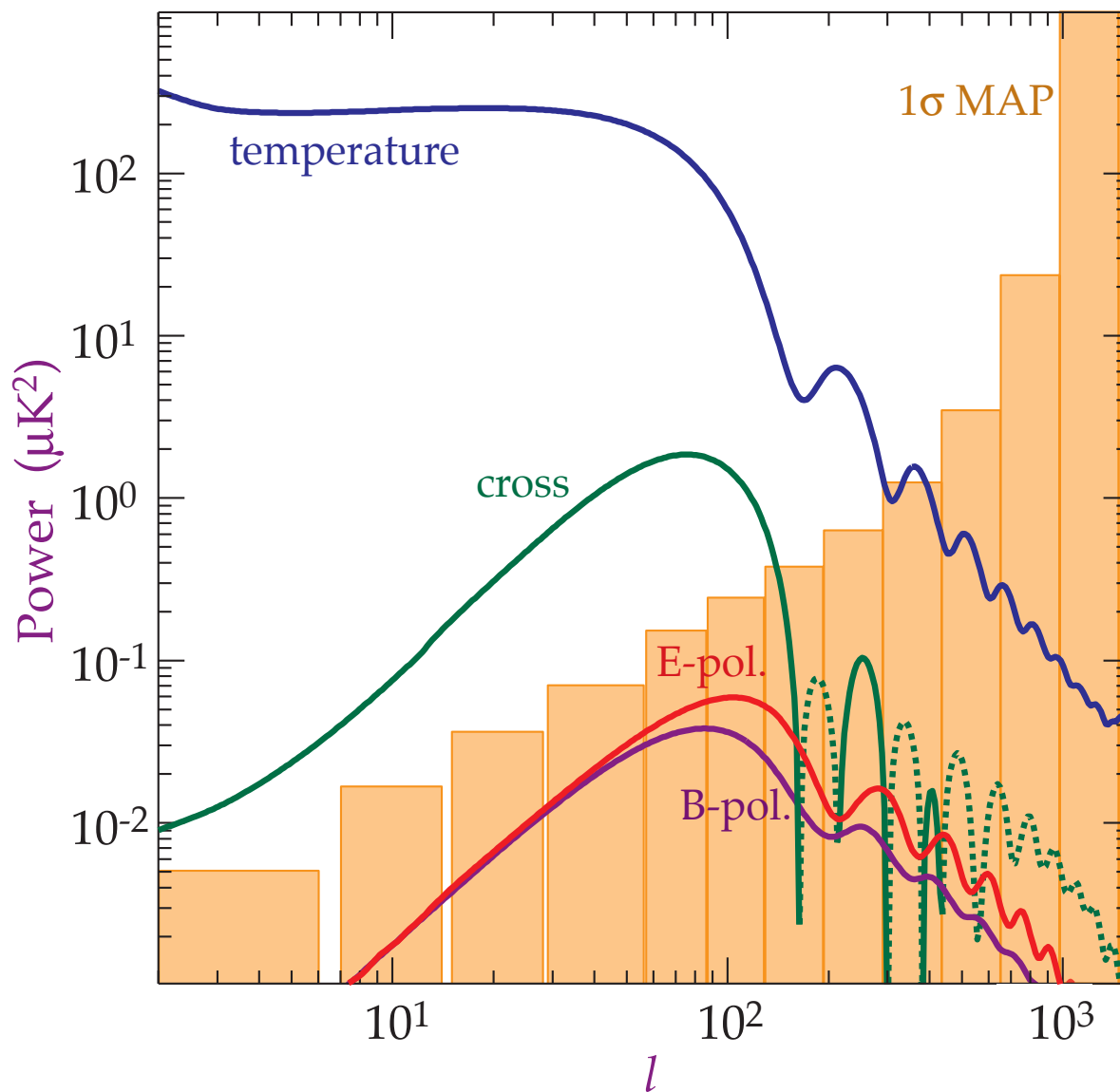
CMB Temperature Anisotropy

- Power spectrum shows characteristic scales where the intensity of variations peak - reveals geometry and contents of the universe:



Tensor Power Spectrum

- Gravitational waves from inflation (yet to be detected)



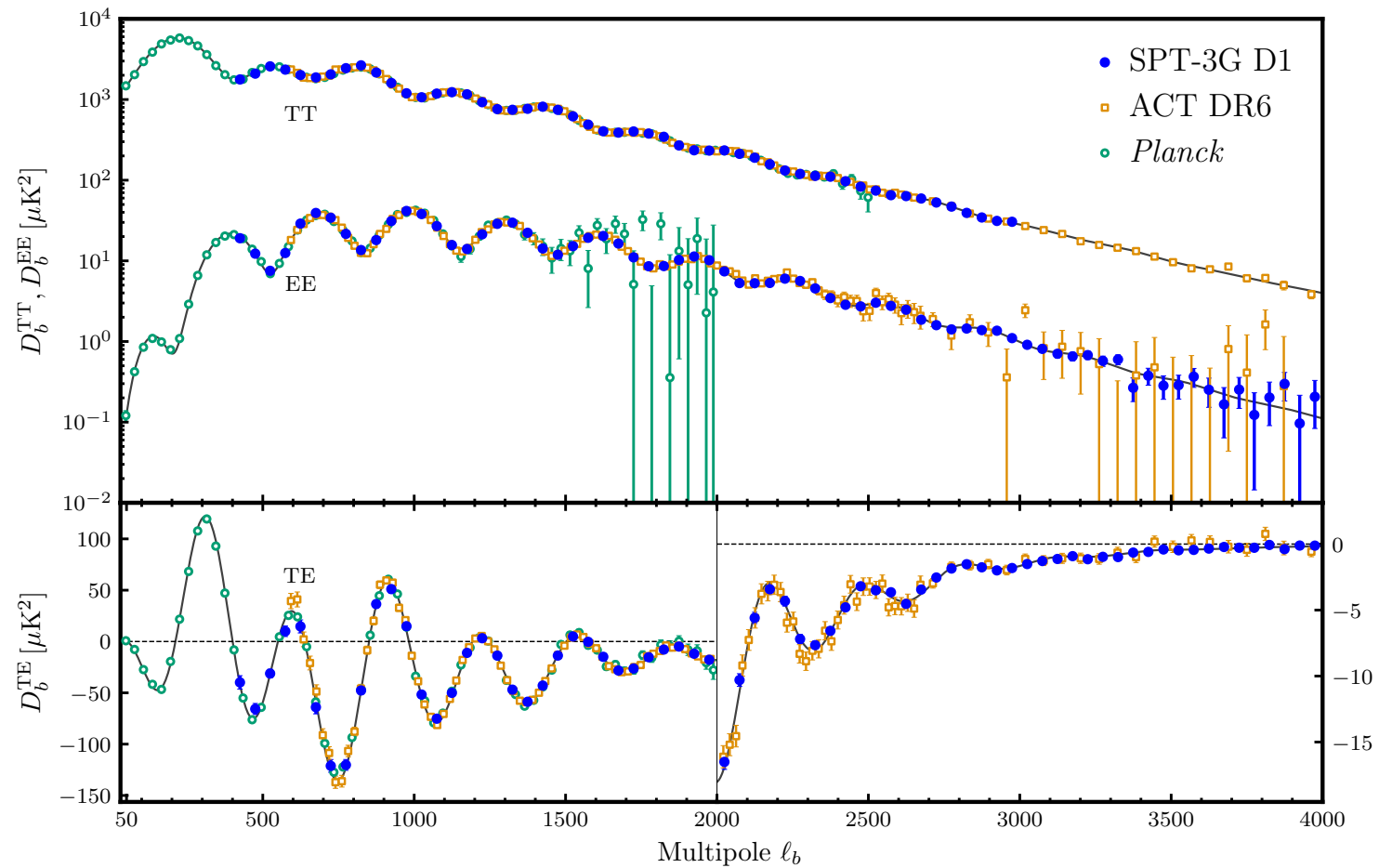
Planck 2018 Predictions

- Observables at lower $z < 1100$ all predicted to high accuracy

Parameter	TT+lowE 68% limits	TE+lowE 68% limits	EE+lowE 68% limits	TT,TE,EE+lowE 68% limits	TT,TE,EE+lowE+lensing 68% limits
$\Omega_b h^2$	0.02212 ± 0.00022	0.02249 ± 0.00025	0.0240 ± 0.0012	0.02236 ± 0.00015	0.02237 ± 0.00015
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τ	0.0522 ± 0.0080	0.0496 ± 0.0085	0.0527 ± 0.0090	$0.0544^{+0.0070}_{-0.0081}$	0.0544 ± 0.0073
$\ln(10^{10} A_s)$	3.040 ± 0.016	$3.018^{+0.020}_{-0.018}$	3.052 ± 0.022	3.045 ± 0.016	3.044 ± 0.014
n_s	0.9626 ± 0.0057	0.967 ± 0.011	0.980 ± 0.015	0.9649 ± 0.0044	0.9649 ± 0.0042
H_0 [km s ⁻¹ Mpc ⁻¹] . .	66.88 ± 0.92	68.44 ± 0.91	69.9 ± 2.7	67.27 ± 0.60	67.36 ± 0.54
Ω_Λ	0.679 ± 0.013	0.699 ± 0.012	$0.711^{+0.033}_{-0.026}$	0.6834 ± 0.0084	0.6847 ± 0.0073
Ω_m	0.321 ± 0.013	0.301 ± 0.012	$0.289^{+0.026}_{-0.033}$	0.3166 ± 0.0084	0.3153 ± 0.0073
$\Omega_m h^2$	0.1434 ± 0.0020	0.1408 ± 0.0019	$0.1404^{+0.0034}_{-0.0039}$	0.1432 ± 0.0013	0.1430 ± 0.0011
$\Omega_m h^3$	0.09589 ± 0.00046	0.09635 ± 0.00051	$0.0981^{+0.0016}_{-0.0018}$	0.09633 ± 0.00029	0.09633 ± 0.00030
σ_8	0.8118 ± 0.0089	0.793 ± 0.011	0.796 ± 0.018	0.8120 ± 0.0073	0.8111 ± 0.0060
$S_8 \equiv \sigma_8(\Omega_m/0.3)^{0.5}$.	0.840 ± 0.024	0.794 ± 0.024	$0.781^{+0.052}_{-0.060}$	0.834 ± 0.016	0.832 ± 0.013
$\sigma_8 \Omega_m^{0.25}$	0.611 ± 0.012	0.587 ± 0.012	0.583 ± 0.027	0.6090 ± 0.0081	0.6078 ± 0.0064
z_{re}	7.50 ± 0.82	$7.11^{+0.91}_{-0.75}$	$7.10^{+0.87}_{-0.73}$	7.68 ± 0.79	7.67 ± 0.73
$10^9 A_s$	2.092 ± 0.034	2.045 ± 0.041	2.116 ± 0.047	$2.101^{+0.031}_{-0.034}$	2.100 ± 0.030
$10^9 A_s e^{-2\tau}$	1.884 ± 0.014	1.851 ± 0.018	1.904 ± 0.024	1.884 ± 0.012	1.883 ± 0.011
Age [Gyr]	13.830 ± 0.037	13.761 ± 0.038	$13.64^{+0.16}_{-0.14}$	13.800 ± 0.024	13.797 ± 0.023
z_*	1090.30 ± 0.41	1089.57 ± 0.42	$1087.8^{+1.6}_{-1.7}$	1089.95 ± 0.27	1089.92 ± 0.25
r_* [Mpc]	144.46 ± 0.48	144.95 ± 0.48	144.29 ± 0.64	144.39 ± 0.30	144.43 ± 0.26
$100\theta_*$	1.04097 ± 0.00046	1.04156 ± 0.00049	1.04001 ± 0.00086	1.04109 ± 0.00030	1.04110 ± 0.00031
z_{drag}	1059.39 ± 0.46	1060.03 ± 0.54	1063.2 ± 2.4	1059.93 ± 0.30	1059.94 ± 0.30
r_{drag} [Mpc]	147.21 ± 0.48	147.59 ± 0.49	146.46 ± 0.70	147.05 ± 0.30	147.09 ± 0.26
k_D [Mpc ⁻¹]	0.14054 ± 0.00052	0.14043 ± 0.00057	0.1426 ± 0.0012	0.14090 ± 0.00032	0.14087 ± 0.00030
z_{eq}	3411 ± 48	3349 ± 46	3340^{+81}_{-92}	3407 ± 31	3402 ± 26
k_{eq} [Mpc ⁻¹]	0.01041 ± 0.00014	0.01022 ± 0.00014	$0.01019^{+0.00025}_{-0.00028}$	0.010398 ± 0.000094	0.010384 ± 0.000081
$100\theta_{s,\text{eq}}$	0.4483 ± 0.0046	0.4547 ± 0.0045	0.4562 ± 0.0092	0.4490 ± 0.0030	0.4494 ± 0.0026

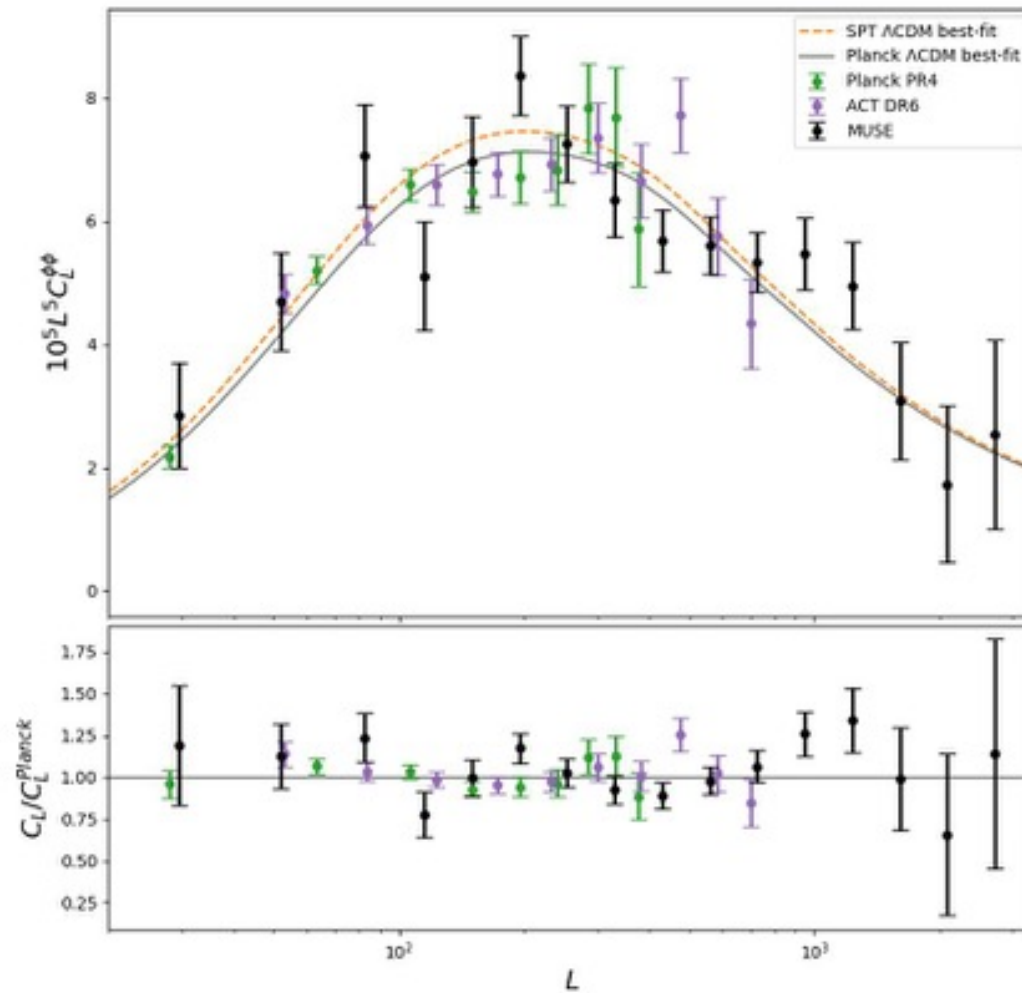
CMB after Planck

- Polarization and higher resolution



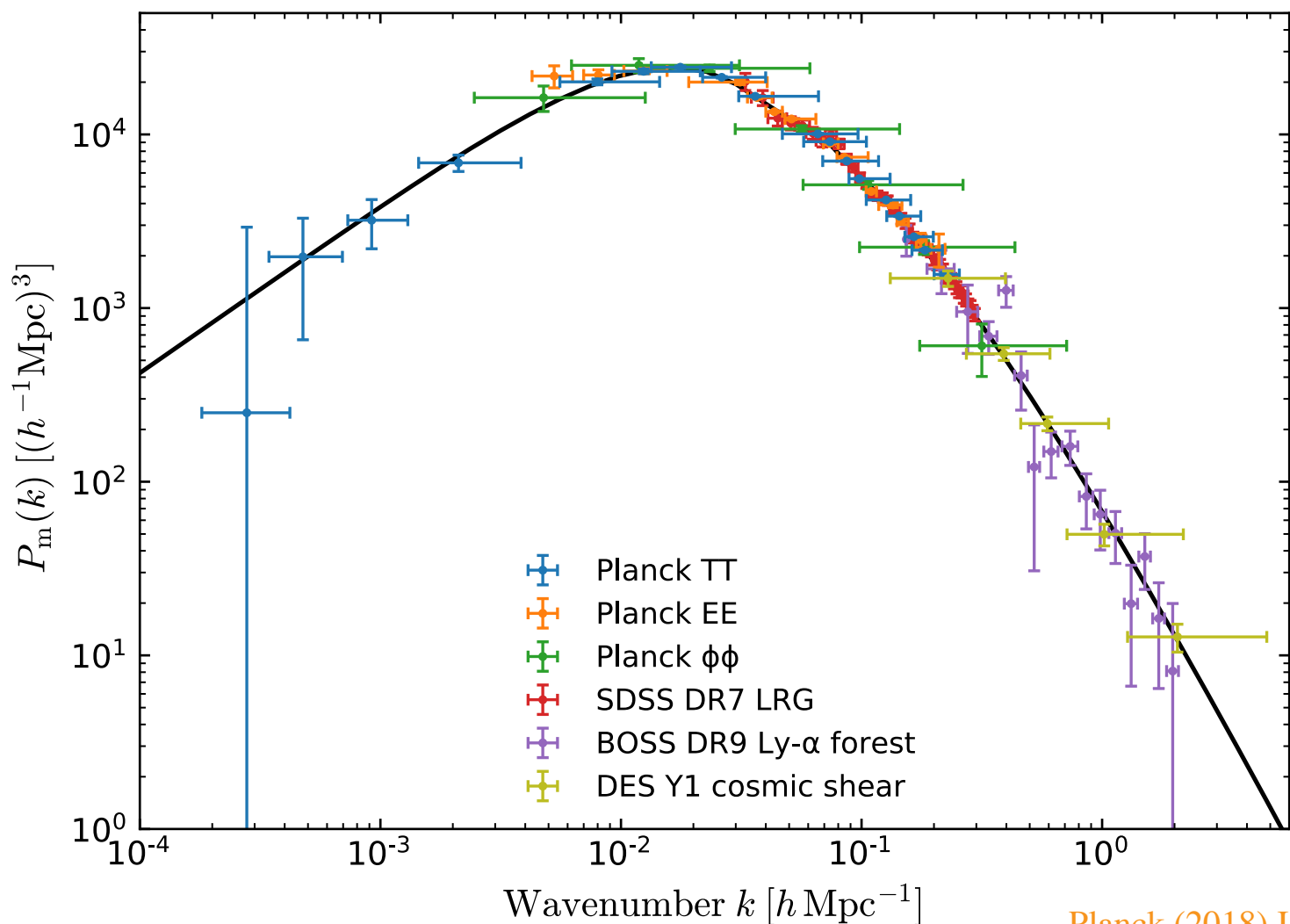
CMB after Planck

- Gravitational lensing power spectrum



Matter Power Spectrum

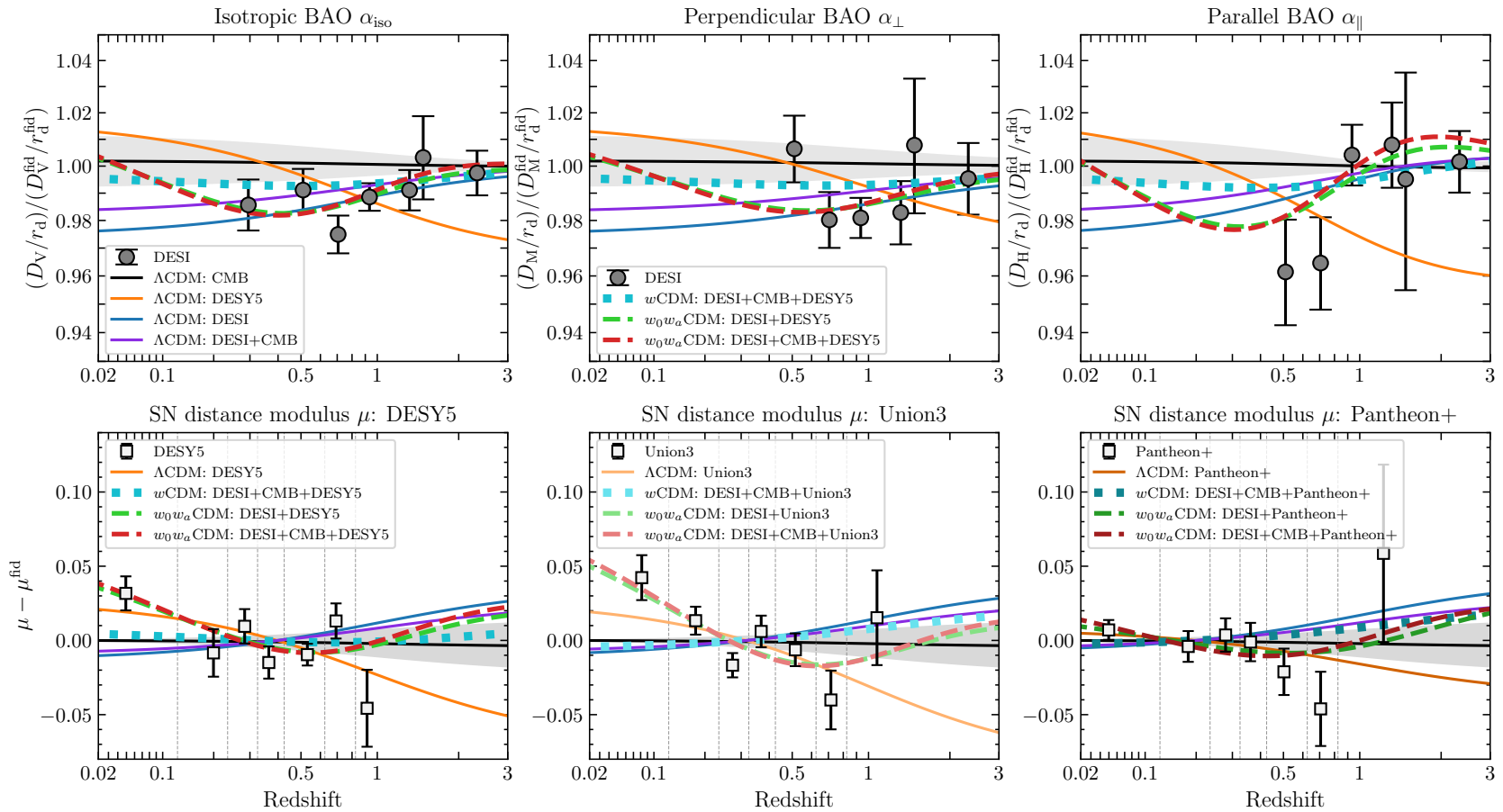
- Compilation of Redshift Surveys, Lensing, CMB



Planck (2018) I

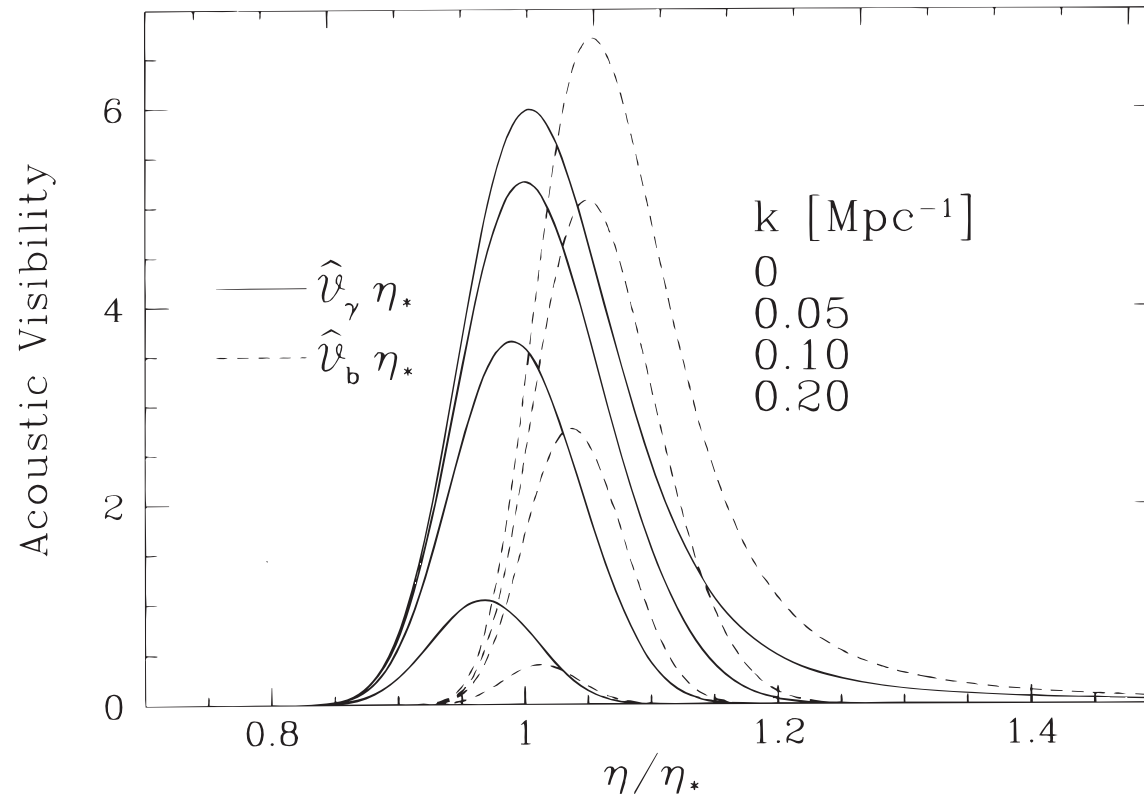
BAO and SN

- Baryon Acoustic Oscillations as transverse and radial standard rulers (DESI DR2)
- SN as standard candles (DES Y5) - is Λ CDM now disfavored?



Drag Aside

- BAO best measure the sound horizon at the drag epoch $\eta_d > \eta_*$, later than recombination due to low baryon momentum ratio R
- A super old plot illustrating this from Hu & Sugiyama astro-ph/9510117



Distance Parameters

- BAO measure both the excess clustering in angular and redshift scales as a function of redshift or scale factor $a = (1 + z)^{-1}$
- Angular dimension measures $D_M = D_A$ the comoving angular diameter distance
- Redshift dimension measures $D_H = 1/H$ the expansion rate
- Recall the FRW line element gives the separation between two points separated by $dz, d\theta, d\phi$

$$ds^2 = a^2[-d\eta^2 + dD^2 + D_A^2(d\theta^2 + \sin^2 \theta d\phi^2)] = a^2[-d\eta^2 + d\Sigma^2]$$

to convert to dz recall that photons travel on null $ds^2 = 0$ radially

$$d\eta = dt/a = (dt/d \ln a) d \ln a / a = d \ln a / aH = dz/H = dD$$

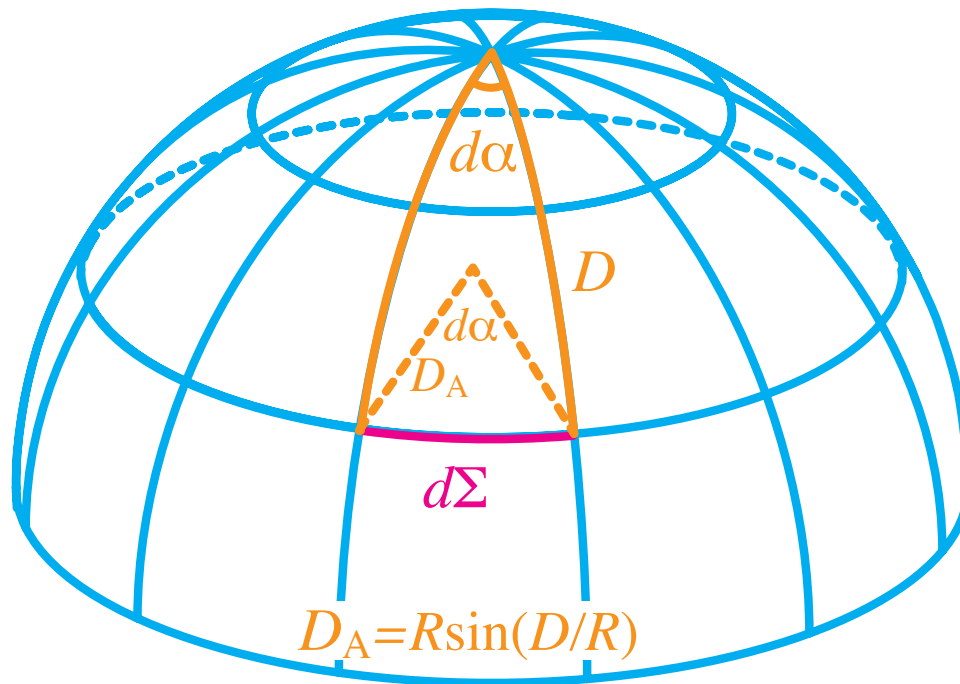
so the separation dz corresponding to the BAO measures $H(z)$

Distance Parameters

- If $K \equiv R^{-2} \neq 0$, $D_A \neq D$ - to visualize suppress one sky dimension so angular separation is $d\alpha^2 = d\theta^2 + \sin^2 \theta d\phi^2$
- Define D_A such that a transverse scale $d\Sigma (= r_d)$ obeys

$$d\Sigma = D_A d\alpha$$

Draw a circle at the distance D , its radius is $D_A = R \sin(D/R)$

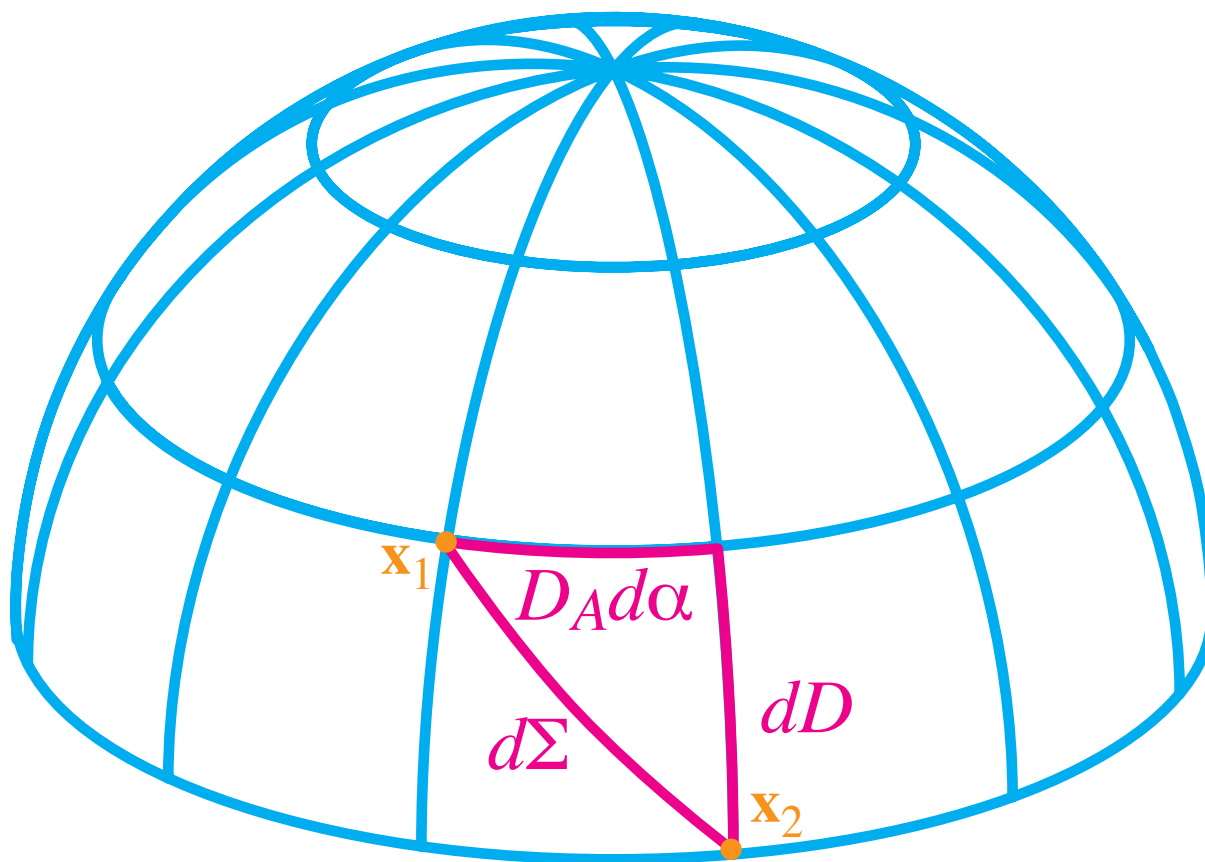


FRW Geometry

- Angular diameter distance
- Positively curved geometry $D_A < D$ and objects are further than they appear
- Negatively curved universe R is imaginary and
$$R \sin(D/R) = i|R| \sin(D/i|R|) = |R| \sinh(D/|R|)$$
and $D_A > D$ objects are closer than they appear
- Flat universe, $R \rightarrow \infty$ and $D_A = D$

FRW Geometry

- What's the distance $d\Sigma$ between $\mathbf{x}_1 = (\theta_1, \phi_1, z_1)$ and $\mathbf{x}_2 = (\theta_2, \phi_2, z_2)$, separated by angle $d\alpha$ and radial distance $dD = dz/H(z)$?



Angular Diameter Distance

- For small angular and radial separations, space is nearly flat so that the Pythagorean theorem holds for differentials

$$d\Sigma^2 = dD^2 + D_A^2 d\alpha^2$$

- Now restore the fact that the angular separation can involve two angles on the sky - the curved sky is just a copy of the spherical geometry with unit radius that we were suppressing before

$$\begin{aligned} d\Sigma^2 &= dD^2 + D_A^2 d\alpha^2 \\ &= dD^2 + D_A^2 (d\theta^2 + \sin^2 \theta d\phi^2) \end{aligned}$$

- D_A useful for describing observables (flux, angular positions)
- D useful for theoretical constructs (causality, relationship to temporal evolution) but also comparisons between photons emitted at different redshifts, e.g. radial BAO

Luminosity Distance

- Photons propagate as in special relativity in comoving/conformal coordinates: ignore scale factor in right units!
- Given a physical luminosity at emission $L = dE/dt$ in comoving coordinates $\mathcal{E} = E/(1+z)$ for the wavelength conversion and $d\eta = (1+z)dt$ for the time interval so the “comoving luminosity”

$$\mathcal{L} = L/(1+z)^2$$

- So flux $F = \mathcal{L}/4\pi D_A^2$ defines luminosity distance $d_L = (1+z)D_A$ and the relative magnitude of high and low redshift supernovae

$$\mu = m - M_{\text{fid}} = 5 \log_{10}[(1+z)D_A/1\text{Mpc}] + 25$$

where M_{fid} is the absolute magnitude at 10pc and can be (controversially... H_0 !) calibrated by the distance ladder through Cepheids, TRGB, etc

On Time

- Despite GRs coordinate invariance, preferred cosmic time is slicing where matter is homogeneous and isotropic on average
- Cosmological constant is a spacetime scalar - with no matter, no preferred time: all 3 FRW metrics
- During inflation, the field and its surface of homogeneity is the clock!

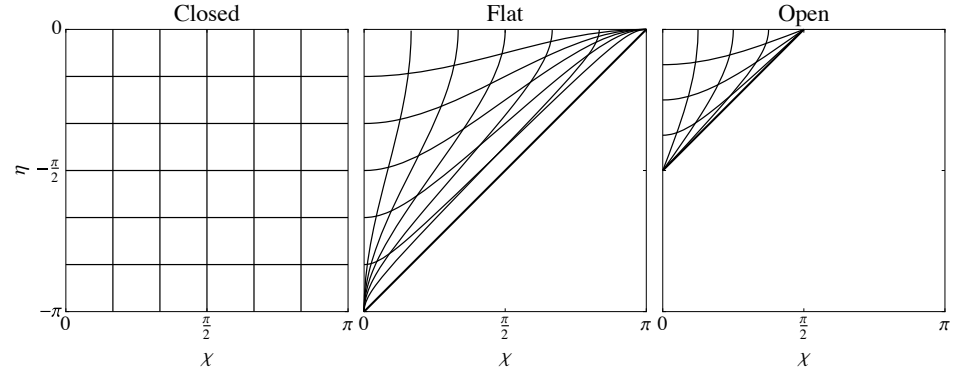


FIG. 1. Conformal diagrams showing portions of de Sitter space charted by closed (left), flat (middle) and open (right) foliations. Thick lines indicate coordinate singularities. Superimposed are lines of constant isotropic time and radius for each foliation.

De Sitter Charts

One special feature of de Sitter space is that there is no preferred temporal coordinate to define a foliation with respect to. Sections of the full spacetime can therefore be charted by isotropic coordinates where the constant time slices have positive, negative or zero spatial curvature. The conformal diagram for de Sitter space can be constructed from the positive curvature (closed) foliation of the spacetime, where the line element takes the form

$$ds^2 = \left(\frac{1}{H \sin \eta} \right)^2 (-d\eta^2 + d\chi^2 + \sin^2 \chi d\Omega_2^2), \quad (22)$$

with the dimensionless conformal time $\eta \in (-\pi, 0)$ and the comoving radial coordinate $\chi \in [0, \pi]$. Here $H^2 = \Lambda_{\text{eff}}/3$. We use the (η, χ) conformal diagram throughout to represent the spacetime.

Closed, flat, and open isotropic coordinates can alternately be used to foliate portions of de Sitter space and are useful in finding solutions to the background massive gravity equations and for investigating perturbations. With the transformations

$$\begin{aligned} \sinh(Ht_c) &= -\cot \eta, \\ Hr_c &= 2 \tan(\chi/2), \end{aligned} \quad (23)$$

the line element (22) takes its closed isotropic form

$$ds^2 = -dt_c^2 + \left[\frac{\cosh(Ht_c)}{1 + (Hr_c)^2/4} \right]^2 (dr_c^2 + r_c^2 d\Omega_2^2), \quad (24)$$

where $t_c \in (-\infty, \infty)$, $r_c \in [0, \infty)$. These coordinates chart the entire de Sitter spacetime. Similarly, defining the coordinates

$$\begin{aligned} e^{Ht_f} &= -\frac{\cos \chi + \cos \eta}{\sin \eta}, \\ Hr_f &= \frac{\sin \chi}{\cos \chi + \cos \eta}, \end{aligned} \quad (25)$$

obtains the flat isotropic form

$$ds^2 = -dt_f^2 + e^{2Ht_f} (dr_f^2 + r_f^2 d\Omega_2^2), \quad (26)$$

where $Ht_f \in (-\infty, \infty)$, $Hr_f \in [0, \infty)$. These coordinates chart the upper left half of the conformal diagram $\eta > \chi - \pi$. Finally, the coordinate definition

$$\begin{aligned} \ln[\tanh(Ht_o/2)] &= \tanh^{-1} \left(\frac{\sin \eta}{\cos \chi} \right), \\ 2 \tanh^{-1}(Hr_o/2) &= \tanh^{-1} \left(\frac{\sin \chi}{\cos \eta} \right), \end{aligned} \quad (27)$$

gives the open isotropic form

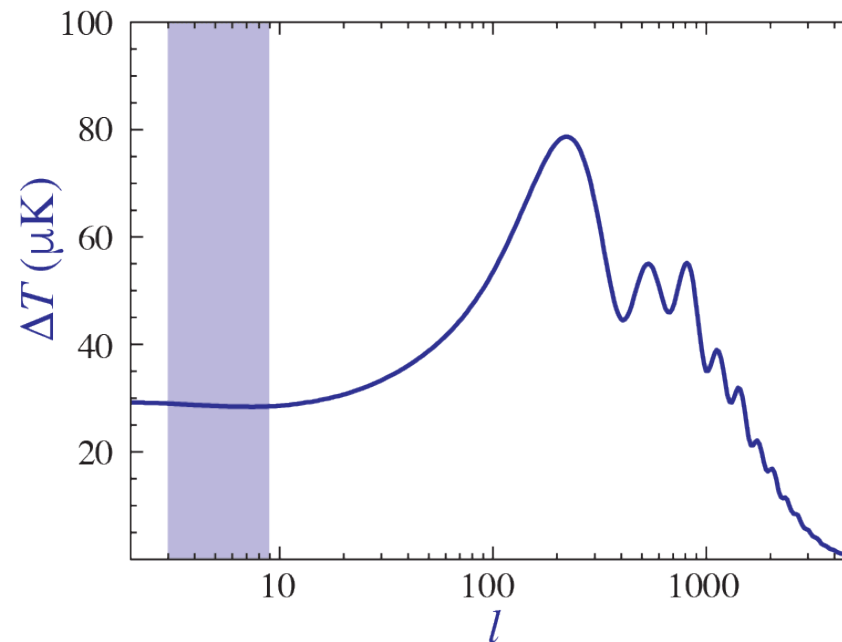
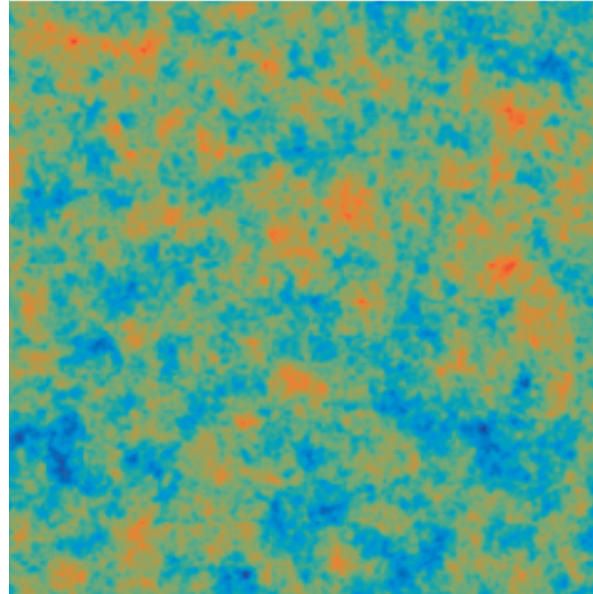
$$ds^2 = -dt_o^2 + \left[\frac{\sinh(Ht_o)}{1 - (Hr_o)^2/4} \right]^2 (dr_o^2 + r_o^2 d\Omega_2^2), \quad (28)$$

where $Ht_o \in (0, \infty)$, $Hr_o \in [0, 2)$. These coordinates chart the upper left wedge of the conformal diagram $\eta > \chi - \pi/2$, corresponding to $1/8$ of the space.

Temperature Anisotropy

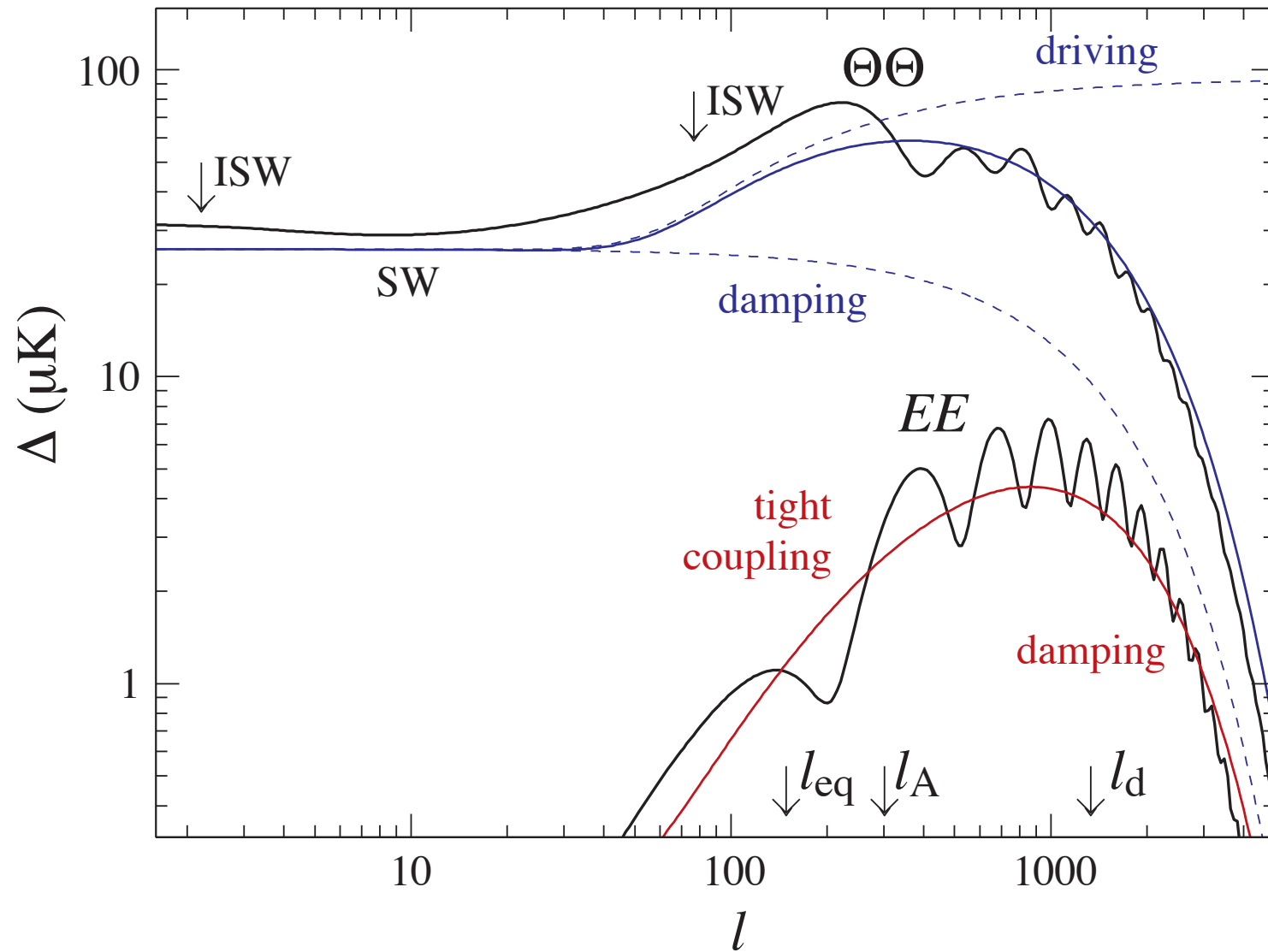
Filtered Maps and Power Spectrum

- Take original $64^\circ \times 64^\circ$ map
- Band filter to a range of multipole moments



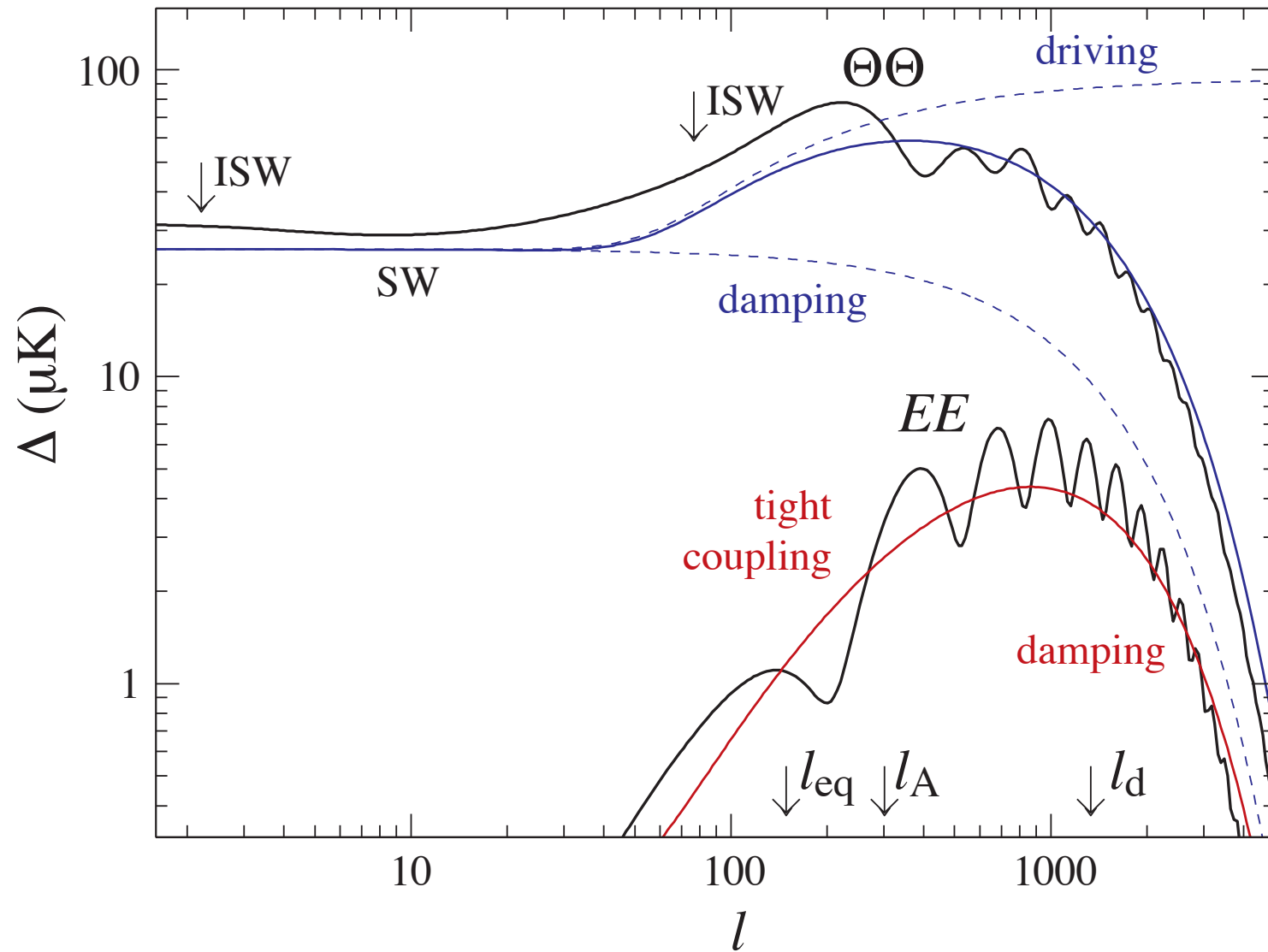
Schematic Outline

- Take apart features in the power spectrum



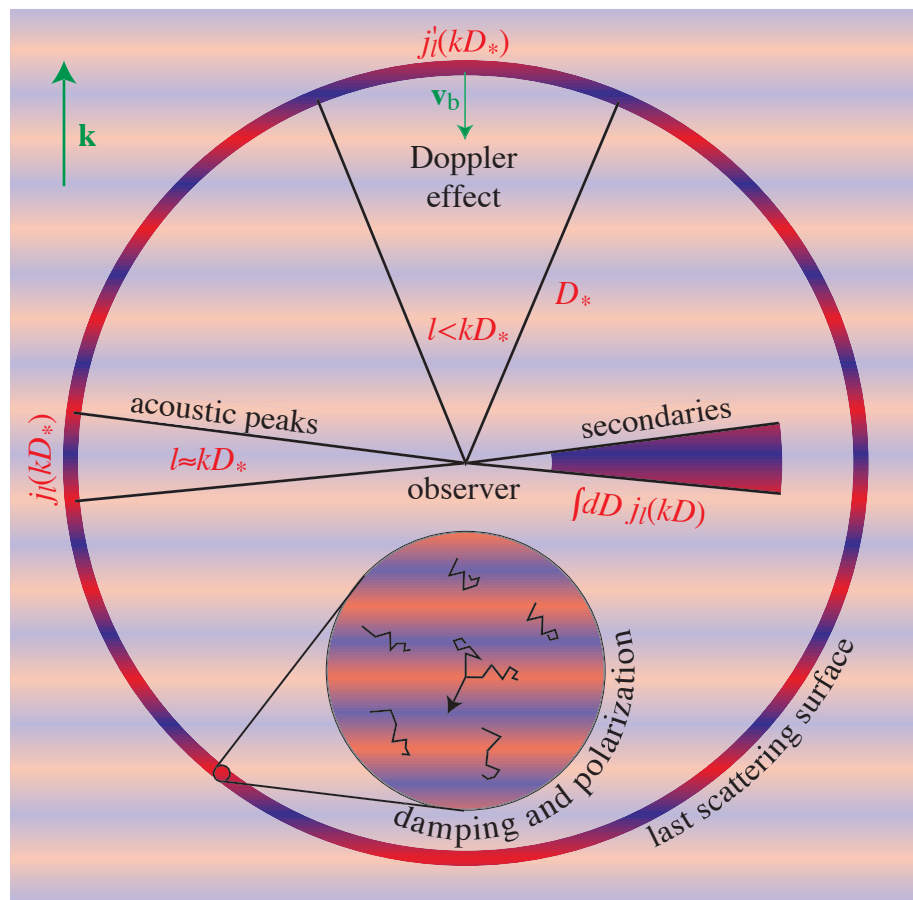
Schematic Outline

- Take apart features in the power spectrum



Last Scattering

- Angular distribution of radiation is the 3D temperature field projected onto a shell - surface of last scattering
- Shell radius is distance from the observer to recombination: called the last scattering surface
- Take the radiation distribution at last scattering to also be described by an isotropic temperature fluctuation field $\Theta(\mathbf{x})$



Astro-Particle Dictionary

Astro and physics use different words to describe same thing:

- Specific intensity $I_\nu = 4\pi\nu^3 f \leftrightarrow$ phase space distribution f

$I_\nu = \Delta E / \Delta t \Delta \nu \Delta \Omega dA$: “energy per unit everything”

Black Body $I_\nu = B_\nu \leftrightarrow f = (e^{E/T} - 1)^{-1}$

- Radiative transfer equation

$$dI_\nu/d\tau = -I_\nu + S_\nu$$

$\leftrightarrow$ Boltzmann equation $Df/Dt = C[f]$

- Formal solution with τ increasing along photon path

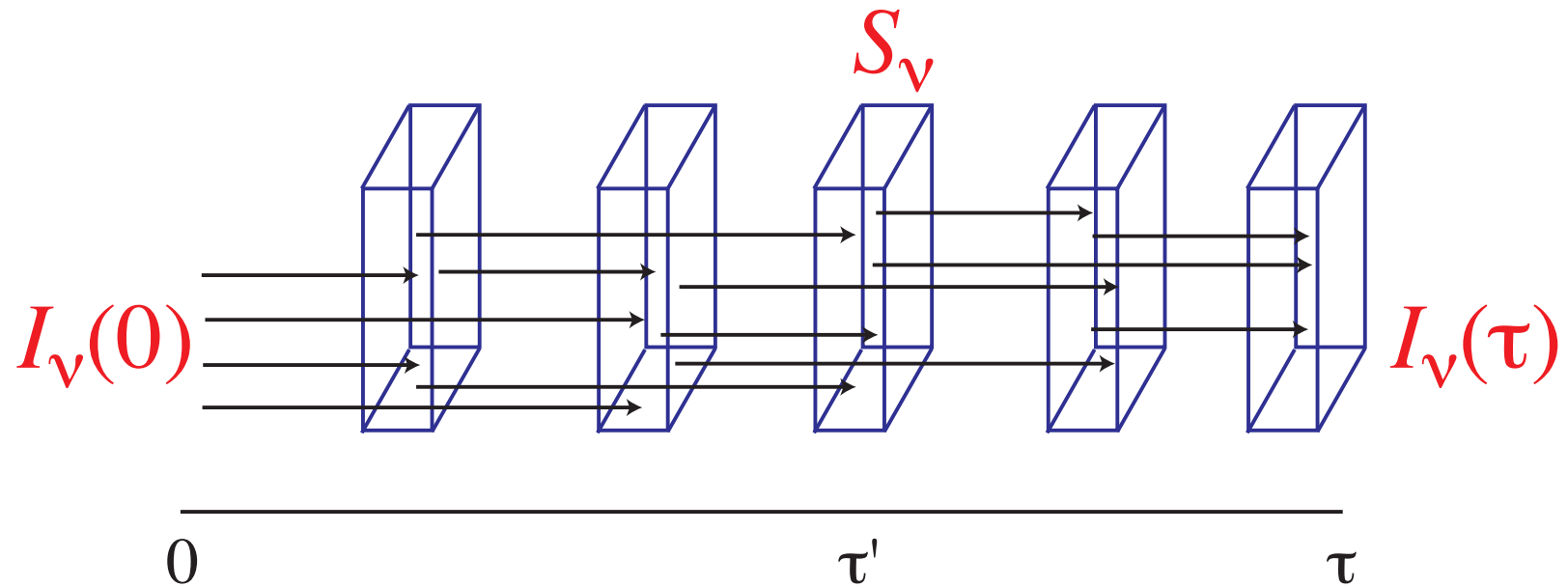
$$I_\nu(\tau) = I_\nu(0)e^{-\tau} + \int_0^\tau d\tau' e^{-\tau+\tau'} S_\nu$$

$\leftrightarrow$ integral approach to Boltzmann equation with τ decreasing,
optical depth in front of source

- Absorption, emission, scattering $\leftrightarrow$ Collision term
- Einstein relations $\leftrightarrow$ Single matrix element
- $dI_\nu/d\tau = 0 \leftrightarrow$ Liouville equation $Df/Dt = 0$

Formal/Integral Solution

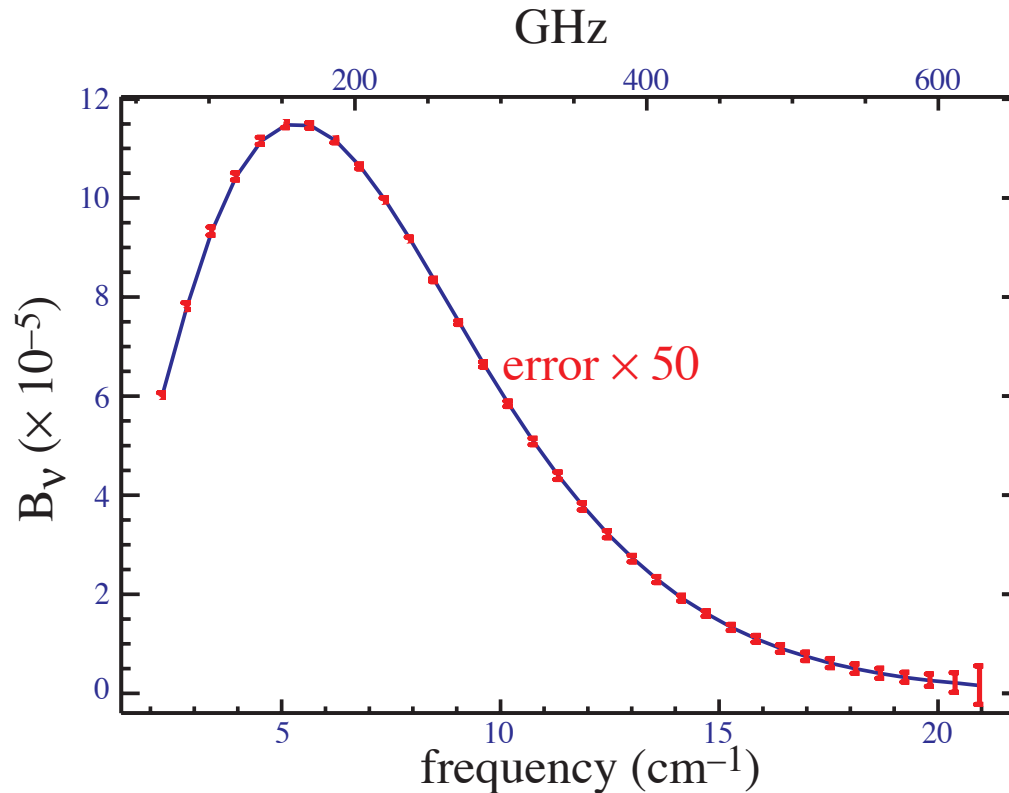
- Initial radiation field $I_\nu(0)$ absorbed and replaced with source emission until observed at $I_\nu(\tau)$



- Observer is on the right at the end of path, optical depth in front of source is $\tau - \tau'$ (in general, frequency ν dependent, but not for Thomson scattering)

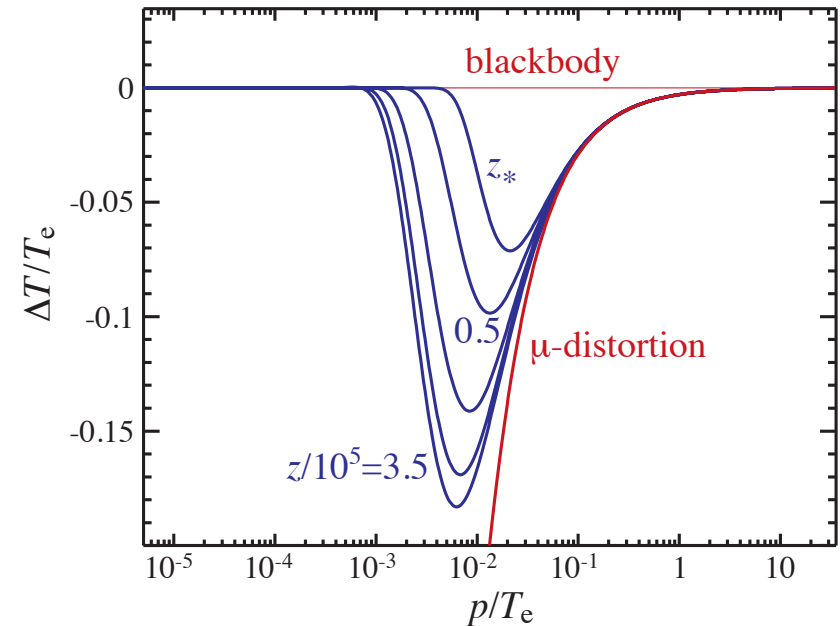
CMB Spectrum

- Modern measurement from COBE satellite of **blackbody spectrum**. $T = 2.725\text{K}$ giving $\Omega_\gamma h^2 = 2.471 \times 10^{-5}$



Black Body Formation

- After $z \sim 10^6$, photon creating processes $\gamma + e^- \leftrightarrow 2\gamma + e^-$ and bremsstrahlung $e^- + p \leftrightarrow e^- + p + \gamma$ drop out of equilibrium for photon energies $E \sim T$.
- Compton scattering remains effective in redistributing energy via exchange with electrons
- Out of equilibrium processes like decays leave residual photon chemical potential imprint
- Observed black body spectrum places tight constraints on any that might dump energy into the CMB



Radiative Processes

- Bremsstrahlung collision term ($k = \hbar = c = 1, x = h\nu/kT_e$)

$$C_{\text{ff}}[f] \sim n_i n_e \sigma_T \left(\frac{T_e}{m} \right)^{-1/2} Z^2 \alpha \frac{e^{-x}}{T_e^3 x^3} [1 - (e^x - 1)f]$$

where $\sim$ absorbs a dimensionless Gaunt factor: note density and temperature dependence

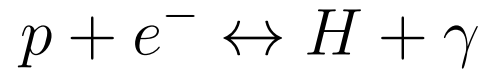
- Double Compton collision term

$$C_{\text{dc}}[f] \sim n_e \sigma_T \left(\frac{T_e}{m} \right)^2 \alpha \frac{e^{-x}}{x^3} [1 - (e^x - 1)f] \int dx x^4 (1 + f) f$$

wins at high T_e and low number densities but requires some seed photons $f \neq 0$

Recombination

- Maxwell-Boltzmann distribution determines the equilibrium distribution for reactions, e.g. big-bang nucleosynthesis, recombination:



$$\frac{n_p n_e}{n_H} \approx e^{-B/T} \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{(\mu_p + \mu_e - \mu_H)/T}$$

where $B = m_p + m_e - m_H = 13.6\text{eV}$ is the binding energy, $g_p = g_e = \frac{1}{2}g_H = 2$, and $\mu_p + \mu_e = \mu_H$ in equilibrium

- Define ionization fraction

$$n_p = n_e = x_e n_b$$

$$n_H = n_b - n_e = (1 - x_e) n_b$$

Recombination

- Saha Equation

$$\begin{aligned}\frac{n_e n_p}{n_H n_b} &= \frac{x_e^2}{1 - x_e} \\ &= \frac{1}{n_b} \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-B/T}\end{aligned}$$

- Naive guess of $T_* = B$ wrong due to the low baryon-photon ratio
– $T_* \approx 0.3\text{eV}$ so recombination at $z_* \approx 1000$
- But the photon-baryon ratio is very low

$$\eta_{b\gamma} \equiv n_b/n_\gamma \approx 3 \times 10^{-8} \Omega_b h^2$$

Recombination

- Eliminate in favor of $\eta_{b\gamma}$ and B/T through

$$n_\gamma = 0.244T^3, \quad \frac{m_e}{B} = 3.76 \times 10^4$$

- Big coefficient

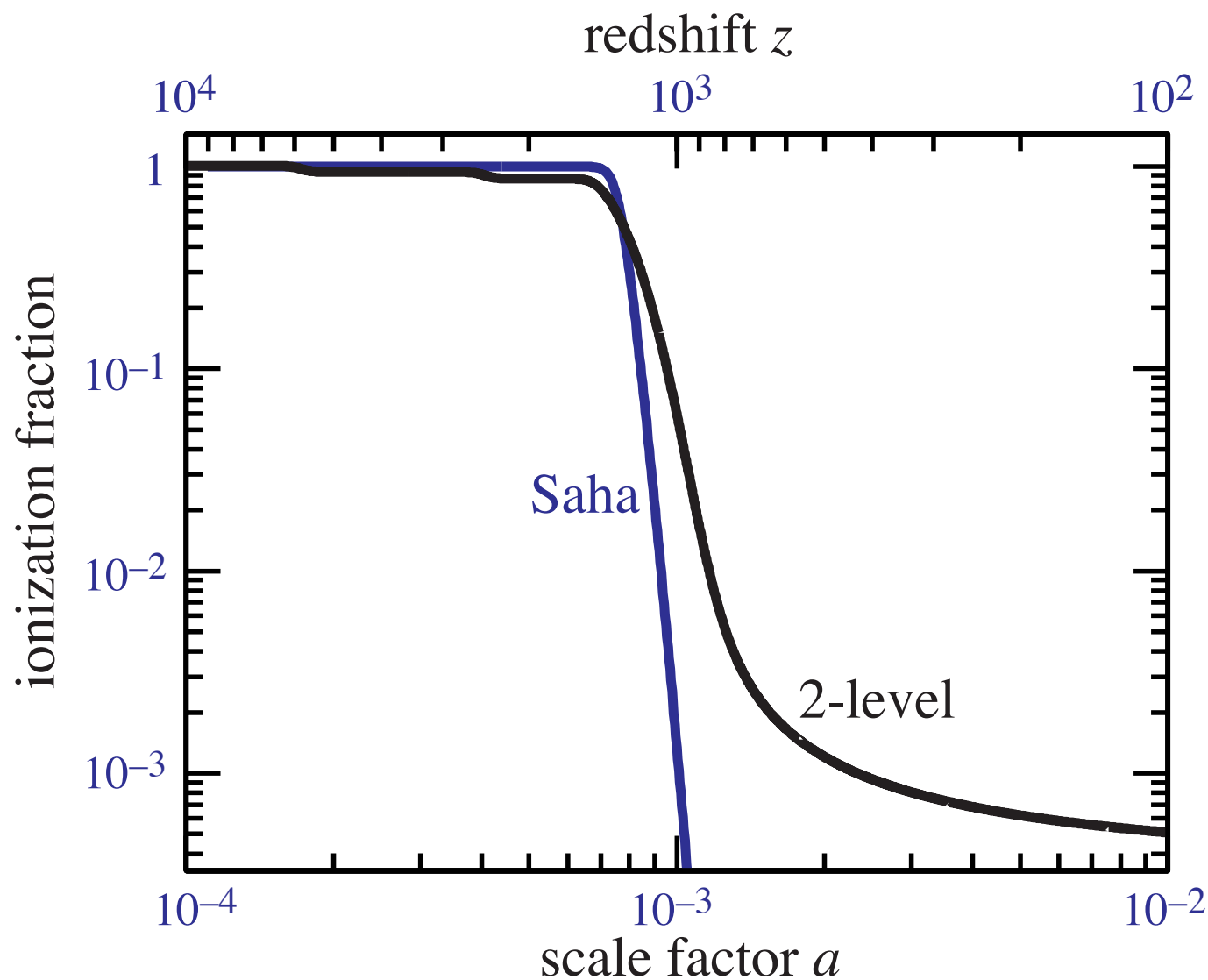
$$\frac{x_e^2}{1 - x_e} = 3.16 \times 10^{15} \left(\frac{B}{T} \right)^{3/2} e^{-B/T}$$

$$T = 1/3\text{eV} \rightarrow x_e = 0.7, T = 0.3\text{eV} \rightarrow x_e = 0.2$$

- Further delayed by inability to maintain equilibrium since net is through 2γ process and redshifting out of line

Recombination

- Relatively sharp transition to $x_e \ll 1$ in less than an efold



Angular Power Spectrum

- Take visibility function $x_e n_e \sigma_T a e^{-\tau} \rightarrow \delta(D - D_*)$

$$\Theta(\hat{\mathbf{n}}) = \int dD \Theta(\mathbf{x}) \delta(D - D_*)$$

where D is the comoving distance and D_* denotes recombination.

- Well under curvature scale, describe $\Theta(\mathbf{x})$ by its Fourier moments

$$\Theta(\mathbf{x}) = \int \frac{d^3 k}{(2\pi)^3} \Theta(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}$$

- Orthogonality and Completeness (forward and inverse transform):

$$\int d^3 x e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{x}} = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')$$

$$\int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} = \delta(\mathbf{x} - \mathbf{x}')$$

Angular Power Spectrum

- Statistical homogeneity and isotropy

$$\langle \Theta(\mathbf{x})\Theta(\mathbf{x}') \rangle = C(|\mathbf{x} - \mathbf{x}'|)$$

function of separation only

$$\langle \Theta(\mathbf{x} + \mathbf{d})\Theta(\mathbf{x}' + \mathbf{d}) \rangle = \langle \Theta(\mathbf{x})\Theta(\mathbf{x}') \rangle$$

$$\begin{aligned} & \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{x}'} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{d}} \langle \Theta^*(\mathbf{k})\Theta(\mathbf{k}') \rangle \\ &= \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{x}'} \langle \Theta^*(\mathbf{k})\Theta(\mathbf{k}') \rangle \end{aligned}$$

requires the 2pt Fourier correlation to be described by a power spectrum

$$\langle \Theta^*(\mathbf{k})\Theta(\mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') P_T(k)$$

Angular Power Spectrum

- Correlation function and power spectrum are Fourier conjugates

$$C(|\mathbf{x} - \mathbf{x}'|) = \langle \Theta(\mathbf{x}) \Theta(\mathbf{x}') \rangle = \int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x})} P_T(k)$$

- Log weighted power spectrum determines variance

$$\langle \Theta(\mathbf{x}) \Theta(\mathbf{x}) \rangle = \int \frac{d^3 k}{(2\pi)^3} P_T(k) = \int \frac{dk}{k} \frac{k^3}{2\pi^2} P_T = \int \frac{dk}{k} \Delta_T^2(k)$$

$$\Delta_T^2 = \frac{k^3}{2\pi^2} P_T [= \mathcal{P}_T(k)]$$

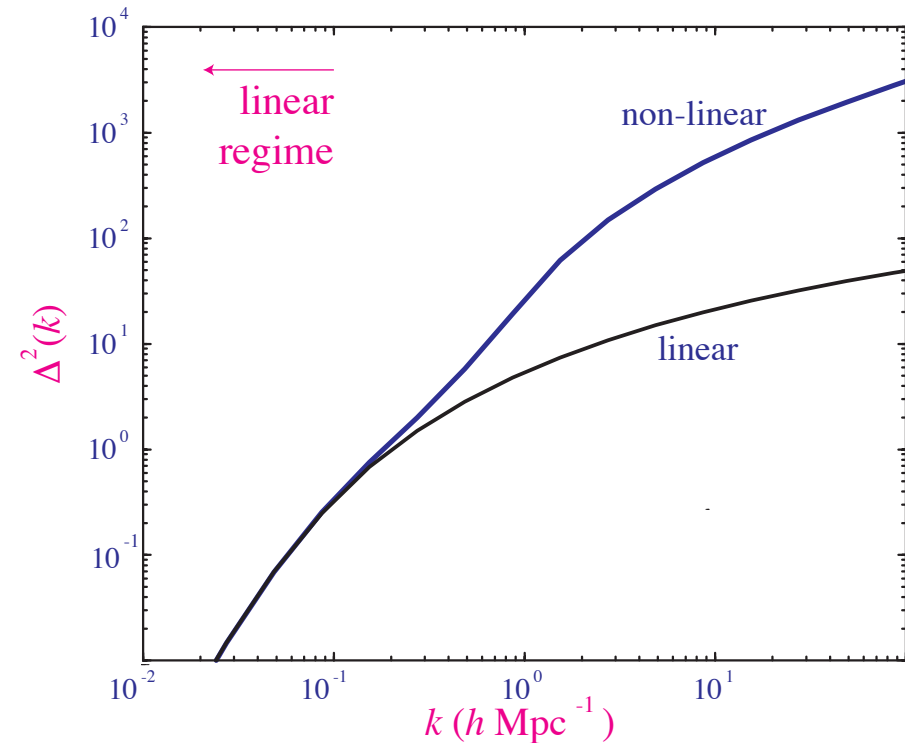
and is the contribution to the total variance per log interval in k

- Δ_T^2 dimensionless, whereas P_T has dimensions of $[L^3]$, e.g. $(h^{-1}\text{Mpc})^3$ for the power spectrum of a redshift survey

Nonlinear regime

- Inflationary initial perturbations provide density perturbations $\delta = \delta\rho/\rho$ that grow as $\delta \propto a$ in the linear regime

- $\Delta^2 = k^3 P(k)/2\pi^2$
contribution to variance
 $\langle\delta^2\rangle$ per $d \ln k$



- Linear theory would predict that for $k > 0.1 h\text{Mpc}^{-1}$, $\langle\delta^2\rangle > 1$.
- Linear approximation breaks down at this point and we must follow the nonlinear equations
- Nonlinearities further enhance the formation of structure

Convolution

- Convolution in real space often occurs, e.g. smoothing of field by telescope beam but also any smoothing $\int d^3x W(\mathbf{x}) = 1$

$$\begin{aligned} F_W(\mathbf{x}) &= \int d^3y W(\mathbf{x} - \mathbf{y}) F(\mathbf{y}) \\ &= \int d^3y \int \frac{d^3k}{(2\pi)^3} W(\mathbf{k}) e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \int \frac{d^3k'}{(2\pi)^3} F(\mathbf{k}') e^{-i\mathbf{k}' \cdot \mathbf{y}} \\ &= \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} W(\mathbf{k}) F(\mathbf{k}') \int d^3y e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{y}} \\ &= \int \frac{d^3k}{(2\pi)^3} W(\mathbf{k}) F(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} \end{aligned}$$

$$F_W(\mathbf{k}) = W(\mathbf{k}) F(\mathbf{k})$$

- Smoothing acts as a low pass filter: if $W(\mathbf{x})$ is a broad function of width L , $W(\mathbf{k})$ suppressed for $k > 2\pi/L$

Convolution

- Filtered Variance

$$\begin{aligned}\langle F_W(\mathbf{x}) F_W(\mathbf{x}) \rangle &= \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{x}} \langle F^*(\mathbf{k}) F(\mathbf{k}') \rangle W^*(\mathbf{k}) W(\mathbf{k}') \\ &= \int \frac{d^3 k}{(2\pi)^3} P_F(k) |W(\mathbf{k})|^2\end{aligned}$$

- Common filter is the spherical tophat:

$$\begin{aligned}W_R(\mathbf{x}) &= V_R^{-1} & x < R \\ W_R(\mathbf{x}) &= 0 & x > R\end{aligned}$$

- Fourier transform

$$W_R(\mathbf{k}) = \frac{3}{y^3} (\sin y - y \cos y), \quad (y = kR)$$

Current Normalization

- Normalization is often quoted as the top hat rms of the density field

$$\sigma_R^2 = \int d \ln k \Delta_\delta^2(k) |W_R(k)|^2$$

where observationally $\sigma_{8h^{-1}\text{Mpc}} \equiv \sigma_8 \approx 1$

- Note that $\Delta_\delta^2(k)$ itself can be thought of as the variance of the field with a filter that has sharp high and low pass filters in k -space
- Convention is that σ_R is defined against the **linear** density field, not the true non-linear density field

Spatial Curvature

- To include spatial curvature: note that the important property is that the k -decomposition is a complete set of modes.
- General prescription: for a curved space is the eigenfunctions of the Laplacian (with covariant derivatives) are a complete set

$$\nabla^2 e^{i\mathbf{k}\cdot\mathbf{x}} = -k^2 e^{i\mathbf{k}\cdot\mathbf{x}} \rightarrow \nabla^2 Q(\hat{\mathbf{n}}, D) = -k^2 Q(\hat{\mathbf{n}}, D)$$

- This is a 3D generalization of the 2D sphere for which we do the same thing next

$$\nabla^2 Y_{\ell m} = -\ell(\ell + 1) Y_{\ell m}$$

- For solving dynamics of k mode and source function, changes due to small curvature currently allowed $|\Omega_K| \lesssim 10^{-2}$ are negligible for all observable modes and only distinction is in mapping of $D, \hat{\mathbf{n}}$ to wavelength $2\pi/k$ through angular diameter distance $D_A(D)$.

Angular Power Spectrum

- Temperature field

$$\Theta(\hat{\mathbf{n}}) = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) e^{i\mathbf{k} \cdot D_* \hat{\mathbf{n}}}$$

Multipole moments $\Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m}$

- Orthogonality:

$$\int d\hat{\mathbf{n}} Y_{\ell m}^*(\hat{\mathbf{n}}) Y_{\ell' m'}(\hat{\mathbf{n}}) = \delta_{\ell\ell'} \delta_{mm'}$$

Completeness:

$$\sum_{\ell m} Y_{\ell m}^*(\hat{\mathbf{n}}) Y_{\ell m}(\hat{\mathbf{n}}') = \delta(\phi - \phi') \delta(\cos \theta - \cos \theta')$$

- Statistical isotropy:

$$\langle \Theta_{\ell m}^* \Theta_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell$$

Angular Power Spectrum

- Expand out plane wave in spherical coordinates

$$e^{i\mathbf{k}D_*\cdot\hat{\mathbf{n}}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kD_*) Y_{\ell m}^*(\hat{\mathbf{k}}) Y_{\ell m}(\hat{\mathbf{n}})$$

- Aside: as in the figure, it will often be convenient when considering a single $\mathbf{k}$ mode to orient the north pole to $\hat{\mathbf{k}}$. This simplifies the decomposition since

$$Y_{\ell m}^*(\hat{\mathbf{k}}) \rightarrow Y_{\ell m}^*(0) = \delta_{m0} \sqrt{\frac{2\ell+1}{4\pi}}$$

Angular Power Spectrum

- Power spectrum

$$\Theta_{\ell m} = \int \frac{d^3 k}{(2\pi)^3} \Theta(\mathbf{k}) 4\pi i^\ell j_\ell(k D_*) Y_{\ell m}^*(\mathbf{k})$$

$$\begin{aligned} \langle \Theta_{\ell m}^* \Theta_{\ell' m'} \rangle &= \int \frac{d^3 k}{(2\pi)^3} (4\pi)^2 i^{\ell-\ell'} j_\ell(k D_*) j_{\ell'}(k D_*) Y_{\ell m}(\mathbf{k}) Y_{\ell' m'}^*(\mathbf{k}) P_T(k) \\ &= \delta_{\ell\ell'} \delta_{mm'} 4\pi \int d \ln k j_\ell^2(k D_*) \Delta_T^2(k) \end{aligned}$$

with $\int_0^\infty j_\ell^2(x) d \ln x = 1/(2\ell(\ell+1))$, slowly varying Δ_T^2

- Angular power spectrum:

$$C_\ell = \frac{4\pi \Delta_T^2(\ell/D_*)}{2\ell(\ell+1)} = \frac{2\pi}{\ell(\ell+1)} \Delta_T^2(\ell/D_*)$$

Angular Power Spectrum

- The log power spectrum (sometimes called $\mathcal{D}_\ell$)

$$\frac{\ell(\ell + 1)}{2\pi} C_\ell \approx \Delta_T^2$$

so that a scale invariant spectrum $\Delta_T^2 = \text{const}$ is scale invariant in the log power spectrum

- Related to the contribution to the variance per log interval in ℓ

$$\langle \Theta(\hat{\mathbf{n}}) \Theta(\hat{\mathbf{n}}) \rangle = \langle \Theta(0) \Theta(0) \rangle = \sum_{\ell} \frac{2\ell + 1}{4\pi} C_\ell = \sum_{\ell} \frac{1}{\ell} \frac{\ell(2\ell + 1)}{4\pi} C_\ell$$

with the two being equivalent if $\ell \gg 1$

Thomson Scattering

- Thomson scattering of photons off of free electrons is the most important CMB process with a cross section (averaged over polarization states) of

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25} \text{cm}^2$$

- Density of free electrons in a fully ionized $x_e = 1$ universe

$$n_e = (1 - Y_p/2)x_e n_b \approx 10^{-5} \Omega_b h^2 (1 + z)^3 \text{cm}^{-3},$$

where $Y_p \approx 0.24$ is the Helium mass fraction, creates a high (comoving) Thomson opacity

$$\dot{\tau} \equiv n_e \sigma_T a$$

where dots are conformal time $\eta \equiv \int dt/a$ derivatives and τ is the optical depth.

Tight Coupling Approximation

- Near **recombination** $z \approx 10^3$ and $\Omega_b h^2 \approx 0.02$, the (comoving) **mean free path** of a photon

$$\lambda_C \equiv \frac{1}{\dot{\tau}} \sim 2.5 \text{Mpc}$$

small by cosmological standards!

- On scales $\lambda \gg \lambda_C$ photons are **tightly coupled** to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions
- Specifically, their bulk velocities are defined by a **single fluid velocity** $v_\gamma = v_b$ and the photons carry **no anisotropy** in the rest frame of the baryons
- $\rightarrow$ No **heat conduction** or **viscosity** (anisotropic stress) in fluid

Full Equations of Motion

- Continuity

$$\dot{\Theta} = -\frac{k}{3}v_{\gamma} - \dot{\Phi}, \quad \dot{\delta}_b = -kv_b - 3\dot{\Phi}$$

which expresses number conservation in the presence of velocity divergence and local expansion, with $\rho_b = m_b n_b$

- Navier-Stokes (Euler + heat conduction, viscosity)

$$\begin{aligned}\dot{v}_{\gamma} &= k(\Theta + \Psi) - \frac{k}{6}\pi_{\gamma} - \dot{\tau}(v_{\gamma} - v_b) \\ \dot{v}_b &= -\frac{\dot{a}}{a}v_b + k\Psi + \dot{\tau}(v_{\gamma} - v_b)/R\end{aligned}$$

where the photon momentum changes due to pressure, gravity and anisotropic stress π_{γ} gradients (from **radiation viscosity**) and a **momentum exchange** term with the baryons and are compensated by the **opposite term** in the baryon Euler equation

Zeroth Order Approximation

- Momentum density of a fluid is $(\rho + p)v$, where p is the pressure
- Neglect the momentum density of the baryons

$$R \equiv \frac{(\rho_b + p_b)v_b}{(\rho_\gamma + p_\gamma)v_\gamma} = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}$$
$$\approx 0.6 \left(\frac{\Omega_b h^2}{0.02} \right) \left(\frac{a}{10^{-3}} \right)$$

since $\rho_\gamma \propto T^4$ is fixed by the CMB temperature $T = 2.73(1 + z)\text{K}$
– OK substantially before recombination

- Neglect radiation in the expansion (not a good approx, just for pedagogical start)

$$\frac{\rho_m}{\rho_r} = 3.6 \left(\frac{\Omega_m h^2}{0.15} \right) \left(\frac{a}{10^{-3}} \right)$$

- Neglect gravity (obviously just for pedagogy)

Fluid Equations

- Density $\rho_\gamma \propto T^4$ so define temperature fluctuation Θ

$$\delta_\gamma = 4 \frac{\delta T}{T} \equiv 4\Theta$$

- Fourier space continuity equation

$$\dot{\delta}_\gamma = -(1 + w_\gamma) k v_\gamma$$

$$\dot{\Theta} = -\frac{1}{3} k v_\gamma$$

- Euler equation (neglecting gravity)

$$\dot{v}_\gamma = -(1 - 3w_\gamma) \frac{\dot{a}}{a} v_\gamma + \frac{k c_s^2}{1 + w_\gamma} \delta_\gamma$$

$$\dot{v}_\gamma = k c_s^2 \frac{3}{4} \delta_\gamma = 3 c_s^2 k \Theta$$

Oscillator: Take One

- Combine these to form the **simple harmonic oscillator** equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = 0$$

where the sound speed is adiabatic

$$c_s^2 = \frac{\delta p_\gamma}{\delta \rho_\gamma} = \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma}$$

here $c_s^2 = 1/3$ since we are photon-dominated

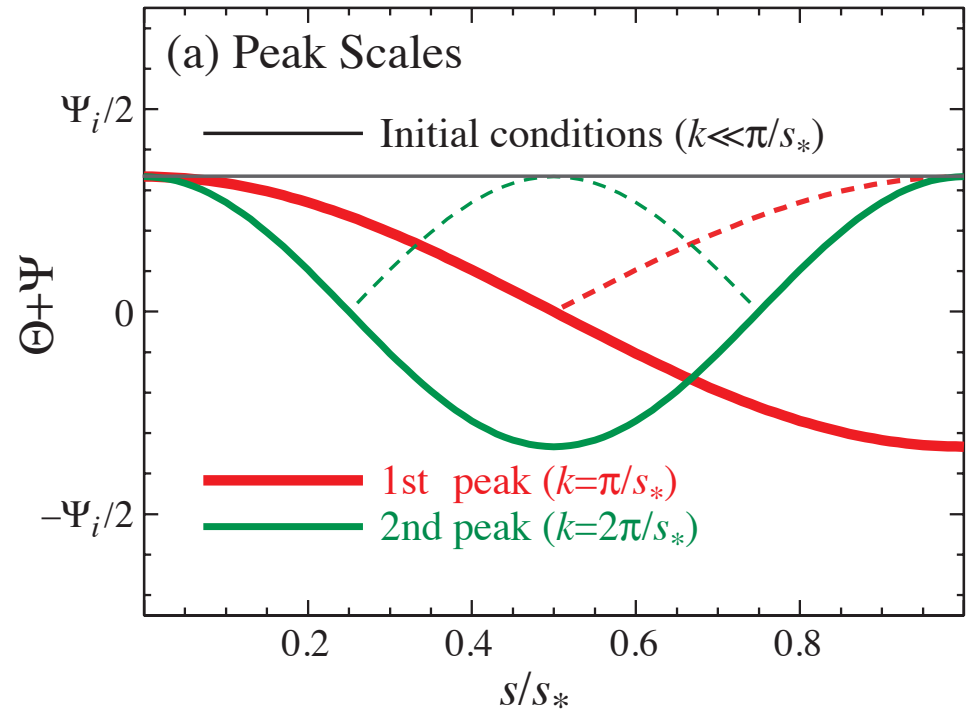
- General solution:

$$\Theta(\eta) = \Theta(0) \cos(k s) + \frac{\dot{\Theta}(0)}{k c_s} \sin(k s)$$

where the **sound horizon** is defined as $s \equiv \int c_s d\eta$

Harmonic Extrema

- All modes are **frozen** in at recombination (denoted with a subscript $*$)
- Temperature perturbations of **different amplitude** for different modes.
- For the adiabatic (curvature mode) initial conditions



$$\dot{\Theta}(0) = 0$$

- So solution

$$\Theta(\eta_*) = \Theta(0) \cos(k s_*)$$

Harmonic Extrema

- Modes caught in the extrema of their oscillation will have enhanced fluctuations

$$k_n s_* = n\pi$$

yielding a fundamental scale or frequency, related to the inverse sound horizon

$$k_A = \pi / s_*$$

and a harmonic relationship to the other extrema as 1 : 2 : 3...

Temperature Anisotropy

- Spatial oscillations frozen at recombination; photons then stream
- Viewed at distance D_* as angular anisotropy $L \approx kD_*$



Peak Location

- The fundamental **physical scale** is translated into a fundamental **angular scale** by simple projection according to the angular diameter distance D_A

$$\theta_A = \lambda_A / D_A$$

$$\ell_A = k_A D_A$$

- In a flat universe, the distance is simply $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$, the horizon distance, and $k_A = \pi / s_* = \sqrt{3}\pi / \eta_*$ so

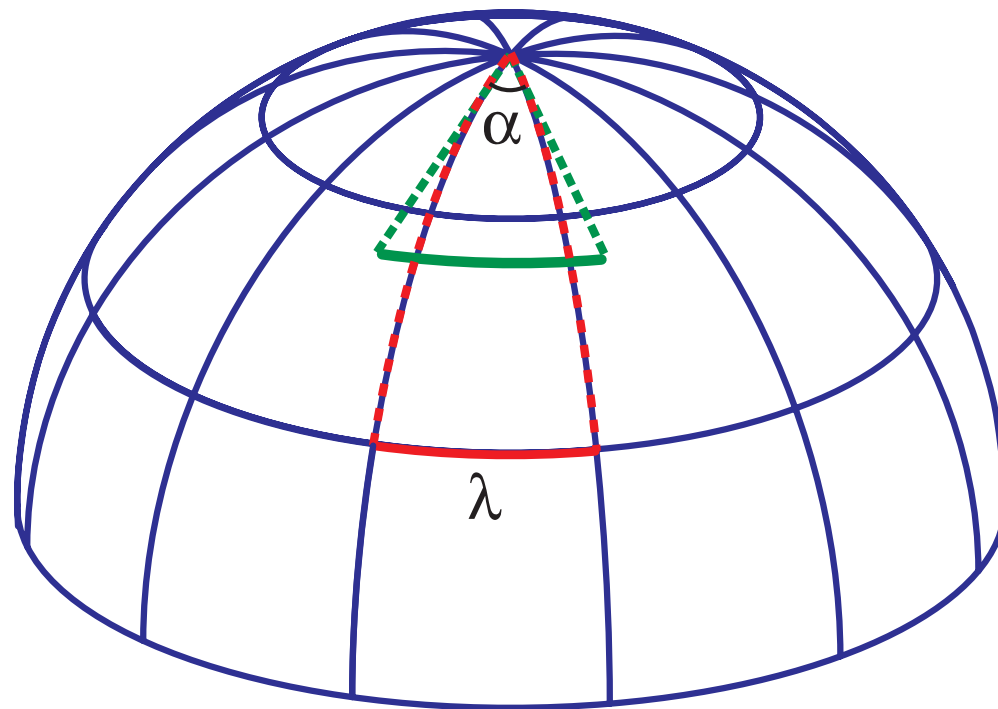
$$\theta_A \approx \frac{\eta_*}{\eta_0}$$

- In a **matter-dominated** universe $\eta \propto a^{1/2}$ so $\theta_A \approx 1/30 \approx 2^\circ$ or

$$\ell_A \approx 200$$

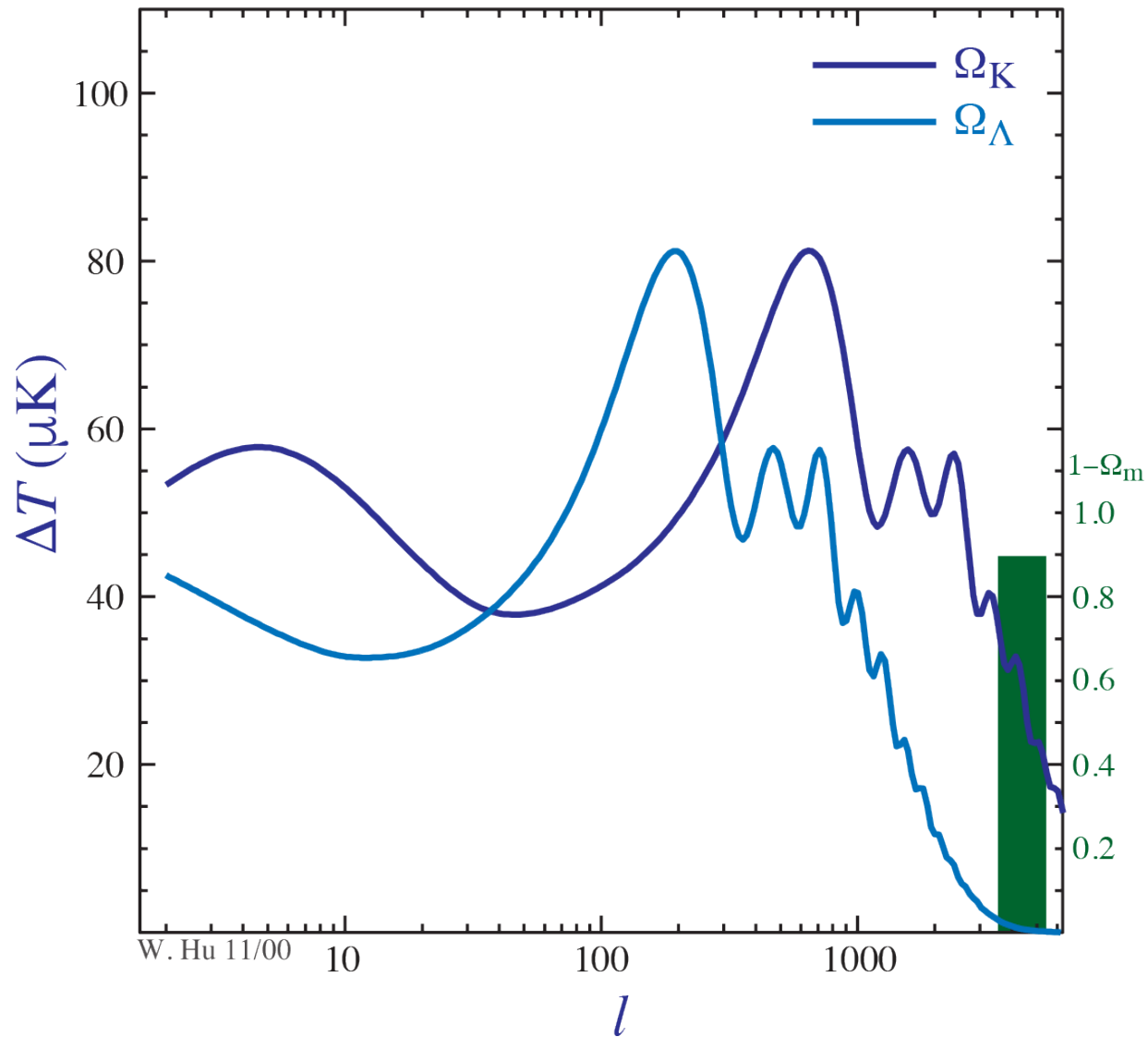
Curvature

- In a curved universe, the apparent or angular diameter distance is no longer the conformal distance
$$D_A = R \sin(D/R) \neq D$$
- Objects in a closed universe are further than they appear! gravitational lensing of the background...
- Curvature scale of the universe must be substantially larger than current horizon



Curvature in the CMB

- Curvature and Λ – consistent with flat Λ CDM



Fixed Deceleration Epoch

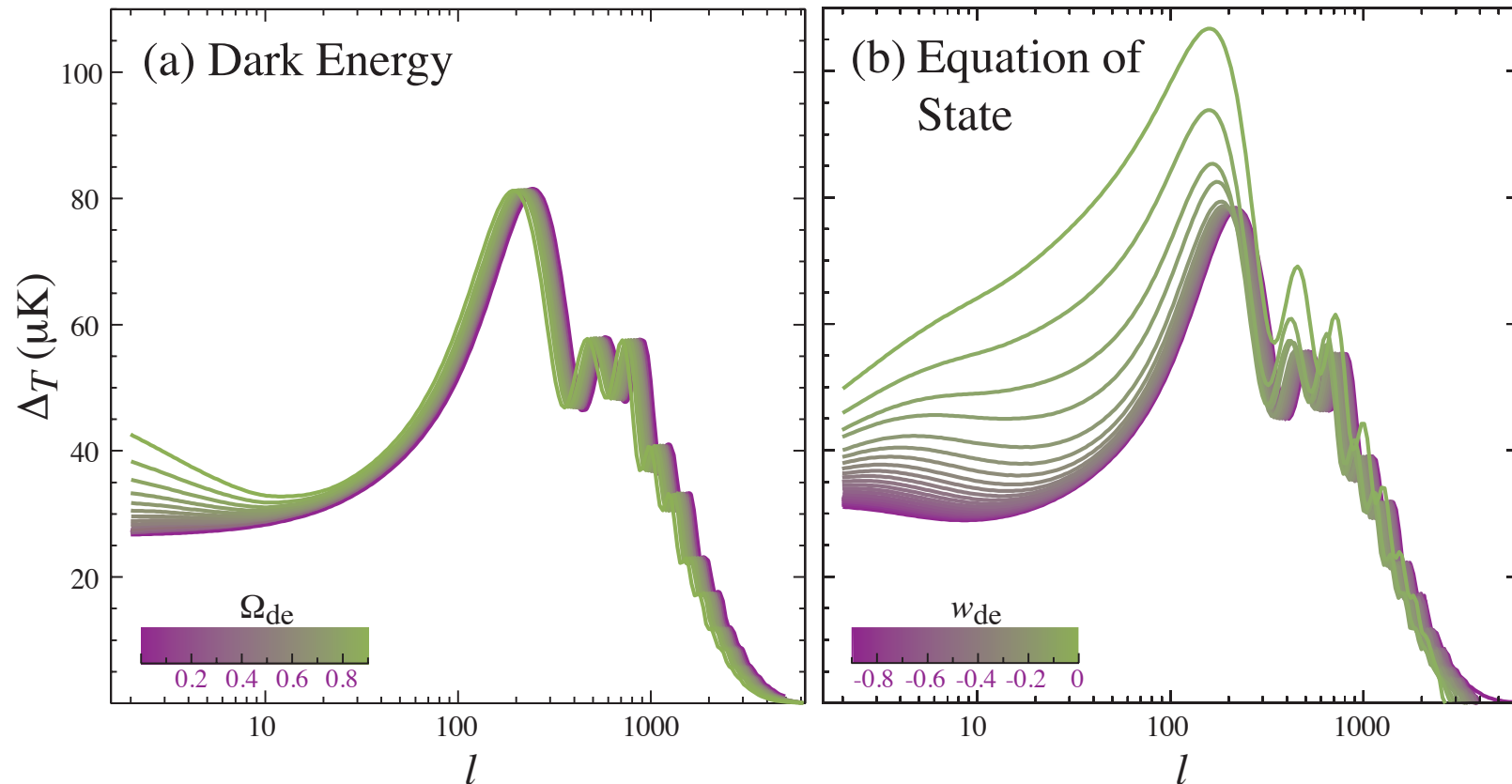
- CMB determination of **matter density** controls all determinations in the **deceleration** (matter dominated) epoch
- **Planck**: $\Omega_m h^2 = 0.1426 \pm 0.0025 \rightarrow 1.7\%$
- **Distance** to recombination D_* determined to $\frac{1}{4}1.7\% \approx 0.43\%$ (Λ CDM result 0.46%; $\Delta h/h \approx -\Delta\Omega_m h^2/\Omega_m h^2$)
[more general: $-0.11\Delta w - 0.48\Delta \ln h - 0.15\Delta \ln \Omega_m - 1.4\Delta \ln \Omega_{\text{tot}} = 0$]
- **Expansion rate** during any redshift in the deceleration epoch determined to $\frac{1}{2}1.7\%$
- **Distance** to **any redshift** in the deceleration epoch determined as

$$D(z) = D_* - \int_z^{z_*} \frac{dz}{H(z)}$$

- **Volumes** determined by a combination $dV = D_A^2 d\Omega dz / H(z)$
- **Structure** also determined by growth of fluctuations from z_*

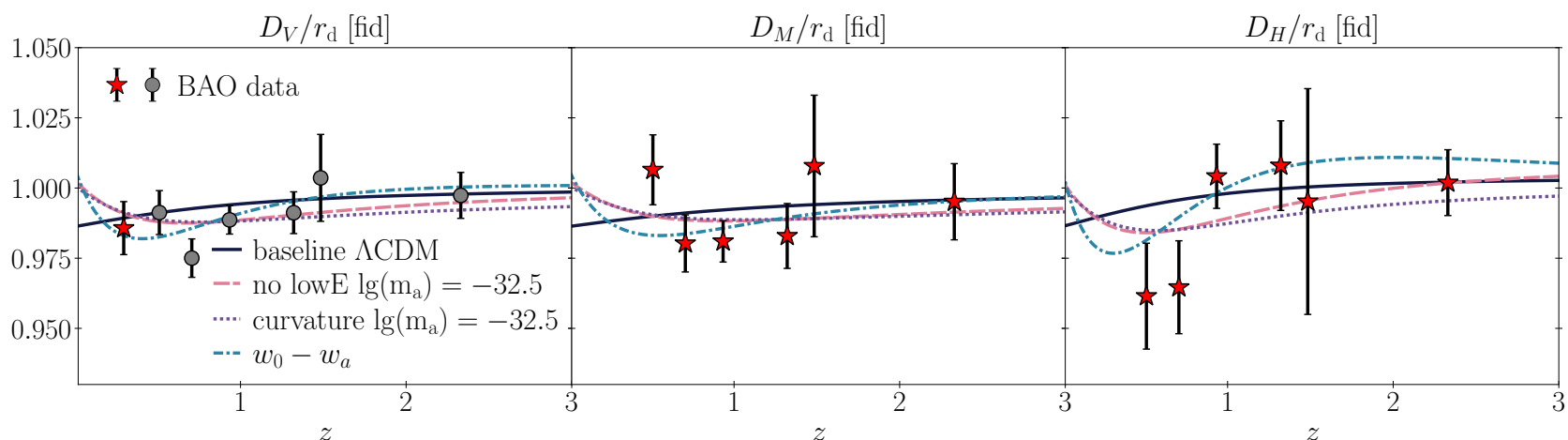
Dark Energy

- Flat universe indicates critical density and implies missing energy given local measures of the matter density “dark energy”
- $D_A = D = \int dz/H(z)$ also depends on dark energy density Ω_{de} and equation of state $w = p_{\text{de}}/\rho_{\text{de}}$.



Curvature and Dark Energy

- Curvature can adjust the relative distance between BAO at $z < 1$ and CMB at $z = 1100$
- $\Omega_K = 0.003$ could alleviate the current percent level discrepancy



- Still requires dark energy to explain SN at $z < 0.1$, here axions, but no phantom $w < -1$
- Expansion rate at recombination or **matter-radiation ratio** enters into calculation of k_A - other possibilities

Doppler Effect

- Bulk motion of fluid changes the observed temperature via Doppler shifts

$$\left(\frac{\Delta T}{T}\right)_{\text{dop}} = \hat{\mathbf{n}} \cdot \mathbf{v}_\gamma$$

- Averaged over directions

$$\left(\frac{\Delta T}{T}\right)_{\text{rms}} = \frac{v_\gamma}{\sqrt{3}}$$

- Acoustic solution

$$\begin{aligned} \frac{v_\gamma}{\sqrt{3}} &= -\frac{\sqrt{3}}{k} \dot{\Theta} = \frac{\sqrt{3}}{k} k c_s \Theta(0) \sin(ks) \\ &= \Theta(0) \sin(ks) \end{aligned}$$

Doppler Peaks?

- Doppler effect for the photon dominated system is of equal amplitude and $\pi/2$ out of phase: extrema of temperature are turning points of velocity
- Effects add in quadrature:

$$\left(\frac{\Delta T}{T}\right)^2 = \Theta^2(0)[\cos^2(ks) + \sin^2(ks)] = \Theta^2(0)$$

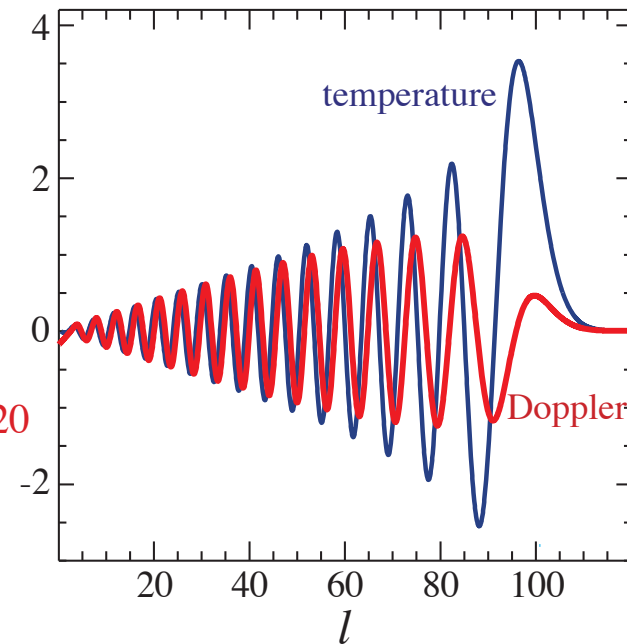
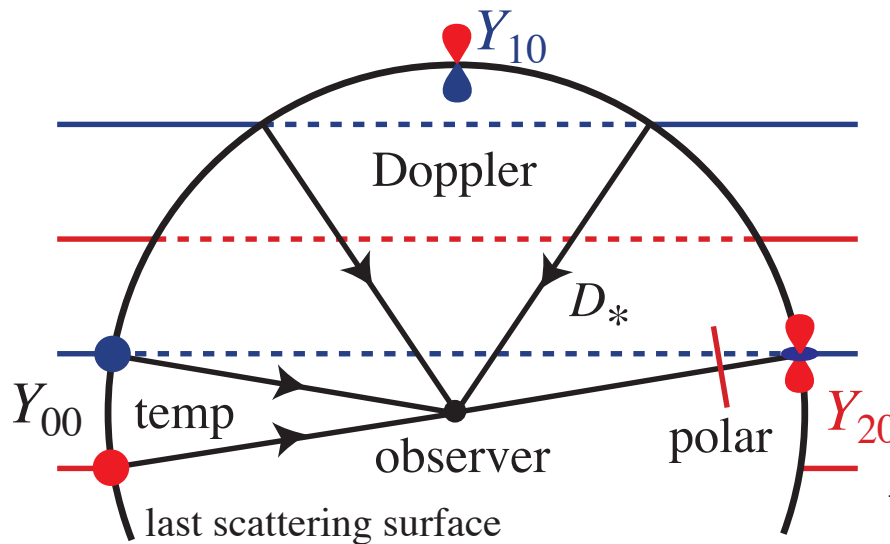
- No peaks in k spectrum! However the Doppler effect carries an angular dependence that changes its projection on the sky
 $\hat{\mathbf{n}} \cdot \mathbf{v}_\gamma \propto \hat{\mathbf{n}} \cdot \hat{\mathbf{k}}$

Doppler Peaks?

- Coordinates where $\hat{\mathbf{z}} \parallel \hat{\mathbf{k}}$

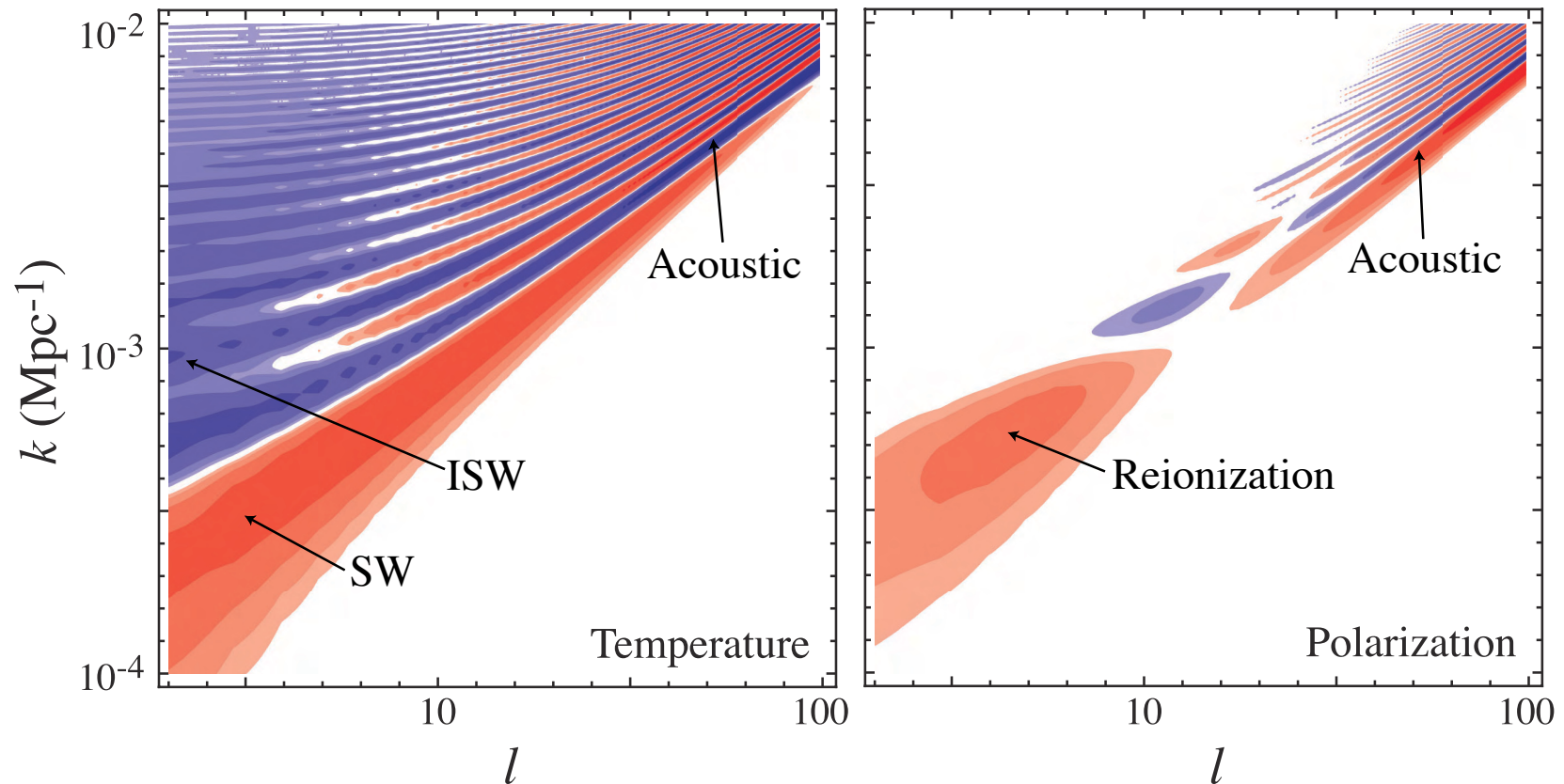
$$Y_{10}Y_{\ell 0} \rightarrow Y_{\ell \pm 1 0}$$

recoupling $j'_\ell Y_{\ell 0}$: no peaks in Doppler effect



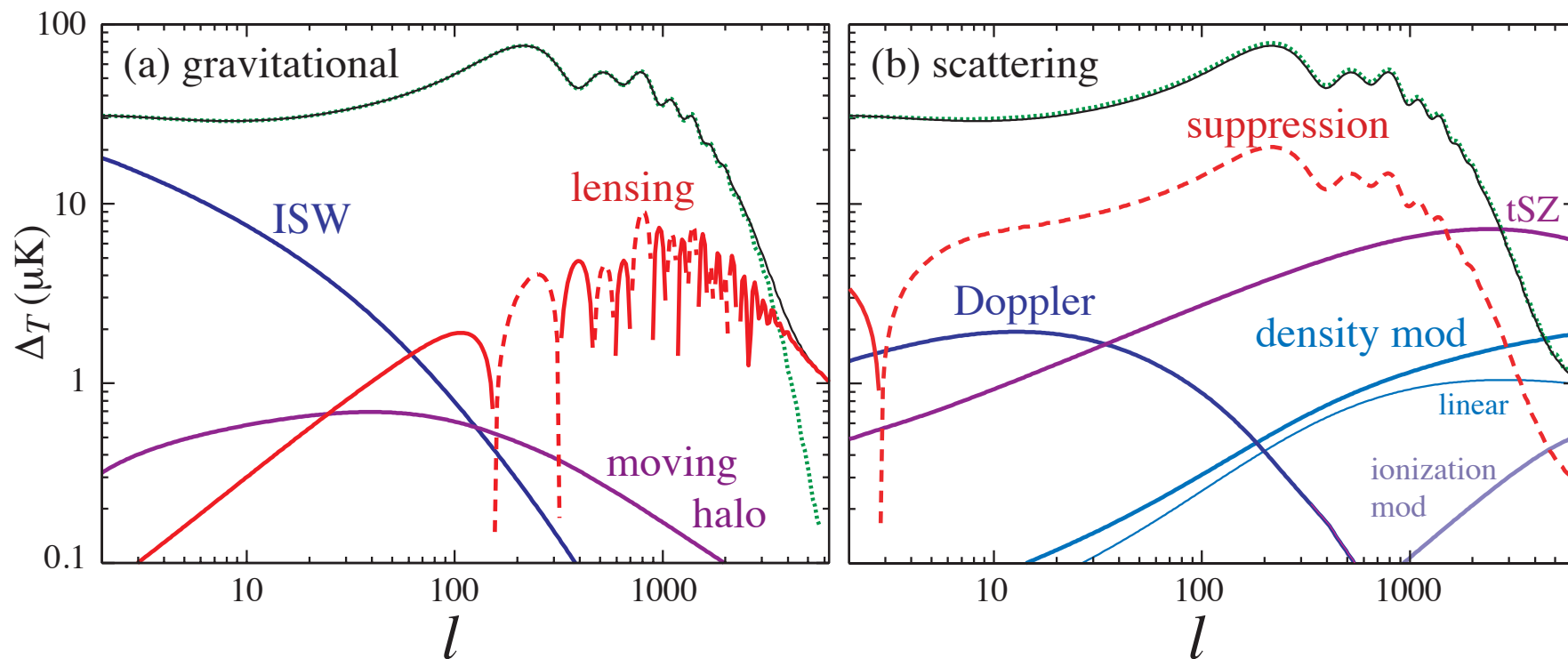
Radiation Transfer Function

- Geometry of projection dictates how power in inhomogeneity (k) transfers to power in anisotropy (ℓ)



Secondary Anisotropy

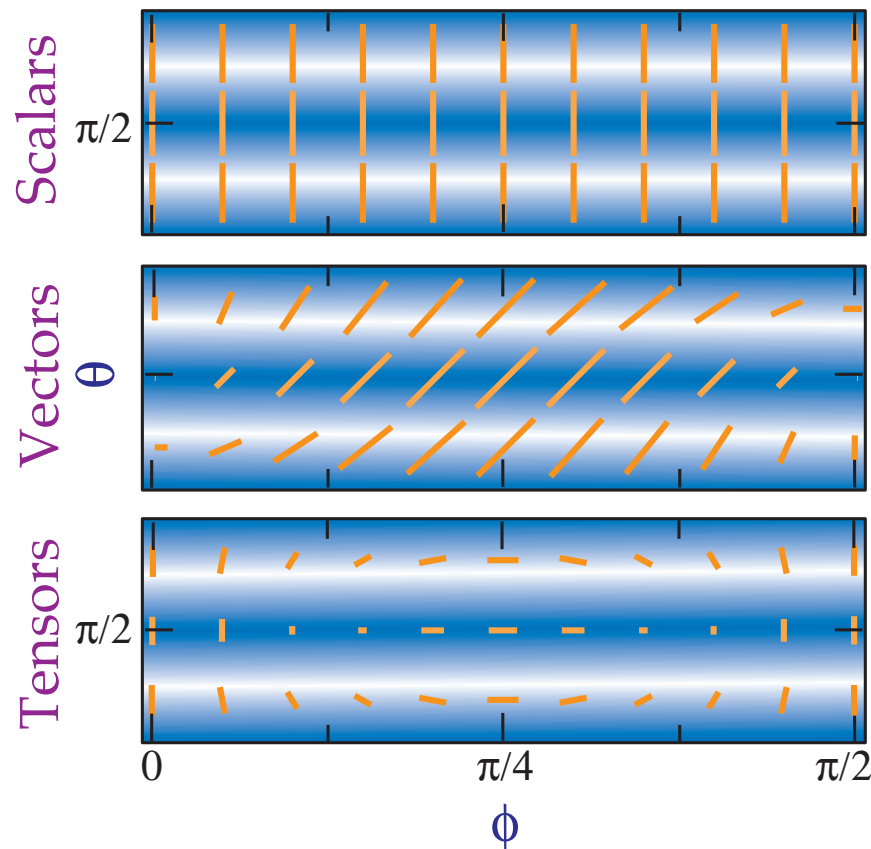
- Even though $\tau \sim 0.06$ and $v \sim 10^{-3}$ and dark energy dominates, scattering and gravitational anisotropy from $z < z_*$ is small
- Can understand most of this in terms of the geometric projection of radiation sources in the “Limber approximation”



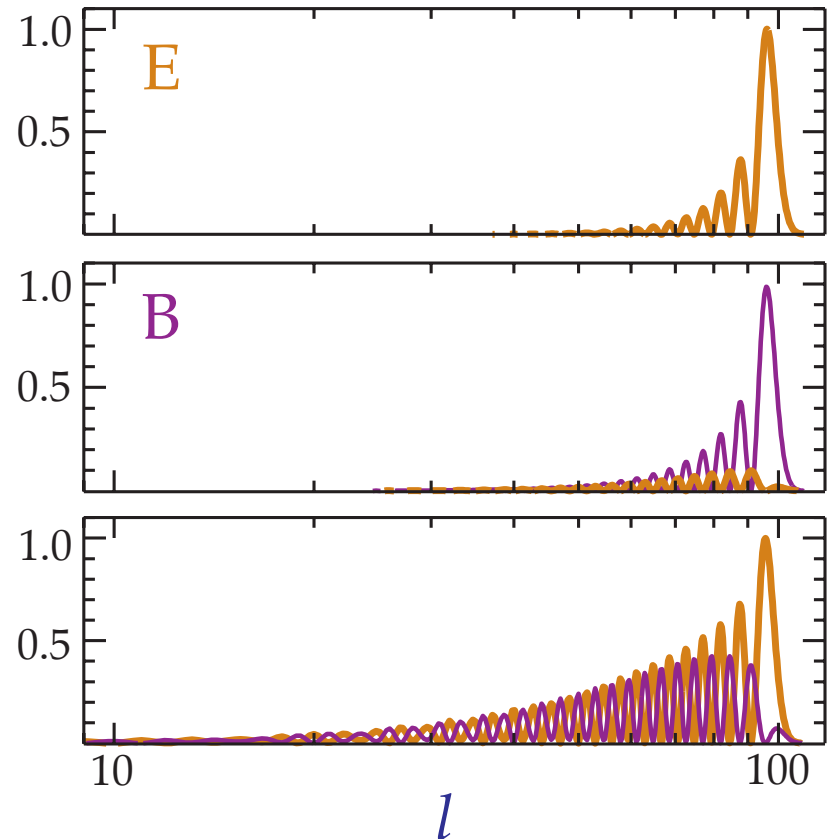
Polarization Transfer (Preview?)

- A polarization source function with $\ell = 2$, modulated with plane wave orbital angular momentum
- Scalars have no B mode contribution, vectors mostly B and tensor comparable B and E

(a) Polarization Pattern



(b) Multipole Power



Restoring Gravity

- Take a simple photon dominated system with gravity
- Continuity altered since a gravitational potential represents a stretching of the spatial fabric that dilutes number densities – formally a spatial curvature perturbation
- Think of this as a perturbation to the scale factor $a \rightarrow a(1 + \Phi)$ so that the cosmological redshift is generalized to

$$\frac{\dot{a}}{a} \rightarrow \frac{\dot{a}}{a} + \dot{\Phi}$$

so that the continuity equation becomes

$$\dot{\Theta} = -\frac{1}{3}k v_{\gamma} - \dot{\Phi}$$

Restoring Gravity

- Gravitational force in momentum conservation $\mathbf{F} = -m\nabla\Psi$ generalized to momentum density modifies the Euler equation to

$$\dot{v}_\gamma = k(\Theta + \Psi)$$

- General relativity says that Φ and Ψ are the relativistic analogues of the Newtonian potential and that $\Phi \approx -\Psi$.
- In our matter-dominated approximation, Φ represents matter density fluctuations through the cosmological Poisson equation

$$k^2\Phi = 4\pi G a^2 \rho_m \Delta_m$$

where the difference comes from the use of comoving coordinates for k (a^2 factor), the removal of the background density into the background expansion ($\rho\Delta_m$) and finally a coordinate subtlety that enters into the definition of Δ_m

Constant Potentials

- In the matter dominated epoch potentials are constant because infall generates velocities as $v_m \sim k\eta\Psi$
- Velocity divergence generates density perturbations as $\Delta_m \sim -k\eta v_m \sim -(k\eta)^2\Psi$
- And density perturbations generate potential fluctuations

$$\Phi = \frac{4\pi G a^2 \rho \Delta}{k^2} \approx \frac{3}{2} \frac{H^2 a^2}{k^2} \Delta \sim \frac{\Delta}{(k\eta)^2} \sim -\Psi$$

keeping them constant. Note that because of the expansion, density perturbations must grow to keep potentials constant.

Constant Curvature

- More generally, if **stress perturbations** are negligible compared with **density perturbations** ($\delta p \ll \delta \rho$) then potential will remain roughly constant
- A variant called the **Bardeen** or **comoving curvature** is strictly constant

$$\mathcal{R} = \text{const} \approx \frac{5 + 3w}{3 + 3w} \Phi$$

where the approximation holds when $w \approx \text{const}$.

- This quantity is the direct prediction from inflation: the curvature or local scale factor fluctuation on surfaces of spatially constant inflaton field
- Geometry: a curvature fluctuation provides local change in space curvature δK and to the local observer looks just like separate FRW universe with $K = \bar{K} + \delta K$, which is constant in time

Oscillator: Take Two

- Combine these to form the **simple harmonic oscillator** equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - \ddot{\Phi}$$

- In a **CDM dominated** expansion $\dot{\Phi} = \dot{\Psi} = 0$. Also for **photon domination** $c_s^2 = 1/3$ so the oscillator equation becomes

$$\ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0$$

- Solution is just an **offset version** of the original

$$[\Theta + \Psi](\eta) = [\Theta + \Psi](0) \cos(ks)$$

- $\Theta + \Psi$ is also the **observed temperature fluctuation** since photons lose energy climbing out of **gravitational potentials** at recombination

Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or effective temperature

$$\Theta + \Psi$$

- Effective temperature oscillates around zero with amplitude given by the initial conditions
- Note: initial conditions are set when the perturbation is outside of horizon, need inflation or other modification to matter-radiation FRW universe.
- GR says that initial temperature is given by initial potential

Sachs-Wolfe Effect and the Magic 1/3

- A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]

$$\frac{\delta t}{t} = \Psi$$

- Convert this to a perturbation in the scale factor,

$$t = \int \frac{da}{aH} \propto \int \frac{da}{a\rho^{1/2}} \propto a^{3(1+w)/2}$$

where $w \equiv p/\rho$ so that during matter domination

$$\frac{\delta a}{a} = \frac{2}{3} \frac{\delta t}{t}$$

- CMB temperature is cooling as $T \propto a^{-1}$ so

$$\Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3}\Psi$$

Sachs-Wolfe Normalization

- Use measurements of $\Delta T/T \approx 10^{-5}$ in the Sachs-Wolfe effect to infer $\Delta_{\mathcal{R}}^2$
- Recall in matter domination $\Psi = -3\mathcal{R}/5$

$$\frac{\ell(\ell+1)C_\ell}{2\pi} \approx \Delta_T^2 \approx \frac{1}{25} \Delta_R^2$$

- Thus, amplitude of initial curvature fluctuations is $\Delta_R \approx 5 \times 10^{-5}$
- Modern usage: acoustic peak measurements plus known radiation transfer function is used to convert $\Delta T/T$ to Δ_R . Best measured at $k = 0.08 \text{ Mpc}^{-1}$ by Planck
- Current convention set in the WMAP era

$$\Delta_R^2(k) \equiv A_s \left(\frac{k}{0.05 \text{ Mpc}^{-1}} \right)^{n_s - 1}$$

$$A_s \sim 2.5 \times 10^{-9} \text{ (slightly smaller since } n_s - 1 \approx -0.03 \sim -0.04)$$

Baryon Loading

- Baryons add extra **mass** to the photon-baryon fluid
- Controlling parameter is the **momentum density ratio**:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}} \right)$$

of order **unity** at recombination

- Momentum density of the **joint system** is conserved

$$\begin{aligned} (\rho_\gamma + p_\gamma)v_\gamma + (\rho_b + p_b)v_b &\approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma)v_\gamma \\ &= (1 + R)(\rho_\gamma + p_\gamma)v_{\gamma b} \end{aligned}$$

New Euler Equation

- Momentum density ratio enters as

$$[(1 + \textcolor{red}{R})v_{\gamma b}]^{\cdot} = k\Theta + (1 + \textcolor{red}{R})k\Psi$$

- Photon continuity remains the same

$$\dot{\Theta} = -\frac{k}{3}v_{\gamma b} - \dot{\Phi}$$

- Modification of oscillator equation

$$[(1 + \textcolor{red}{R})\dot{\Theta}]^{\cdot} + \frac{1}{3}k^2\Theta = -\frac{1}{3}k^2(1 + \textcolor{red}{R})\Psi - [(1 + \textcolor{red}{R})\dot{\Phi}]^{\cdot}$$

Oscillator: Take Three

- Combine these to form the not-quite-so simple harmonic oscillator equation

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})$$

where $c_s^2 \equiv \dot{p}_{\gamma b} / \dot{\rho}_{\gamma b}$

$$c_s^2 = \frac{1}{3} \frac{1}{1 + R}$$

- In a CDM dominated expansion $\dot{\Phi} = \dot{\Psi} = 0$ and the adiabatic approximation $\dot{R}/R \ll \omega = kc_s$

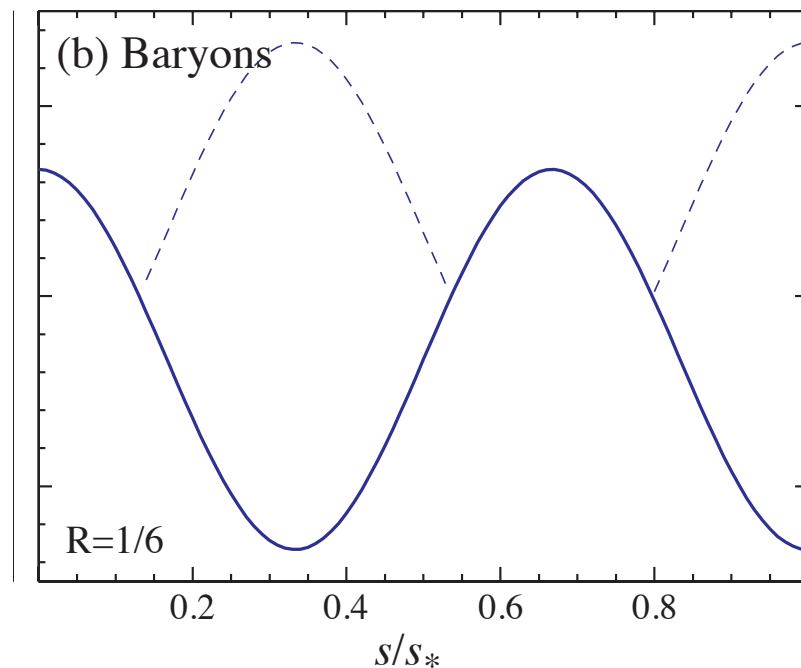
$$[\Theta + (1 + R)\Psi](\eta) = [\Theta + (1 + R)\Psi](0) \cos(k s)$$

Baryon Peak Phenomenology

- Photon-baryon ratio enters in **three** ways
- Overall larger **amplitude**:

$$[\Theta + (1 + R)\Psi](0) = \frac{1}{3}(1 + 3R)\Psi(0)$$

- Even-odd peak **modulation** of effective temperature



$$[\Theta + \Psi]_{\text{peaks}} = [\pm(1 + 3R) - 3R] \frac{1}{3} \Psi(0)$$

$$[\Theta + \Psi]_1 - [\Theta + \Psi]_2 = [-6R] \frac{1}{3} \Psi(0)$$

- Shifting of the **sound horizon** down or ℓ_A up

$$\ell_A \propto \sqrt{1 + R}$$

Photon Baryon Ratio Evolution

- Actual effects **smaller** since R evolves
- Oscillator equation has time **evolving mass**

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = 0$$

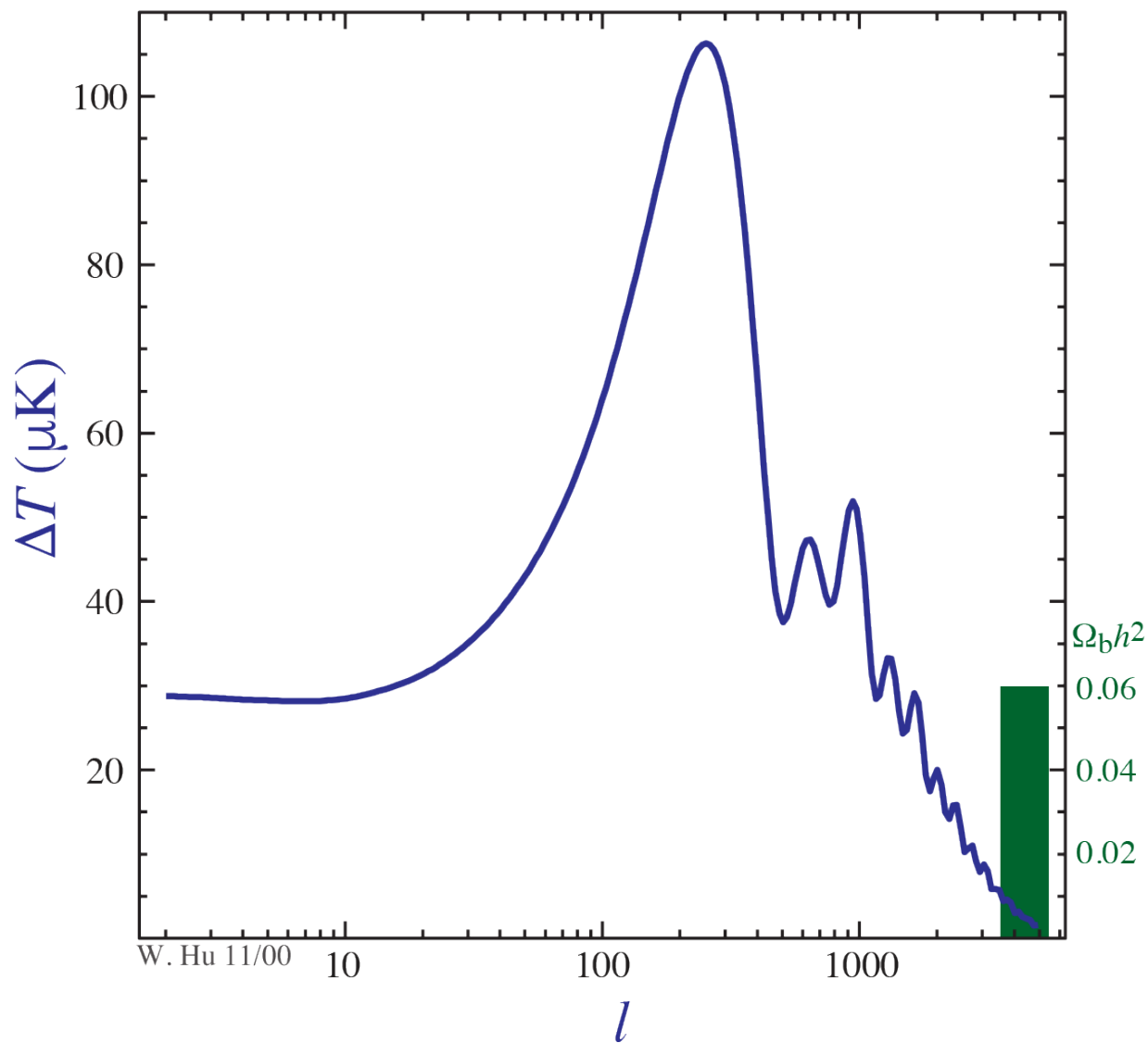
- Effective mass is $m_{\text{eff}} = 3c_s^{-2} = (1 + R)$
- **Adiabatic invariant**

$$\frac{E}{\omega} = \frac{1}{2} m_{\text{eff}} \omega A^2 = \frac{1}{2} 3c_s^{-2} k c_s A^2 \propto A^2 (1 + R)^{1/2} = \text{const.}$$

- Amplitude of oscillation $A \propto (1 + R)^{-1/4}$ **decays adiabatically** as the photon-baryon ratio changes

Baryons in the CMB

- Modulation, amplitude, sound horizon scale



Oscillator: Take Three and a Half

- The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \Phi)$$

changes in the gravitational potentials alter the form of the acoustic oscillations

- If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator
- Term involving Ψ is the ordinary gravitational force
- Term involving Φ involves the $\dot{\Phi}$ term in the continuity equation as a (curvature) perturbation to the scale factor

Potential Decay

- Matter-to-radiation ratio

$$\frac{\rho_m}{\rho_r} \approx 24 \Omega_m h^2 \left(\frac{a}{10^{-3}} \right)$$

of order **unity** at recombination in a low Ω_m universe

- Radiation is not stress free and so **impedes** the growth of structure

$$k^2 \Phi = 4\pi G a^2 \rho_r \Delta_r$$

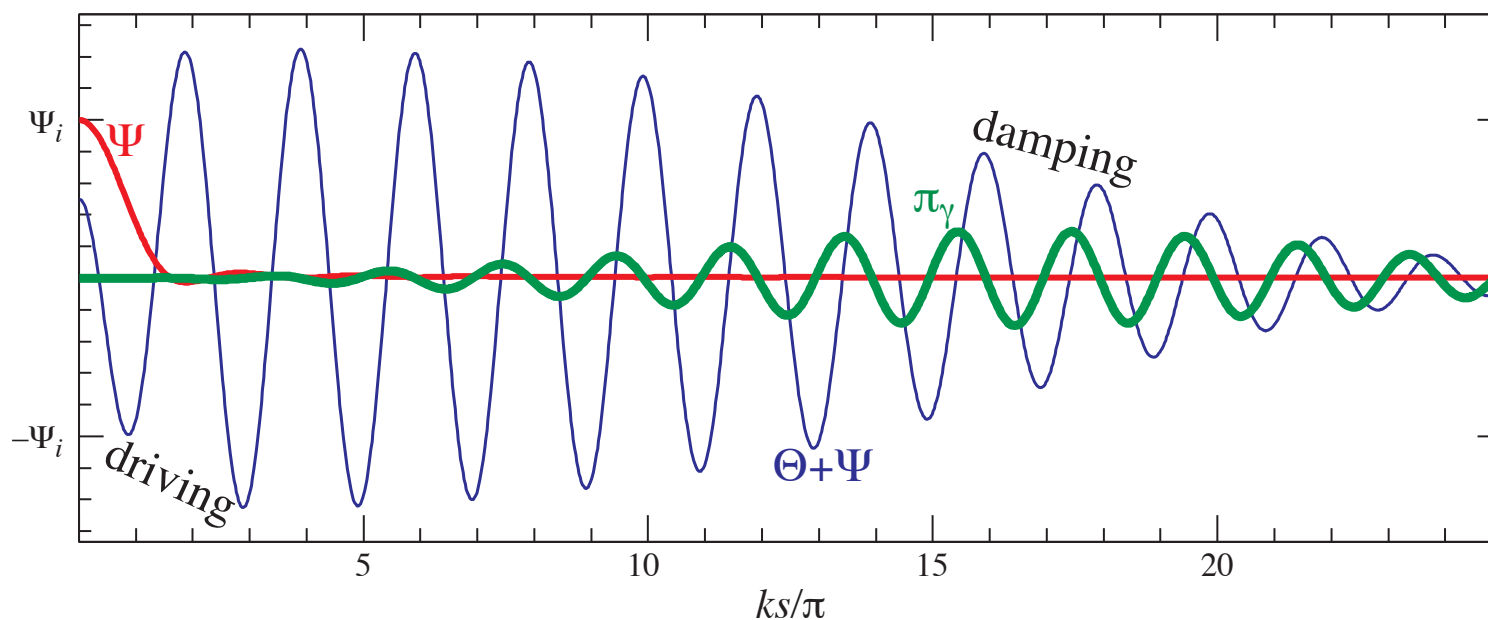
$\Delta_r \sim 4\Theta$ **oscillates** around a constant value, $\rho_r \propto a^{-4}$ so the Newtonian **curvature decays**.

- General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale

Radiation Driving

- Decay is timed precisely to **drive** the oscillator - close to fully coherent

$$\begin{aligned}
 |[\Theta + \Psi](\eta)| &= |[\Theta + \Psi](0) + \Delta\Psi - \Delta\Phi| \\
 &= \left| \frac{1}{3}\Psi(0) - 2\Psi(0) \right| = \left| \frac{5}{3}\Psi(0) \right|
 \end{aligned}$$



- $5\times$ the amplitude of the Sachs-Wolfe effect!

External Potential Approach

- Solution to homogeneous equation

$$(1 + R)^{-1/4} \cos(ks), \quad (1 + R)^{-1/4} \sin(ks)$$

- Give the general solution for an external potential by propagating impulsive forces

$$(1 + R)^{1/4} \Theta(\eta) = \Theta(0) \cos(ks) + \frac{\sqrt{3}}{k} \left[\dot{\Theta}(0) + \frac{1}{4} \dot{R}(0) \Theta(0) \right] \sin ks \\ + \frac{\sqrt{3}}{k} \int_0^\eta d\eta' (1 + R')^{3/4} \sin[ks - ks'] F(\eta')$$

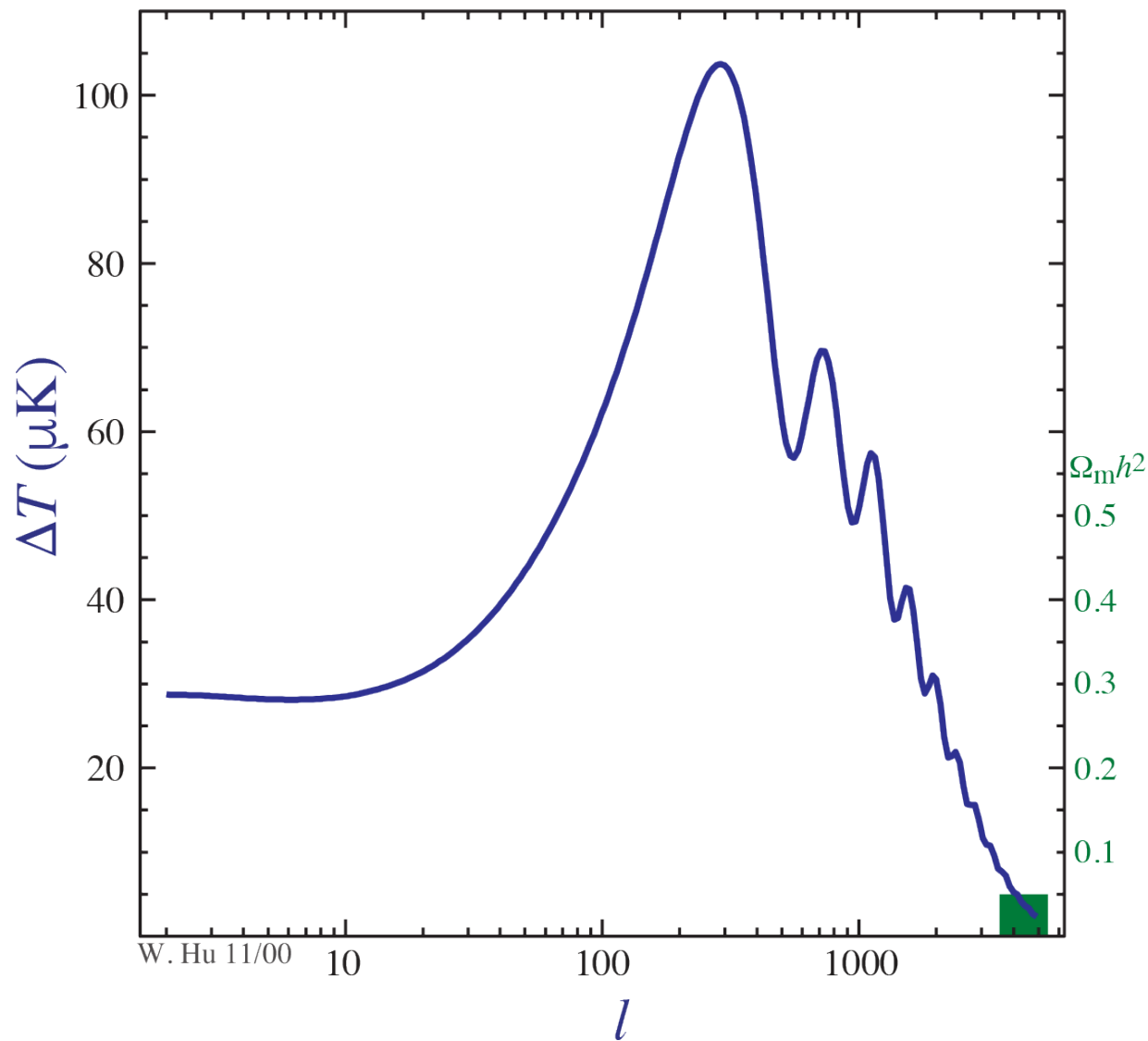
where

$$F = -\ddot{\Phi} - \frac{\dot{R}}{1 + R} \dot{\Phi} - \frac{k^2}{3} \Psi$$

- Useful if general form of potential evolution is known

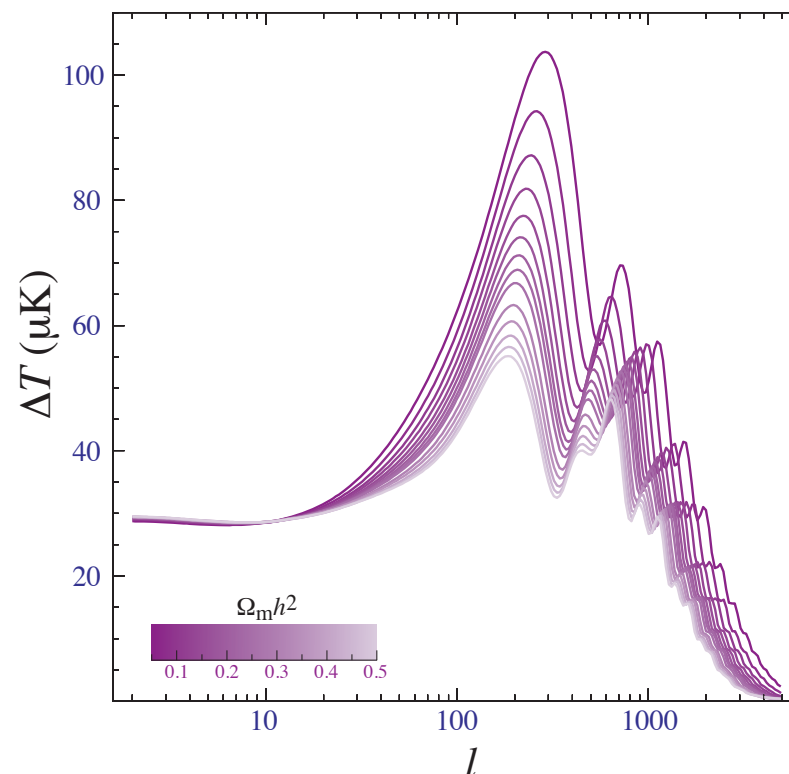
Cold Dark Matter in the CMB

- Hydrostatic equilibrium, oscillation forcing, damping



Matter-Radiation in the Power Spectrum

- Coherent approximation is **exact** for a photon-baryon fluid but reality is reduced to $\sim 4\times$ because **neutrino contribution** is free streaming not fluid like
- Neutrinos drive the oscillator less efficiently and also slightly change the phase of the oscillation
- Actual **initial conditions** are $\Theta + \Psi = \Psi/2$ for radiation domination but comparison to matter dominated SW correct
- With 3 peaks, it is possible to solve for both the baryons and dark matter densities, providing a calibration for the sound horizon
- Higher peaks check consistency with assumptions: e.g. extra relativistic d.o.f.s



Damping

- Tight coupling equations assume a **perfect fluid**: no **viscosity**, no **heat conduction**
- Fluid imperfections are related to the **mean free path** of the **photons in the baryons**

$$\lambda_C = \dot{\tau}^{-1} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a$$

is the conformal opacity to **Thomson scattering**

- Dissipation related to **diffusion length**: random walk approx

$$\lambda_D = \sqrt{N} \lambda_C = \sqrt{\eta / \lambda_C} \lambda_C = \sqrt{\eta \lambda_C}$$

the **geometric mean** between the horizon and mean free path

- $\lambda_C / \eta_* \sim \%$, so expect **peaks** > 3 to be affected by **dissipation**
- $\sqrt{\eta}$ enters here and η in the acoustic scale $\rightarrow$ expansion rate and extra relativistic species

Equations of Motion

- Continuity

$$\dot{\Theta} = -\frac{k}{3}v_\gamma - \dot{\Phi}, \quad \dot{\delta}_b = -kv_b - 3\dot{\Phi}$$

where the photon equation remains unchanged and the baryons follow number conservation with $\rho_b = m_b n_b$

- Navier-Stokes (Euler + heat conduction, viscosity)

$$\begin{aligned}\dot{v}_\gamma &= k(\Theta + \Psi) - \frac{k}{6}\pi_\gamma - \dot{\tau}(v_\gamma - v_b) \\ \dot{v}_b &= -\frac{\dot{a}}{a}v_b + k\Psi + \dot{\tau}(v_\gamma - v_b)/R\end{aligned}$$

where the photons gain an anisotropic stress term π_γ from radiation viscosity and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation

Viscosity

- **Viscosity** is generated from radiation **streaming** from hot to cold regions
- Expect

$$\pi_\gamma \sim v_\gamma \frac{k}{\dot{\tau}}$$

generated by streaming, suppressed by **scattering** in a wavelength of the fluctuation. **Radiative transfer** says

$$\pi_\gamma \approx 2A_v v_\gamma \frac{k}{\dot{\tau}}$$

where $A_v = 16/15$

$$\dot{v}_\gamma = k(\Theta + \Psi) - \frac{k}{3} A_v \frac{k}{\dot{\tau}} v_\gamma$$

Oscillator: Penultimate Take

- Adiabatic approximation ($\omega \gg \dot{a}/a$)

$$\dot{\Theta} \approx -\frac{k}{3}v_\gamma$$

- Oscillator equation contains a $\dot{\Theta}$ damping term

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} A_v \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})$$

- Heat conduction term similar in that it is proportional to v_γ and is suppressed by scattering $k/\dot{\tau}$. Expansion of Euler equations to leading order in $k\dot{\tau}$ gives

$$A_h = \frac{R^2}{1 + R}$$

since the effects are only significant if the baryons are dynamically important

Oscillator: Final Take

- Final oscillator equation

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} [A_v + A_h] \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})$$

- Solve in the adiabatic approximation

$$\Theta \propto \exp(i \int \omega d\eta)$$

$$-\omega^2 + \frac{k^2 c_s^2}{\dot{\tau}} (A_v + A_h) i\omega + k^2 c_s^2 = 0$$

Dispersion Relation

- Solve

$$\begin{aligned}\omega^2 &= k^2 c_s^2 \left[1 + i \frac{\omega}{\dot{\tau}} (A_v + A_h) \right] \\ \omega &= \pm k c_s \left[1 + \frac{i}{2} \frac{\omega}{\dot{\tau}} (A_v + A_h) \right] \\ &= \pm k c_s \left[1 \pm \frac{i}{2} \frac{k c_s}{\dot{\tau}} (A_v + A_h) \right]\end{aligned}$$

- Exponentiate

$$\begin{aligned}\exp(i \int \omega d\eta) &= e^{\pm i k s} \exp\left[-k^2 \int d\eta \frac{1}{2} \frac{c_s^2}{\dot{\tau}} (A_v + A_h)\right] \\ &= e^{\pm i k s} \exp\left[-(k/k_D)^2\right]\end{aligned}$$

- Damping is exponential under the scale k_D

Diffusion Scale

- Diffusion wavenumber

$$k_D^{-2} = \int d\eta \frac{1}{\dot{\tau}} \frac{1}{6(1+R)} \left(\frac{16}{15} + \frac{R^2}{(1+R)} \right)$$

- Limiting forms

$$\lim_{R \rightarrow 0} k_D^{-2} = \frac{1}{6} \frac{16}{15} \int d\eta \frac{1}{\dot{\tau}}$$

$$\lim_{R \rightarrow \infty} k_D^{-2} = \frac{1}{6} \int d\eta \frac{1}{\dot{\tau}}$$

- Geometric mean between horizon and mean free path as expected from a random walk

$$\lambda_D = \frac{2\pi}{k_D} \sim \frac{2\pi}{\sqrt{6}} (\eta \dot{\tau}^{-1})^{1/2}$$

Source Separation

- Decomposition into local temperature + SW, Doppler, ISW (no dark energy) in this simple semianalytic description

